Estimates for order of Nevanlinna matrices

Raphael Pruckner

raphael.pruckner@tuwien.ac.at
Vienna University of Technology

23.05.2017
Outline

Hamburger moment problem
- Moment sequences
- Jacobi matrices
- Hamburger Hamiltonians

Results concerning the order
- Livšic
- Berg, Szwarc
- Beresanskii

Estimates via the Square transform
- Square transform
- Beresanskii-type theorem
Hamburger moment problem

3 viewpoints
Moment sequences

Given a sequence of real numbers \((s_n)_{n=0}^\infty\), does there exist a positive Borel measure \(\mu\) on \(\mathbb{R}\) such that

\[ s_n = \int_{\mathbb{R}} x^n \, d\mu(x), \]

for all \(n \in \mathbb{N} \cup \{0\}\)?
Moment sequences

Given a sequence of real numbers \((s_n)_{n=0}^{\infty}\), does there exist a positive Borel measure \(\mu\) on \(\mathbb{R}\) such that

\[ s_n = \int_{\mathbb{R}} x^n \, d\mu(x), \]

for all \(n \in \mathbb{N} \cup \{0\}\)?

In the indeterminate case, i.e. when there is more than one solution \(\mu\), all solutions can be parametrized:
Theorem (Nevanlinna, 1922)

Let \((s_n)_{n=0}^{\infty}\) be an indeterminate moment sequence. Then there exists a Nevanlinna matrix \(W(z) = (w_{ij}(z))_{i,j=1}^{2}\) such that a measure \(\mu\) is a solution of the HMP if and only if

\[
\int_{\mathbb{R}} \frac{1}{t-z} d\mu(t) = -\frac{w_{11}(z)\phi(z) - w_{12}(z)}{w_{21}(z)\phi(z) - w_{22}(z)},
\]

for some Nevanlinna function \(\phi\).
### Definition

The *order* of an entire function $f$ is defined as

$$
\rho(f) := \inf \left\{ d > 0 \mid \exists c_1, c_2 > 0 : |f(z)| \leq c_1 \exp(c_2 |z|^d), z \in \mathbb{C} \right\}
$$
Definition

The *order* of an entire function $f$ is defined as

$$\rho(f) := \inf \left\{ d > 0 \mid \exists c_1, c_2 > 0 : |f(z)| \leq c_1 \exp \left( c_2 |z|^d \right), z \in \mathbb{C} \right\}$$
Definition

The order of an entire function $f$ is defined as

$$\rho(f) := \inf \left\{ d > 0 \mid \exists c_1, c_2 > 0 : |f(z)| \leq c_1 \exp \left( c_2 |z|^d \right), z \in \mathbb{C} \right\}$$

Theorem (M. Riesz, 1923, [Rie23])

The order of each entry of $W(z)$ is at most 1.
**Definition**

The *order* of an entire function $f$ is defined as

$$\rho(f) := \inf \left\{ d > 0 \mid \exists c_1, c_2 > 0 : |f(z)| \leq c_1 \exp(c_2|z|^d), \ z \in \mathbb{C} \right\}$$

**Theorem (M. Riesz, 1923, [Rie23])**

*The order of each entry of $W(z)$ is at most 1.*

**Theorem (Berg, Pedersen, 1994, [BP94])**

*All four entries of $W(z)$ have the same order.*

We refer to this common value as the order of the HMP, $\rho$. 

Jacobi matrices

Denote by \( P_n(z) \) the orthogonal polynomials of the first kind. There exists two sequences \((q_n)_{n=0}^{\infty}, (\rho_n)_{n=0}^{\infty}\) with \( q_n \in \mathbb{R}, \rho_n > 0 \) such that

\[
 zP_n(z) = \rho_n P_{n+1}(z) + q_n P_n(z) + \rho_{n-1} P_{n-1}(z), \quad n \in \mathbb{N} \\
zP_0(z) = \rho_0 P_1(z) + q_0 P_0(z).
\]
Jacobi matrices

Denote by $P_n(z)$ the orthogonal polynomials of the first kind. There exists two sequences $(q_n)_{n=0}^\infty$, $(\rho_n)_{n=0}^\infty$ with $q_n \in \mathbb{R}$, $\rho_n > 0$ such that

$$zP_n(z) = \rho_n P_{n+1}(z) + q_n P_n(z) + \rho_{n-1} P_{n-1}(z), \quad n \in \mathbb{N}$$

$$zP_0(z) = \rho_0 P_1(z) + q_0 P_0(z).$$

\[
\begin{pmatrix}
q_0 & \rho_0 & 0 \\
\rho_0 & q_1 & \rho_1 \\
\rho_1 & q_2 & \ddots \\
0 & \ddots & \ddots
\end{pmatrix}
\]

Jacobi matrices are in a one-to-one correspondence with moment sequences.
Canonical systems

We consider the canonical system

\[ y'(x) = zJH(x)y(x), \quad x \in [0, L), \]

where \( L \in (0, \infty], \ J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ z \in \mathbb{C}. \)
Canonical systems

We consider the *canonical system*

\[ y'(x) = z J H(x) y(x), \quad x \in [0, L), \]

where \( L \in (0, \infty] \), \( J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), \( z \in \mathbb{C} \).

The *Hamiltonian* \( H : [0, L) \to \mathbb{R}^{2 \times 2} \) satisfies

- \( H(x) \geq 0 \) for a.e. \( x \)
- \( H \in L^1_{\text{loc}}[0, L) \)
- \( \text{tr} \, H(x) = 1 \) for a.e. \( x \)
Canonical systems

We consider the *canonical system*

$$y'(x) = z J H(x) y(x), \quad x \in [0, L),$$

where $L \in (0, \infty)$, $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $z \in \mathbb{C}$.

The *Hamiltonian* $H : [0, L) \to \mathbb{R}^{2 \times 2}$ satisfies

- $H(x) \geq 0$ for a.e. $x$
- $H \in L^1_{\text{loc}}[0, L)$
- $\text{tr} \, H(x) = 1$ for a.e. $x$

The *fundamental solution* is the unique solution of

$$\begin{cases} \frac{d}{dx} W(x, z) J = z W(x, z) H(x), & x \in [0, L) \\ W(0, z) = I. \end{cases}$$
In the limit circle case (lcc), i.e. \( L < \infty \), the limit

\[
W(L, z) := \lim_{x \to L} W(x, z)
\]

exists locally uniformly and is called the monodromy matrix.
In the limit circle case (lcc), i.e. $L < \infty$, the limit

$$W(L, z) := \lim_{x \to L} W(x, z)$$

exists locally uniformly and is called the *monodromy matrix*.

Again, all four entries of the monodromy matrix are entire functions of the same order.

We denote this common value by $\rho(H)$, the order of the Hamiltonian.
In the limit point case (lpc), i.e. $L = \infty$, the limit

$$Q_H(z) := \lim_{x \to \infty} \frac{w_{1,1}(x, z)}{w_{2,1}(x, z)}$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and is called *Titchmarsh-Weyl coefficient*. 
In the limit point case (lpc), i.e. $L = \infty$, the limit

$$Q_H(z) := \lim_{x \to \infty} \frac{w_{1,1}(x, z)}{w_{2,1}(x, z)}$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ and is called *Titchmarsh-Weyl coefficient*.

If $Q_H$ is meromorphic, denote by $(\omega_n)$ the sequence of non-zero poles of $Q_H$, and define $\rho(H)$ as the convergence exponent of this sequence, i.e.

$$\rho(H) := \inf\{\alpha > 0 : \sum \omega_n^{-\alpha} < \infty\}.$$ 

Otherwise, set $\rho(H) := \infty$. 
Definition (Hamburger Hamiltonian)

Let \( \vec{l} = (l_n)_{n=1}^{\infty} \) and \( \vec{\phi} = (\phi_n)_{n=1}^{\infty} \) be sequences of real numbers with \( l_n > 0 \) and \( \phi_{n+1} \neq \phi_n \mod \pi, n \in \mathbb{N} \).
Definition (Hamburger Hamiltonian)

Let \( \vec{l} = (l_n)_{n=1}^{\infty} \) and \( \vec{\phi} = (\phi_n)_{n=1}^{\infty} \) be sequences of real numbers with \( l_n > 0 \) and \( \phi_{n+1} \not\equiv \phi_n \mod \pi, n \in \mathbb{N} \).

Set

\[
\begin{align*}
    x_0 &:= 0, \\
    x_n &:= \sum_{k=1}^{n} l_k, \ n \in \mathbb{N}, \\
    L &:= \sum_{k=1}^{\infty} l_k \in (0, \infty], \\
    \xi_{\phi} &:= (\cos(\phi), \sin(\phi))^T.
\end{align*}
\]
**Definition (Hamburger Hamiltonian)**

Let \( \vec{l} = (l_n)_{n=1}^{\infty} \) and \( \vec{\phi} = (\phi_n)_{n=1}^{\infty} \) be sequences of real numbers with \( l_n > 0 \) and \( \phi_{n+1} \not\equiv \phi_n \mod \pi, \ n \in \mathbb{N} \).

Set

\[
x_0 := 0, \quad x_n := \sum_{k=1}^{n} l_k, \ n \in \mathbb{N}, \quad L := \sum_{k=1}^{\infty} l_k \in (0, \infty],
\]

\[
\xi_\phi := (\cos(\phi), \sin(\phi))^T.
\]

Then we call

\[
H_{\vec{l}, \vec{\phi}}(x) := \xi_{\phi_n} \xi_{\phi_n}^T, \quad x \in [x_{n-1}, x_n), \ n \in \mathbb{N},
\]

the *Hamburger Hamiltonian* with lengths \( \vec{l} \) and angles \( \vec{\phi} \).
Theorem (Kac, [Kac99])

There is a one-to-one correspondence between the set of all HMPs and the set of all Hamburger Hamiltonians. The HMP is indeterminate if and only if the corresponding Hamburger Hamiltonian is in the lcc. In this case, the Nevanlinna matrix which describes all solutions of the HMP coincides with the monodromy matrix.
Theorem (Kac, [Kac99])

There is a one-to-one correspondence between the set of all HMPs and the set of all Hamburger Hamiltonians.
Theorem (Kac, [Kac99])

There is a one-to-one correspondence between the set of all HMPs and the set of all Hamburger Hamiltonians.

The HMP is indeterminate if and only if the corresponding Hamburger Hamiltonian is in the lcc.
Theorem (Kac, [Kac99])

There is a one-to-one correspondence between the set of all HMPs and the set of all Hamburger Hamiltonians.
The HMP is indeterminate if and only if the corresponding Hamburger Hamiltonian is in the lcc.
In this case, the Nevanlinna matrix which describes all solutions of the HMP coincides with the monodromy matrix.
Task:

Assume that one of the following parameters are given:

- the moment sequence $(s_n)_{n=0}^\infty$
- the parameters of the Jacobi-matrix $(\rho_n)_{n=0}^\infty$, $(q_n)_{n=0}^\infty$
- the parameters of the Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$, i.e.
  $(l_n)_{n=1}^\infty$, $(\phi_n)_{n=1}^\infty$
Task:

Assume that one of the following parameters are given:

- the moment sequence \((s_n)_{n=0}^{\infty}\)
- the parameters of the Jacobi-matrix \((\rho_n)_{n=0}^{\infty}, (q_n)_{n=0}^{\infty}\)
- the parameters of the Hamburger Hamiltonian \(H_{\vec{l},\vec{\phi}}\), i.e. \((l_n)_{n=1}^{\infty}, (\phi_n)_{n=1}^{\infty}\)

How large is the order \(\rho(H)\) of the corresponding Hamiltonian?
Results concerning the order
Theorem (Livšic 1939, [Liv39])

Let \((s_n)_{n=0}^\infty\) be an indeterminate moment sequence. Then

\[
\rho(H) \geq \limsup_{n \to \infty} \frac{2n \ln n}{\ln s_{2n}}.
\]
Theorem (Livšic 1939, [Liv39])

Let \((s_n)_{n=0}^{\infty}\) be an indeterminate moment sequence. Then

\[
\rho(H) \geq \limsup_{n \to \infty} \frac{2n \ln n}{\ln s_{2n}}.
\]

The gap between the left and right hand side can be arbitrarily close to 1, [PRW16].
### Theorem (Berg, Szwarc [BS14])

Let \((s_n)_{n=0}^{\infty}\) be an indeterminate moment sequence, and denote by

\[ P_n(z) = \sum_{k=0}^{n} b_{k,n} z^k, \quad n \in \mathbb{N} \cup \{0\}, \]

the orthogonal polynomials of the first kind.

\[ \rho(H) = \limsup_{k \to \infty} -\frac{2}{k \ln k} \ln \sum_{n=0}^{\infty} b_{k,n}^2. \]

This formula requires knowledge of all coefficients of all orthogonal polynomials.

Livşic' estimate follows since

\[ \sum_{n=0}^{\infty} b_{k,n}^2 \geq b_{k,k}^2 \geq s_k^2. \]
**Theorem (Berg, Szwarc [BS14])**

Let \((s_n)_{n=0}^{\infty}\) be an indeterminate moment sequence, and denote by
\[ P_n(z) = \sum_{k=0}^{n} b_{k,n} z^k, \quad n \in \mathbb{N} \cup \{0\}, \] the orthogonal polynomials of the first kind.
Then
\[
\rho(H) = \limsup_{k \to \infty} \frac{-2k \ln k}{\ln \sum_{n=k}^{\infty} b_{k,n}^2}.
\]
### Theorem (Berg, Szwarc [BS14])

Let \((s_n)_{n=0}^{\infty}\) be an indeterminate moment sequence, and denote by 

\[ P_n(z) = \sum_{k=0}^{n} b_{k,n} z^k, \quad n \in \mathbb{N} \cup \{0\}, \text{ the orthogonal polynomials of the first kind.} \]

Then 

\[ \rho(H) = \limsup_{k \to \infty} \frac{-2k \ln k}{\ln \sum_{n=k}^{\infty} b_{k,n}^2}. \]

This formula requires knowledge of all coefficients of all orthogonal polynomials.
Theorem (Berg, Szwarc [BS14])

Let \((s_n)_{n=0}^\infty\) be an indeterminate moment sequence, and denote by 
\[ P_n(z) = \sum_{k=0}^n b_{k,n} z^k, \quad n \in \mathbb{N} \cup \{0\}, \] the orthogonal polynomials of the first kind.

Then 
\[ \rho(H) = \limsup_{k \to \infty} \frac{-2k \ln k}{\ln \sum_{n=k}^\infty b_{k,n}^2}. \]

This formula requires knowledge of all coefficients of all orthogonal polynomials.

Livšic’ estimate follows since 
\[ \sum_{n=k}^\infty b_{k,n}^2 \geq b_{k,k}^2 \geq \frac{1}{s_{2k}}. \]
Theorem (Beresanskii 1968, [Ber68], [BS14])

Let $\rho_n > 0$, $q_n \in \mathbb{R}$ be the parameters of a Jacobi matrix.
Theorem (Beresanskii 1968, [Ber68], [BS14])

Let $\rho_n > 0$, $q_n \in \mathbb{R}$ be the parameters of a Jacobi matrix. Assume

\[
\sum_{n=1}^{\infty} \frac{1}{\rho_n} < \infty \quad \text{(Carleman condition)}
\]

\[
\rho_n^2 \geq \rho_{n-1}\rho_{n+1} \quad \text{or} \quad \rho_n^2 \leq \rho_{n-1}\rho_{n+1} \quad \text{(log-concave/convex)}
\]

\[
\left(\frac{q_n}{\rho_n}\right)_{n=1}^{\infty} \in \ell^1 \quad \text{(small diagonal)}
\]
Theorem (Beresanskii 1968, [Ber68], [BS14])

Let $\rho_n > 0$, $q_n \in \mathbb{R}$ be the parameters of a Jacobi matrix. Assume

$\sum_{n=1}^{\infty} \frac{1}{\rho_n} < \infty$ \hspace{1cm} (Carleman condition)

$\rho_n^2 \geq \rho_{n-1}\rho_{n+1}$ \hspace{1cm} or $\rho_n^2 \leq \rho_{n-1}\rho_{n+1}$ \hspace{1cm} (log-concave/convex)

$\left(\frac{q_n}{\rho_n}\right)_{n=1}^{\infty} \in \ell^1$ \hspace{1cm} (small diagonal)

Then the corresponding moment problem is indeterminate and

$\rho(H) = \inf \left\{ \alpha > 0 : (\rho_n^{-1})_{n=1}^{\infty} \in l^\alpha \right\}$,

i.e. the order is equal to the convergence exponent of $(\rho_n)_{n=1}^{\infty}$. 
Estimates via the Square transform

Joint work with Harald Woracek
A Hamburger Hamiltonians $H_{\vec{l},\vec{\phi}}$ is a.e. diagonal if $\phi_n \in \{0, \frac{\pi}{2}\}$ modulo $\pi$. After a normalisation we can assume

$$\vec{\phi} = (0, \frac{\pi}{2}, 0, \frac{\pi}{2} \ldots) := \vec{\delta}.$$
A Hamburger Hamiltonians $H_{\vec{l},\vec{\phi}}$ is a.e. diagonal if $\phi_n \in \{0, \frac{\pi}{2}\}$ modulo $\pi$. After a normalisation we can assume

$$\vec{\phi} = (0, \frac{\pi}{2}, 0, \frac{\pi}{2} \ldots) := \vec{\delta}.$$ 

Diagonal Hamiltonians can be viewed as Krein strings. In this case the order is in principal known due to Kacs, cf. [Kac86, Theorems A–C].
A Hamburger Hamiltonians $H_{\vec{l},\vec{\phi}}$ is a.e. diagonal if $\phi_n \in \{0, \frac{\pi}{2}\}$ modulo $\pi$. After a normalisation we can assume

$$\vec{\phi} = (0, \frac{\pi}{2}, 0, \frac{\pi}{2}, \ldots) := \vec{\delta}.$$ 

Diagonal Hamiltonians can be viewed as Krein strings. In this case the order is in principal known due to Kacs, cf. [Kac86, Theorems A–C].

**Idea in a nutshell**

Transform an arbitrary Hamburger Hamiltonian to a diagonal one, use Kacs’ Theorem to get the order, and try to drag the information back to the original Hamiltonian.
How to transform?

A Stieltjes string can be written as a canonical system in two different ways, either with a

- Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$, with monotonic increasing angles $\vec{\phi}$,
A Stieltjes string can be written as a canonical system in two different ways, either with a

- Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$, with monotonic increasing angles $\vec{\phi}$,
- or with a diagonal Hamburger Hamiltonian.
How to transform?

A Stieltjes string can be written as a canonical system in two different ways, either with a

- Hamburger Hamiltonian $H_{\vec{l},\vec{\phi}}$, with monotonic increasing angles $\vec{\phi}$,
- or with a diagonal Hamburger Hamiltonian.

It is possible to transform these Hamiltonian into each other, cf. [KWW07].
Given $H_{l,\phi}$ with monotonic increasing angles $\vec{\phi}$. Set

\[ m_n := \cot \phi_n - \cot \phi_{n-1}, \quad h_n := l_n \sin^2(\phi_n), \]
Given $\vec{H}_{\vec{l},\vec{\phi}}$ with monotonic increasing angles $\vec{\phi}$. Set

$$m_n := \cot \phi_n - \cot \phi_{n-1}, \quad h_n := l_n \sin^2(\phi_n),$$

and,

$$\vec{m} : \vec{h} := (m_1, h_1, m_2, h_2, \ldots), \quad \vec{\delta} := (0, \frac{\pi}{2}, 0, \ldots).$$

$\vec{H}_{\vec{m} : \vec{h}, \vec{\delta}}$ is called the Square transform of $\vec{H}_{\vec{l},\vec{\phi}}$. 
Given $H_{\vec{l}, \vec{\phi}}$ with monotonic increasing angles $\vec{\phi}$. Set

$$m_n := \cot \phi_n - \cot \phi_{n-1}, \quad h_n := l_n \sin^2(\phi_n),$$

and,

$$\vec{m} : \vec{h} := (m_1, h_1, m_2, h_2, \ldots), \quad \vec{\delta} := (0, \frac{\pi}{2}, 0, \ldots).$$

$H_{\vec{m} : \vec{h}, \vec{\delta}}$ is called the Square transform of $H_{\vec{l}, \vec{\phi}}$.

Idea

Use this formula to define the Square transform for general Hamburger Hamiltonians $H_{\vec{l}, \vec{\phi}}$. 
**Definition (Hamburger Hamiltonian)**

Let \( \vec{l} = (l_n)_{n=1}^{\infty} \) and \( \vec{\phi} = (\phi_n)_{n=1}^{\infty} \) be sequences of real numbers with \( l_n > 0 \) and \( \phi_{n+1} \not\equiv \phi_n \mod \pi, \ n \in \mathbb{N} \). Set

\[
x_0 := 0, \quad x_n := \sum_{k=1}^{n} l_k, \ n \in \mathbb{N}, \quad L := \sum_{k=1}^{\infty} l_k \in (0, \infty].
\]

Then we call

\[
H_{\vec{l},\vec{\phi}}(x) := \xi_{\phi_n} \xi_{\phi_n}^T, \quad x \in [x_{n-1}, x_n), \ n \in \mathbb{N},
\]

the *Hamburger Hamiltonian* with *lengths* \( \vec{l} \) and *angles* \( \vec{\phi} \).
Definition (signed Hamburger Hamiltonian)

Let $\vec{l} = (l_n)_{n=1}^{\infty}$ and $\vec{\phi} = (\phi_n)_{n=1}^{\infty}$ be sequences of real numbers with $l_n \neq 0$ and $\phi_{n+1} \not\equiv \phi_n \mod \pi$, $n \in \mathbb{N}$. Set

\[ x_0 := 0, \quad x_n := \sum_{k=1}^{n} |l_k|, \quad n \in \mathbb{N}, \quad L := \sum_{k=1}^{\infty} |l_k| \in (0, \infty]. \]

Then we call

\[ H_{\vec{l},\vec{\phi}}(x) := \text{sgn}(l_n) \xi_{\phi_n} \xi_{\phi_n}^T, \quad x \in [x_{n-1}, x_n), \quad n \in \mathbb{N}, \]

the *signed* Hamburger Hamiltonian with lengths $\vec{l}$ and angles $\vec{\phi}$. 
$H_{\vec{l}, \vec{\phi}}: x_0 \mid l_1 \mid x_1 \mid l_2 \mid x_2 \mid l_3 \mid x_3 \mid L$

$\text{sgn}(l_1)\xi_1 \xi_1^* \text{sgn}(l_2)\xi_2 \xi_2^* \text{sgn}(l_3)\xi_3 \xi_3^*$
Definition (Square transform)

Let $H_{\vec{l},\vec{\phi}}$ be a Hamburger Hamiltonian. Set

$$m_n := \cot \phi_n - \cot \phi_{n-1}, \quad h_n := l_n \sin^2(\phi_n),$$

and, as before,

$$\vec{m} : \vec{h} := (m_1, h_1, m_2, h_2, \ldots), \quad \vec{\delta} := (0, \frac{\pi}{2}, 0, \ldots).$$

We call the signed Hamburger Hamiltonian $H_{\vec{m} : \vec{h}, \vec{\delta}}$ the Square transform of $H_{\vec{l},\vec{\phi}}$. 
Theorem (1; P., Woracek)

Let $H_{\vec{l},\vec{\phi}}$ be a Hamburger Hamiltonian in the lcc with $\phi_n \not\equiv 0 \mod \pi, n \in \mathbb{N}$.

Denote by $H_{\vec{m},\vec{\delta}}$ be the Square transform of $H_{\vec{l},\vec{\phi}}$. 
**Theorem (1; P., Woracek)**

Let $H_{\vec{l},\vec{\phi}}$ be a Hamburger Hamiltonian in the lcc with $\phi_n \not\equiv 0 \mod \pi$, $n \in \mathbb{N}$.

Denote by $H_{\vec{m} : \vec{h}, \vec{\delta}}$ be the Square transform of $H_{\vec{l},\vec{\phi}}$, and set

$$|\vec{m} : \vec{h}| := (|m_1|, h_1, |m_2|, h_2, \ldots).$$
Theorem (1; P., Woracek)

Let $H_{\vec{l}, \vec{\phi}}$ be a Hamburger Hamiltonian in the lcc with $\phi_n \not\equiv 0 \mod \pi$, $n \in \mathbb{N}$.

Denote by $H_{\vec{m} : \vec{h}, \vec{\delta}}$ be the Square transform of $H_{\vec{l}, \vec{\phi}}$, and set

$$|\vec{m} : \vec{h}| := (|m_1|, h_1, |m_2|, h_2, \ldots).$$

Then

$$\rho(H_{\vec{l}, \vec{\phi}}) \leq \frac{1}{2} \rho(H_{|\vec{m} : \vec{h}|, \vec{\delta}}).$$
### sketch of proof

| Hamiltonian | $H_{\vec{l},\vec{\phi}}$ | $H_{\vec{m}:\vec{h},\vec{\delta}}$ | $H_{|\vec{m}:\vec{h}|,\vec{\delta}}$ |
|-------------|----------------|-------------------------------|----------------------------------|
| fundamental solution | $W(x, z)$ | $V(x, z)$ | $V^+(x, z)$ |
sketch of proof

| Hamiltonian                  | $H_{\vec{l}, \vec{\phi}}$ | $H_{\vec{m}:\vec{h}, \vec{\delta}}$ | $H_{|\vec{m}:\vec{h}|, \vec{\delta}}$ |
|-----------------------------|-----------------------------|---------------------------------------|---------------------------------------|
| fundamental solution        | $W(x, z)$                   | $V(x, z)$                             | $V^+(x, z)$                           |

\[ |W_{11}(x_n, z^2)| = |V_{11}(y_{2n}, z)| \leq \]
\[ \leq V_{11}^+(y_{2n}, i|z|) \leq C \exp \left( |z|^{\rho(H_{|\vec{m}:\vec{h}|, \vec{\delta}}) + \epsilon} \right) \]
### sketch of proof

| Hamiltonian       | $H_{\vec{l},\vec{\phi}}$ | $H_{\vec{m}:\vec{h},\vec{\delta}}$ | $H_{|\vec{m}:\vec{h}|,\vec{\delta}}$ |
|-------------------|---------------------------|------------------------------------|------------------------------------|
| fundamental solution | $W(x,z)$                 | $V(x,z)$                           | $V^+(x,z)$                         |

\[ |W_{11}(x_n, z^2)| = *|V_{11}(y_{2n}, z)| \leq \]
\[ \leq V_{11}^+(y_{2n}, i|z|) \leq C \exp \left(|z|^{\rho(H_{|\vec{m}:\vec{h}|,\vec{\delta}})+\epsilon}\right) \]

*: We know how the fundamental solution changes when applying the square transform, cf. [KWW06].
### Sketch of Proof

| Hamiltonian          | $H_{\phi,\bar{\phi}}$ | $H_{\bar{m}:\bar{h},\bar{\delta}}$ | $H_{|\bar{m}:\bar{h}|,\bar{\delta}}$ |
|----------------------|------------------------|-------------------------------------|----------------------------------------|
| Fundamental Solution | $W(x, z)$              | $V(x, z)$                           | $V^+(x, z)$                           |

$$\left| W_{11}(x_n, z^2) \right| = \left| V_{11}(y_{2n}, z) \right| \leq \ast$$

$$\leq \ast V_{11}^+ (y_{2n}, z) \leq C \exp \left( |z|^{\rho(H_{|\bar{m}:\bar{h}|,\bar{\delta}}) + \epsilon} \right)$$

*: Based on a purely algebraic property, seen by inductively calculating the fundamental solutions at $y_{2n}$. 
Hamburger moment problem
Results concerning the order
Estimates via the Square transform
References

---

**Sketch of Proof**

| Hamiltonian          | $H_{\vec{l},\vec{\phi}}$ | $H_{\vec{m}:\vec{h},\vec{\delta}}$ | $H_{|\vec{m}:\vec{h}|,\vec{\delta}}$ |
|----------------------|---------------------------|-------------------------------------|-------------------------------------|
| **Fundamental solution** | $W(x, z)$                | $V(x, z)$                           | $V^+(x, z)$                        |

\[
|W_{11}(x_n, z^2)| = |V_{11}(y_{2n}, z)| \leq \\
\leq V_{11}^+(y_{2n}, i|z|) \leq * C \exp \left( |z|^{\rho(H_{|\vec{m}:\vec{h}|,\vec{\delta}})+\epsilon} \right)
\]

*: We use an operator theoretic argument.
**sketch of proof**

| Hamiltonian | $H_{\vec{r}, \vec{\phi}}$ | $H_{\vec{m} : \vec{h}, \vec{\delta}}$ | $H_{|\vec{m} : \vec{h}|, \vec{\delta}}$ |
|-------------|----------------------------|-------------------------------------|-------------------------------------|
| fundamental solution | $W(x, z)$ | $V(x, z)$ | $V^+(x, z)$ |

$$|W_{11}(x_n, z^2)| = |V_{11}(y_{2n}, z)| \leq$$

$$\leq V_{11}^+(y_{2n}, i|z|) \leq C \exp \left( |z|^{\rho(H_{|\vec{m} : \vec{h}|, \vec{\delta}}) + \epsilon} \right)$$

Taking the limit $n \to \infty$ completes the proof.
The order of a general string can be calculated using Kac’ formula, [Kac86, Theorems A–C].
The order of a general string can be calculated using Kac’ formula, [Kac86, Theorems A–C].

For Stieltjes strings the expressions are simpler, cf. [Kac90, p.31 (15)], but still very complicated.
The order of a general string can be calculated using Kac’ formula, [Kac86, Theorems A–C].

For Stieltjes strings the expressions are simpler, cf. [Kac90, p.31 (15)], but still very complicated.

Under some additional regularity assumptions, it was shown in [Kac90] that the expression can be handled.

Combining Theorem 1 with this formula gives rise to the following theorem.
Theorem (2; P., Woracek)

Let $H_{l,\phi}$ be a Hamburger Hamiltonian in the lcc.
Assume $\phi_n \not\equiv 0 \mod \pi$, $n \in \mathbb{N}$,

- $|\cot(\phi_{n+1}) - \cot(\phi_n)|$ is nondecreasing and bounded,
- $(\ln \sin^2 \phi_n)_{n=1}^{\infty}$ is monotonically decreasing, and

$$\sum_{n=1}^{\infty} [\ln \sin^2 \phi_n]^\frac{1}{2} \ln n < \infty.$$
Theorem (2; P., Woracek)

Let $H_{\vec{l},\vec{\phi}}$ be a Hamburger Hamiltonian in the lcc.
Assume $\phi_n \not\equiv 0 \mod \pi$, $n \in \mathbb{N}$,

- $|\cot(\phi_{n+1}) - \cot(\phi_n)|$ is nondecreasing and bounded,
- $(l_n \sin^2 \phi_n)_{n=1}^\infty$ is monotonically decreasing, and

$$\sum_{n=1}^\infty [l_n \sin^2 \phi_n]^{\frac{1}{2}} \ln n < \infty.$$

Then

$$\rho(H_{\vec{l},\vec{\phi}}) = \inf \{ \alpha > 0 : (l_n \sin^2 \phi_n)_{n=1}^\infty \in l^\alpha \},$$

i.e., the order of $H_{\vec{l},\vec{\phi}}$ equals the convergence exponent of

$$([l_n \sin^2 \phi_n]^{-1})_{n=1}^\infty.$$
In the present paper we establish an upper estimate for the order of a Hamburger Hamiltonian; see Theorem 4.1, which is our first main result. This estimate is incomparable with the one obtained recently in [PRW16]; in some cases it is better and in some others it is worse, cf. Proposition 4.

The proof of Theorem 4.1 is achieved by associating with the given Hamburger Hamiltonian a certain (singular) Krein-string. During this process several different types of arguments come into play. Our method relies on an operator theoretic limiting argument (Proposition 2.5), some purely algebraic computations and transformations (§3), and estimates for canonical products by means of the density of their zeroes. Moreover, on the way, we leave the positive definite scheme and encounter Hamiltonians which may take negative semidefinite matrices as values.

Our second main result is Theorem 4.3 where we discuss a class of Hamiltonians whose order can be determined. We consider a Hamburger Hamiltonian $H$ whose angles $\phi_n$ (up to a small deviation) walk on the grid $\arccot(Z)$:

$\phi_n = \arccot(Zn) \pmod{\pi}$

and assume that lengths $l_n := x_n - x_{n-1}$ and angles together decay sufficiently rapidly (the series $\sum_{n=1}^{\infty} \left[l_n \sin^2 \phi_n\right]^{1/2} \ln n$ should converge) and regularly (the sequence $l_n \sin^2 \phi_n$ should be nonincreasing). The conclusion then is that the order of $w_{ij}(L, z)$ is equal to the convergence exponent of $(l_n^{-1})_n$.

The proof is obtained by evaluating the upper estimate Theorem 4.1 with help of [Kac90], and combining this with a lower estimate from [PRW16]. Theorem 4.3 can be seen as a generalisation (for orders $\leq 1/2$) of a theorem of Yu.M.Berezanskii. In the language of Hamburger Hamiltonians the essence of Berezanskii's theorem can be phrased as follows: Consider a Hamburger Hamiltonian whose angles alternate between two values:

$\psi_1 \psi_2$

If lengths decay regularly (the sequence $l_n^{-2/l_n}$ should be monotone), then the order of $w_{ij}(L, z)$ is equal to the convergence exponent of $(l_n^{-1})_n$.2
The main case of Beresanskii’s Theorem is the case of a zero diagonal.
The main case of Beresanskii’s Theorem is the case of a zero diagonal.

Theorem (essence of Beresanskii, Hamiltonian version)

Let $H_{\vec{l},\vec{\phi}}$ be a Hamburger Hamiltonian in the lcc, such that

- $\vec{\phi} = (0, \frac{\pi}{2}, 0, \frac{\pi}{2}, 0, \ldots)$ ("jumping angles")
- $\frac{l_{n+1}}{l_{n-1}}$ is increasing or decreasing.
The main case of Beresanskii’s Theorem is the case of a zero diagonal.

**Theorem (essence of Beresanskii, Hamiltonian version)**

Let \( H_{\vec{l},\vec{\phi}} \) be a Hamburger Hamiltonian in the lcc, such that

- \( \vec{\phi} = (0, \frac{\pi}{2}, 0, \frac{\pi}{2}, 0, \ldots) \) ("jumping angles")
- \( \frac{l_{n+1}}{l_{n-1}} \) is increasing or decreasing.

Then

\[
\rho(H_{\vec{l},\vec{\phi}}) = \inf \{ \alpha > 0 : (l_n)^\infty_{n=1} \in l^\alpha \},
\]

i.e., the order of \( H_{\vec{l},\vec{\phi}} \) equals the convergence exponent of \( ((l_n)^{-1})^\infty_{n=1} \).
Estimates via the Square transform

References

Hamburger moment problem

Results concerning the order

Estimates via the Square transform

References
Following I.S. Kac, we call such Hamiltonians Hamburger Hamiltonians. In the present paper we establish an upper estimate for the order of a Hamburger Hamiltonian. If lengths decay regularly (the sequence $\psi_2$) is equal to the convergence exponent of $\sum_{n=1}^{\infty} l^{-n}$, then the order of $w_{ij}(L, z)$ is equal to the convergence exponent of $\left(\frac{1}{n}\right)^{\infty}_{n=1}$. If lengths decay regularly (the series $\phi_2$) is equal to the convergence exponent of $\left[\begin{array}{c} 1 \\ \pi \\ 0 \end{array}\right]$ can be seen as a generalisation (for orders $\leq 1$) of a theorem $3$ where we discuss a class of Hamiltonians whose order can be determined. We consider a Hamburger Hamiltonian $H$ whose angles alternate between two values: $\phi_n$. Moreover, on the way, we leave the positive definite scheme and encounter Hamiltonians which may take negative semidefinite matrices as values. The proof of Theorem 4 is obtained by evaluating the upper estimate Theorem 4. Our second main result is Theorem 4, which is our first main result. This estimate is incomparable with the one obtained recently in [PRW16]; in some cases it is better and in some others it is worse, cf. Proposition 4 estimate.
It would be nice to be able to

- generalize this to general Hamiltonians.
It would be nice to be able to

- generalize this to general Hamiltonians.
- find a way to really work with the negativity.


