Order and type of canonical systems. A survey.

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In this talk we understand by a canonical system an equation of the form

\[ y'(x) = zJH(x)y(x), \quad x \in (0, L), \]

where \( H : (0, L) \to \mathbb{R}^{2 \times 2} \) is locally integrable, \( H(x) \geq 0 \) a.e., where \( J \) is the symplectic matrix \( J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), and where \( z \in \mathbb{C} \) is the eigenvalue parameter.

Assuming that \( H \) is integrable up to the endpoint 0, there exists a fundamental solution matrix \( W(x, z) = (w_{ij}(x, z))_{i,j=1}^{2}, x \in [0, L] \), starting from the initial value \( W(0, z) = I \). The entries of \( W(x, \cdot) \) are entire functions of bounded type in the half-planes \( \mathbb{C}^{\pm} \), in particular they are of finite exponential type. Their type can be computed by means of the Krein–de Branges formula

\[
\text{exponential type of } w_{ij}(x, \cdot) = \int_{0}^{x} \sqrt{\det H(y)} \, dy, \quad i, j = 1, 2,
\]

in fact their exact asymptotics w.r.t. exponential growth are

\[
\lim_{r \to \infty} \frac{1}{r} \log |w_{ij}(x, re^{i\theta})| = \left( \int_{0}^{x} \sqrt{\det H(y)} \, dy \right) \cdot \sin \theta, \quad \theta \in (0, \pi) \cup (\pi, 2\pi).
\]

A much more difficult problem than computing exponential type is to determine the exact order of the functions \( w_{ij}(x, \cdot) \), or to compute their type w.r.t. a growth function different from \( r \). Of course, the interesting case is that \( \det H(x) = 0 \) a.e., since otherwise the above formula completely solves the problem.

In this talk I give an introduction to this circle of ideas, present several theorems (of different authors), and give some examples. Results range from a lower estimate for order of M.S.Livšič dating back to 1939 to an upper estimate of R.V.Romanov published in 2016. Particular attention is paid to canonical systems arising from Hamburger moment problems and Krein strings. For such there is, most prominently, work of I.S.Kac from the 1980’s giving a formula for the order of a string, and recent work of C.Berg and R.Szwarc about order of moment problems and birth-and-death processes. Concerning my own contribution to the topic, I present several recent result obtained partially in collaboration with A.D.Baranov, R.V.Romanov, and R.Pruckner.