Definitizable normal linear operators on Krein spaces

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Outline

Preliminaries
- Spectral Theorem of normal operators on Hilbert Spaces
- Krein spaces
- Selfadjoint definitizable operators

Dragging operators
- Embedding of Hilbert space in Krein space
- Embedding induced by definitizable operator

Zero-dimensional Ideals
- Structure of zero-dimensional Ideals
- Proper Function Class

Functional Calculus for normal, definitizable operators
- A Function class
- Functional Calculus
- Spectrum of $N$
Preliminaries

Dragging operators

Zero-dimensional Ideals

Functional Calculus for normal, definitizable operators
Spectral Theorem of normal operators on Hilbert Spaces

Recall: Given a normal $N = A + iB \in B(\mathcal{H})$, i.e. $NN^* = N^*N$, 

• there exists unique spectral measure $E: \{\text{Borel-subsets}\} \rightarrow B(H)$ with support $\sigma(N)$ and $N = \int zdE(z)$;

• for $\phi: \sigma(N) \rightarrow \mathbb{C}$ bounded and measurable define, i.e. $\phi \in B(\sigma(N))$

\[ \phi(N) := \int \phi(z) dE(z) \in \{N, N^*\}' \subseteq B(H) ; \]

• $\phi \mapsto \phi(N)$ is $\ast$-homomorphism s.t. for $\phi(z) = s(\text{Re}z, \text{Im}z)$ with $s \in \mathbb{C}[x,y]$ we have $\phi(N) = s(A,B)$, hence functional calculus;

• reobtain $E$ by $E(\Delta) = \frac{1}{\Delta}(N)(\Delta)$.
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- reobtain $E$ by $E(\Delta) = 1_\Delta(N)$. 
Krein Spaces

Recall:

- \((\mathcal{K}, \langle ., . \rangle)\) is Krein space, if \([., .] \) is hermitian sesquilinear form on vector space \(\mathcal{K}\) over \(\mathbb{C}\) such that
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  \mathcal{K} = \mathcal{K}_+ [+] \mathcal{K}_- ,
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  where \((\mathcal{K}_+, [., .])\) and \((\mathcal{K}_-, [-., .])\) are Hilbert spaces;
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- such decomposition induces Hilbert space scalar product by
  \[(x, y)_{\mathcal{K}_+, \mathcal{K}_-} := [x_+, y_+] - [x_-, y_-] \text{ on } \mathcal{K} ;\]
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- there are many such decomposition; corresponding \((.,.)_{\mathcal{K}_+,\mathcal{K}_-}\) are equivalent to each other;
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- unique Hilbert space topology on \(\mathcal{K}\);
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- there are many such decomposition; corresponding \((., .)_{\mathcal{K}_+ , \mathcal{K}_-}\) are equivalent to each other;

- unique Hilbert space topology on \(\mathcal{K}\);

- for \(N \in B(\mathcal{K}_1, \mathcal{K}_2)\) Krein space adjoint \(N^[*] \in B(\mathcal{K}_2, \mathcal{K}_1)\) defined by
  \[
  [Nx, y]_2 = [x, N^[*]y]_1.
  \]
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- few hope even for general selfadjoint operators in Krein spaces;
- in the early 1980’s Heinz Langer gave some sort of Spectral Theorem for definitizable selfadjoint operators:
Selfadjoint definitizable operators

Theorem (Heinz Langer)

Let \( A \in B(K) \) be selfadjoint and definitizable, i.e. \( A = A^\star \) and
\[ p(A)x, x \geq 0 \text{ for all } x \in K \text{ for some } 0 \neq p \in \mathbb{R}[z]. \] Then:
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- $\sigma(A) \subseteq \mathbb{R} \cup p^{-1}\{0\}$; $\sigma(A)$ is symmetric wrt. $\mathbb{R}$;
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- for Borel-subset $\Delta \subseteq \mathbb{R}$ with $p^{-1}\{0\} \cap \partial \Delta = \emptyset$ there exists $E(\Delta) = E(\Delta)^{[*]} \subseteq \{A\}'' \subseteq B(\mathcal{K})$ (bi-commutant of $\{A\}$) with $\sigma(A|_{\text{ran } E(\Delta)}) \subseteq \overline{\Delta}$;
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- Riesz-projection $E(\{z\})$ corresponding to $z \in p^{-1}\{0\} \setminus \mathbb{R}$ satisfies $E(\{z\})^{[*]} = E(\{\overline{z}\})$ and $\text{ran } E(\{z\})$ is neutral, i.e. $[x, x] = 0$, $x \in \text{ran } E(\{z\})$. 

Main idea of the proof: $E(\Delta)$ were obtained with contour integrals around $\Delta$. 
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What is the proper definition for definitizability for normal operators?
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First try:

**Definition**

A normal \( N \in B(\mathcal{K}) \) is called definitizable, if the selfadjoint operators \( A := \frac{N+N^*}{2} \) and \( B := \frac{N-N^*}{2i} \) are definitizable in the sense that there exist \( p, q \in \mathbb{R}[z] \setminus \{0\} \) such that

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[p(A)x, x] \geq 0 \quad \text{and} \quad [q(B)x, x] \geq 0 \quad \text{for all} \quad x \in \mathcal{K}.
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With this straightforward definition it was possible to derive a functional calculus.
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With this straight forward definition it was possible to derive a functional calculus.

There are unsatisfactory phenomenons with this concept of definitizability. For example, it could be that a bijective $N = A + iB$ is definitizable in the above sense, but $N^{-1}$ is not.
Second try:

**Definition**

For a normal $N \in B(K)$ we call $p(x, y) \in \mathbb{R}[x, y]$ a definitizing polynomial for $N$, if

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and $N$ definitizable if there exist non-zero definitizing polynomials.
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*Rest of the talk is devoted to a Spectral Theorem for definitizable normal operators with a zero-dimensional $I$, i.e. \( \dim \mathbb{C}[x, y]/I < \infty \).*
Method of Dragging Operators
Embedding of Hilbert space in Krein space

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{K}$ be a Krein space, and let $T : \mathcal{H} \to \mathcal{K}$ be bounded linear embedding (injective).
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$$
\begin{array}{c}
\mathcal{H} \\
\downarrow^T \\
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\end{array}
\]

**Definition**

For \( C \in B(\mathcal{K}) \) define

\[
\Theta(C) := T^{-1}CT = (T \times T)^{-1}(C),
\]

where \( C \) is identified with its graph viewed as a subspace of \( \mathcal{K} \times \mathcal{K} \), i.e. as a linear relation.
Embedding of Hilbert space in Krein space

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Embedding of Hilbert space in Krein space

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**Theorem (R.Pruckner,K.;M.Dritschel,J.Rovnyak)**

$\Theta : C \mapsto T^{-1}CT$ constitutes $\ast$-Algebra Homomorphism

$$\Theta : \{TT^*\}' \to \{T^*[T]\}' \subset B(K) \subset B(H)$$
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satisfying $\Theta(I_{\mathcal{K}}) = I_{\mathcal{H}}$, $\Theta(TT[*]) = T[*]T$ and $\Theta(C)T[*] = T[*]C$ for all $C \in \{TT[*]\}'$. 
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In general $\Theta(C')$ is a not everywhere defined operator. But if $C(TT^\ast) = (TT^\ast)C$, then $\Theta(C) \in B(\mathcal{H})$.

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for all $C \in \{TT^\ast\}'$.

*In particular:* $\Theta(N)$ is normal if $N \in \{TT^\ast\}'$ is normal.
Embedding of Hilbert space in Krein space

There is a mapping in the other direction:

**Theorem**

\[ \Xi : B(\mathcal{H}) \rightarrow B(\mathcal{K}) \]  
\[ defined \ by \: \Xi(D) := TDT^\ast \]  
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- **is linear and satisfies** \[ \Xi(D)^* = \Xi(D^*) \];
- **maps** \( \{T^*T\}' \) **into** \( \{TT^*\}' \) **and** \( \{T^*T\}'' \) **into** \( \{TT^*\}'' \).
Embedding of Hilbert space in Krein space

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For \( D, D_1 \in \{T[*]T\}' \), \( C \in \{TT[*]\}' \) we have:
- \( \Xi(DD_1 T[*]T) = \Xi(D) \Xi(D_1) \);
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- \( \Xi(D \Theta(C)) = \Xi(D)C, \Xi(\Theta(C) \ D) = C\Xi(D) \);
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- \( \Xi \circ \Theta(C) = TT^* C = C TT^* \).
Embedding of Hilbert space in Krein space
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\[ \mathcal{H} \xrightarrow{T} \mathcal{K} \]

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- Given a definitizable normal $N = A + iB \in B(K)$ with definitizing $p_1, \ldots, p_m \in \mathbb{R}[z]$, define

$$\langle x, y \rangle := \left[ \sum_{k} p_k(A, B) \right] x, y, \quad x, y \in K;$$
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- Let \( \mathcal{H} \) be the completion of \( \mathcal{K}/\{x \in \mathcal{K} : \langle x, x \rangle = 0\} \) wrt. \( \langle ., . \rangle \);
- \( \iota : \mathcal{K} \rightarrow \mathcal{H} \) shall be \( x \mapsto x + \{x \in \mathcal{K} : \langle x, x \rangle = 0\} \);
- \( T := \iota^* : \mathcal{H} \rightarrow \mathcal{K} \);
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• Given a definitizable normal \( N = A + iB \in B(\mathcal{K}) \) with definitizing \( p_1, \ldots, p_m \in \mathbb{R}[z] \), define

\[
\langle x, y \rangle := \left( \sum_k p_k(A, B) \right)x, y, \quad x, y \in \mathcal{K} ;
\]

• \( \langle ., . \rangle \) positive semidefinite scalar product on \( \mathcal{K} \), continuous wrt. natural Hilbert space topology on \( \mathcal{K} \);

• Let \( \mathcal{H} \) be the completion of \( \mathcal{K}/\{x \in \mathcal{K} : \langle x, x \rangle = 0\} \) wrt. \( \langle ., . \rangle \);

• \( \iota : \mathcal{K} \to \mathcal{H} \) shall be \( x \mapsto x + \{x \in \mathcal{K} : \langle x, x \rangle = 0\} \);

• \( T := \iota^* : \mathcal{H} \to \mathcal{K} \);

• \( T \) is injective due to

\[
\ker T = \iota(\mathcal{K})^\perp = \mathcal{K}/\{x \in \mathcal{K} : \langle x, x \rangle = 0\}^\perp = \{0\}.
\]
Embedding induced by definitizable operator

\[ \mathcal{H} \text{ and } T \text{ constructed from } [\left( \sum_k p_k(A, B) \right),. , .] \]
Embedding induced by definitizable operator

Construct $\mathcal{H}_k$ and $T_k$ in the same way from $[p_k(A, B), .]$
Embedding induced by definitizable operator

exist contractions \( R_k : \mathcal{H}_k \rightarrow \mathcal{H} \) with \( \sum_k R_k R_k^* = I_\mathcal{H} \)
Embedding induced by definitizable operator

Proposition

\[ TT^*[\ast] = \sum_k p_k(A, B) \ (\in B(\mathcal{K})) \text{ and } N = A + iB \in \{TT^*[\ast]\}' \]
\[ T_k T_k^*[\ast] = p_k(A, B) \ (\in B(\mathcal{K})) \text{ and } N = A + iB \in \{T_k T_k^*[\ast]\}' \]
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Therefore:

- \( \Theta(N) \) and \( \Theta_k(N) = T_k^{-1} NT_k \) bounded and normal on \( \mathcal{H}, \mathcal{H}_k \);
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- \( \Theta_k(N) \) has spectral measure
  \[ E_k : \{\text{‘Borel-subsets’ of } \mathbb{C}\} \rightarrow B(\mathcal{H}_k); \]
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- \( \Theta_k(N) \) has spectral measure \( E_k : \{'Borel-subsets' \text{ of } \mathbb{C}\} \rightarrow B(\mathcal{H}_k) \);
- \( \phi \mapsto \int \phi(z) dE_k(z) \) is \(*\)-algebra homomorphism from \( \{ \phi : \mathbb{C} \rightarrow \mathbb{C} | \phi \text{ bounded, measurable} \} \) into \( B(\mathcal{H}_k) \).
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- \( \phi \mapsto \int \phi(z) \, dE_k(z) \) is *-algebra homomorphism from \( \{\phi : \mathbb{C} \rightarrow \mathbb{C}|\phi \text{ bounded, measurable}\} \) into \( B(\mathcal{H}_k) \).

**Question**

How to drag spectral measure or functional calculus from the spaces \( \mathcal{H}_k \) to \( \mathcal{K} \)?
Structure of zero-dimensional Ideals
Primary decomposition

Applying the the Noether-Lasker Theorem from ring theory to the special situation of a zero-dimensional ideal $I$ in $\mathbb{C}[x, y]$ we obtain minimal primary decomposition

$$I = Q_1 \cap \cdots \cap Q_m$$
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- $Q_j$ are primary ideals, i.e. $fg \in Q_j$ implies $f \in Q_j$ or $g^k \in Q$ for some $k \in \mathbb{N}$. 
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- $Q_j \nsubseteq \bigcap_{i \neq j} Q_i$ for all $j = 1, \ldots, m$.
- $P_j \neq P_i$ for $i \neq j$, where $P_j := \sqrt{Q_j}$ denotes the radical $\{ f \in \mathbb{C}[x, y] : f^k \in Q_j \text{ for some } k \in \mathbb{N} \}$. 
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- zero-dimensionality implies uniqueness of the above decomposition, and maximality of the $P_j$’s.
Primary decomposition

- Thus, $P_j = \{ p \in \mathbb{C}[x, y] : p(a_j) = 0 \}$ for unique and pairwise distinct $a_j \in \mathbb{C}^2$. 
Primary decomposition

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- For the variety $V(I) = \{ a \in \mathbb{C}^2 : f(a) = 0 \text{ for all } f \in I \}$ induced by $I$, we have $V(I) = \{ a_1, \ldots, a_m \}$. 
Primary decomposition

• Thus, \( P_j = \{ p \in \mathbb{C}[x, y] : p(a_j) = 0 \} \) for unique and pairwise distinct \( a_j \in \mathbb{C}^2 \).

• For the variety \( V(I) = \{ a \in \mathbb{C}^2 : f(a) = 0 \text{ for all } f \in I \} \) induced by \( I \), we have \( V(I) = \{ a_1, \ldots, a_m \} \).

• We write \( Q(a) := Q_j \), \( P(a) := P_j \), if \( a = a_j \in V(I) \). Hence,

\[
I = \bigcap_{a \in V(I)} Q(a)
\]
Primary decomposition, real Ideal

Our $I$, which is the ideal generated in $\mathbb{C}[x, y]$ by all definitizing polynomials $p \in \mathbb{R}[x, y]$, satisfies $I = I^\#$, where $p^\#(x, y) = p(\bar{x}, \bar{y})$. 
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- We consider $V_{\mathbb{R}}(I) := \{a_j : a_j \in \mathbb{R}^2\}$ as a subset of $\mathbb{C}$ and $V(I) \setminus \mathbb{R}^2$ as a subset of $\mathbb{C}^2$; $V(I) \setminus \mathbb{R}^2$ invariant under $(\xi, \eta) \mapsto (\bar{\xi}, \bar{\eta})$. 
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- For \(w \in V_\mathbb{R}(I)\) algebra \(A(w) := \mathbb{C}[x, y]/(P(w) \cdot Q(w))\) is finite dimensional, because with \(Q(w)\) also \(P(w) \cdot Q(w)\) is primary and zero-dimensional.
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- For $w \in V_{\mathbb{R}}(I)$ algebra $\mathcal{A}(w) := \mathbb{C}[x, y]/(P(w) \cdot Q(w))$ is finite dimensional, because with $Q(w)$ also $P(w) \cdot Q(w)$ is primary and zero-dimensional.
- For $\zeta \in V(I) \setminus \mathbb{R}^2$ consider the finite-dimensional algebra $\mathcal{B}(\zeta) := \mathbb{C}[x, y]/Q(\zeta)$. 


A Function class

Define Function class $\mathcal{M}_N$ consisting of functions

$$\phi : (\sigma(\Theta(N)) \cup V_R(I)) \cup V(I) \setminus \mathbb{R}^2 \to$$

$$\subseteq \mathbb{C} \cup \bigcup_{w \in V_R(I)} A(w) \cup \bigcup_{\zeta \in V(I) \setminus \mathbb{R}^2} B(\zeta)$$
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$$\phi : \left( \sigma(\Theta(N)) \cup V_{\mathbb{R}}(I) \right) \cup V(I) \setminus \mathbb{R}^2 \rightarrow \mathbb{C}$$

such that

$$\phi(z) \in \begin{cases} 
\mathcal{B}(z) & \text{for } z \in V(I) \setminus \mathbb{R}^2 , \\
\mathcal{A}(z) & \text{for } z \in V_{\mathbb{R}}(I) , \\
\mathbb{C} & \text{otherwise} .
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$$\phi(z) \in \begin{cases} B(z) & \text{for } z \in V(I) \setminus \mathbb{R}^2, \\ A(z) & \text{for } z \in V_\mathbb{R}(I), \\ \mathbb{C} & \text{otherwise}. \end{cases}$$

$\mathcal{M}_N$ becomes $\ast$-algebra when provided with pointwise with multiplication, addition and With $\phi \mapsto \phi^\#$, where $\phi^\#(z) := \phi(z^\#)^\#$. 
Function class $\mathcal{F}_N$

$I$ generated by finitely many definitizable $p_1, \ldots, p_m \in \mathbb{R}[z]$. 

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$$\beta_w(x + iy) := \max_{k=1,\ldots,m} |p_k(x, y)|.$$
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We denote by $\mathcal{F}_N$ the set of all elements $\phi \in \mathcal{M}_N$ such that
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- for each $w \in V_{\mathbb{R}}(I)$ which is non-isolated in $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$

$$\phi(z) - \phi(w)|_{x = \Re z, y = \Im z} = O(\beta_w(z)) \quad \text{as} \quad \sigma(\Theta(N)) \setminus V_{\mathbb{R}}(I) \ni z \to w.$$
Function class $\mathcal{F}_N$

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Lemma

$\mathcal{F}_N$ 'contains' $\mathbb{C}[x, y]$ in the following sense:
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Lemma

$\mathcal{F}_N$ 'contains' $\mathbb{C}[x, y]$ in the following sense:
For $s \in \mathbb{C}[x, y]$ the function $s_N \in \mathcal{M}_N$ defined by

$$s_N(z) = \begin{cases} 
  s(\text{Re} \ z, \text{Im} \ z), & z \in \sigma(N) \setminus V_{\mathbb{R}}(I), \\
  s + (P(z) \cdot Q(z)) \in \mathcal{A}(z), & z \in V_{\mathbb{R}}(I), \\
  s + Q(z) \in \mathcal{B}(z), & z \in V(I) \setminus \mathbb{R}^2.
\end{cases}$$

belongs to $\mathcal{F}_N$. 
Function class $\mathcal{F}_N$

Lemma

For each $\phi \in \mathcal{F}_N$ there exists a $p(x, y) \in \mathbb{C}[x, y]$ and $f_1, \ldots, f_m \in \mathcal{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I))$ with $f_k(w) = 0$ for $w \in V_{\mathbb{R}}(I)$ such that

$$\phi(z) = p_N(z) + \sum_k f_k(z) \cdot (p_k)_N(z)$$

for all $z \in \sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)$, and that $\phi(\zeta) = p_N(\zeta)$ for all $\zeta \in V(I) \setminus \mathbb{R}^2$. 
**FUNCTIONAL CALCULUS FOR NORMAL, DEFINITIZABLE OPERATORS**
Functional Calculus

Definition

For $\phi \in \mathcal{F}_N$ let $p(x, y) \in \mathbb{C}[x, y]$ and $f_1, \ldots, f_m \in \mathcal{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I))$ be as in the previous lemma.
Definition

For \( \phi \in \mathcal{F}_N \) let \( p(x, y) \in \mathbb{C}[x, y] \) and \( f_1, \ldots, f_m \in \mathfrak{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(I)) \) be as in the previous lemma. Then we define

\[
\phi(N) := p(A, B) + \sum_{k=1}^{m} \Xi_k \left( \int_{\sigma(\Theta_k(N))} f_k dE_k \right).
\]
Functional Calculus

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For $\phi \in \mathcal{F}_N$ let $p(x, y) \in \mathbb{C}[x, y]$ and $f_1, \ldots, f_m \in \mathcal{B}(\sigma(\Theta(N)) \cup V_\mathbb{R}(I))$ be as in the previous lemma. Then we define

$$\phi(N) := p(A, B) + \sum_{k=1}^{m} \Xi_k \left( \int_{\sigma(\Theta_k(N))} f_k \, dE_k \right).$$

Recall: $\Xi_k : B(H_k) \to B(\mathcal{K})$ such that $\Xi_k(D) = T_k DT_k^{[*]}$. 
Functional Calculus

Theorem (Functional Calculus for normal operators)

- $\phi(N)$ independent of particular choice of $p \in \mathbb{C}[x, y]$ and $f_1, \ldots, f_m$ as long as the assertion of the previous lemma is satisfied.
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- $\phi(N)$ independent of particular choice of $p \in \mathbb{C}[x, y]$ and $f_1, \ldots, f_m$ as long as the assertion of the previous lemma is satisfied.
- The mapping $\phi \mapsto \phi(N)$ constitutes *-homomorphism from $\mathcal{F}_N$ into $\{N, N^*\}'' (\subseteq B(\mathcal{K}))$.
Functional Calculus

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- \( \phi(N) \) independent of particular choice of \( p \in \mathbb{C}[x, y] \) and \( f_1, \ldots, f_m \) as long as the assertion of the previous lemma is satisfied.

- The mapping \( \phi \mapsto \phi(N) \) constitutes \( \ast \)-homomorphism from \( \mathcal{F}_N \) into \( \{N, N^*\}'' \) (\( \subseteq B(K) \)).

- For \( s \in \mathbb{C}[x, y] \) and \( \phi = s_N \in \mathcal{F}_N \) we have \( \phi(N) = s(A, B) \).
Spectrum of $N$

Proposition

$$\sigma(N) = \sigma(\Theta(N)) \cup (V_{\mathbb{R}}(I) \cap \sigma(N)) \cup \\
\{\alpha + i\beta : (\alpha, \beta) \in V(I) \setminus \mathbb{R}^2, \alpha + i\beta, \alpha + i\beta \in \sigma(N)\}$$
Spectrum of $N$

Proposition

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Accordingly, Functional Calculus is supported on

\[\sigma(\Theta(N)) \cup (V_{\mathbb{R}}(I) \cap \sigma(N)) \cup \{\alpha + i\beta : (\alpha, \beta) \in V(I) \setminus \mathbb{R}^2, \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\}\]