The order of canonical systems associated with indeterminate moment sequences

Raphael Pruckner

joint work with Harald Woracek and Roman Romanov

Let $H : [0, L] \to \mathbb{R}^{2 \times 2}$ be a trace-normed positive semidefinite Hamiltonian, and let $W(x, z)$ be the fundamental solution of the canonical system with Hamiltonian $H$, i.e., the initial value problem

$$\frac{\partial}{\partial x} W(x, z) = -z J H(x) W(x, z), \quad x \in [0, L]$$

$$W(0, z) = I,$$

where $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $z$ is a complex parameter. All entries of $W(L, z)$ are entire functions of $z$ and belong to the Cartwright class. Furthermore, they have the same exponential type, given by the classical Krein-de Branges formula

$$\text{type of } W_{i,j}(L, \cdot) = \int_0^L \sqrt{\det H(x)} \, dx.$$

If $\det H(x) = 0$ for a.e. $x \in [0, L]$, the order of $W_{i,j}(x, \cdot)$ may be less than 1. It is known that these four functions have the same order, call it $\rho(H)$. We want to determine or estimate $\rho(H)$ for given $H$.

A recent result of R. Romanov gives rise to upper bounds for $\rho(H)$ if the Hamiltonian can be approximated in some sense by finite rank Hamiltonians.

In this talk we consider Hamiltonians corresponding to indeterminate moment problems, i.e., there is a partition of $[0, L]$ consisting of countable subintervals accumulating only at $L$, such that $H$ is constant on each subinterval. We present lower and upper bounds for $\rho(H)$, and show that, under weak regularity assumptions, these bounds coincide. In general, neither the upper nor the lower bound coincides with the order.