Killip-Simon problem and the Jacobi flow on GMP matrices

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Let $\ell^2 = \ell^2(\mathbb{Z})$, $\ell^2_+ = \overline{\text{span}}\{e_k : k \geq 0\}$ and $\ell^2_- = \ell^2 \ominus \ell^2_+$. For $a, b \in \ell^\infty(\mathbb{Z})$, we define a two-sided Jacobi matrix by

$$J_{e_n} = a(n)e_{n-1} + b(n)e_n + a(n+1)e_{n+1}, \quad a(n) > 0.$$ 

Moreover, we set

$$J_{\pm} = P_{\pm} J|_{\ell^2_{\pm}}.$$ 

$$J = \begin{bmatrix}
\vdots & \ddots & \ddots & \\
\ddots & \ddots & a(-1) & \\
\ddots & a(-1) & b(-1) & a(0) \\
a(0) & b(0) & a(1) & \ddots \\
a(1) & \ddots & \ddots & \ddots \\
\end{bmatrix}$$
Theorem

Let $J_+$ be a one-sided Jacobi matrix and let its spectral measure be defined by

$$\langle (J_+ - z)^{-1}e_0, e_0 \rangle = \int \frac{1}{x - z} d\sigma(x).$$

Then the triples $(J_+, \ell_+^2, e_0)$ and $(x, L^2_{d\sigma}, 1)$ are unitarily equivalent.

- That is, there exists a unitary map $U : \ell_+^2 \to L^2_{d\sigma}$ such that $UJ_+ = xU$ and $Ue_0 = 1$.
- Thus, $J_+$ is unitarily equivalent to the multiplication operator in $L^2_{d\sigma}$. 
Since $J_+$ is a bounded self adjoint operator, $\text{supp}(\sigma) \subset \mathbb{R}$ and bounded.

Let $\{P_n(x)\}_{n \in \mathbb{N}}$ be the orthonormal polynomials w.r.t. $d\sigma$. Then they satisfy

$$xP_n(x) = a(n - 1)P_{n-1}(x) + b(n)P_n(x) + a(n + 1)P_{n+1}(x),$$

where $\{a(n), b(n)\}$ are the Jacobi parameters of $J_+$. 
Perturbation Theory

Theorem (von Neumann)

Let $A$ be an arbitrary self adjoint operator having a nontrivial absolutely continuous (a.c.) component of the spectrum. Then there exists a self-adjoint perturbation $B$ of Hilbert-Schmidt (HS) class such that $A + B$ has pure point spectrum.

- Thus, the operator $B$ completely destroys the a.c.-part of the spectrum of $A$. 
BUT, let $A$ be the free 1D discrete Schrödinger operator and $B$ be a potential, i.e.,

$$A = J_+ = \begin{bmatrix} 0 & 1 & 1 & \ldots \\ 1 & 0 & 1 & \ldots \\ \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad B = \begin{bmatrix} V(0) \\ V(1) \\ \vdots \end{bmatrix}$$

Theorem (Deift-Killip, 1999)

If $V \in \ell^2_+ (B \in HS)$, then $\sigma_{a.c.}(A + B) = \sigma_{a.c.}(A) = [-2, 2]$. 
Theorem

Let $d\sigma(x) = \sigma'(x)dx + d\sigma_{\text{sing}}(x)$ be the spectral measure of a one-sided Jacobi matrix $J_+$. Then the following are equivalent:

1. $(\text{op})$ \[ J_+ - J_+ \in HS \]
2. $(\text{sp})$ The spectral measure $d\sigma$ is supported on $X \cup [-2, 2]$ and
   \[ \int_{-2}^{2} |\log \sigma'(x)| \sqrt{4 - x^2}dx + \sum_{x_k \in X} \sqrt{x_k^2 - 4}^3 < \infty. \]

- If $\{a(n), b(n)\}$ are the Jacobi parameters of $J_+$, then $(\text{op})$ is equivalent to the condition that $b(n), (a(n) - 1) \in \ell_+^2$.
- $\sigma_{\text{a.c.}} = [-2, 2]$.
- The measure may have at most countably many mass points outside of $[-2, 2]$, which can only accumulate to $\{-2, 2\}$.
Let $E = [b_0, a_0] \setminus \bigcup_{j=1}^{g} (a_j, b_j)$ and for a two-sided Jacobi matrix $J$ we define

$$r_{\pm}(z; J) = \left\langle (J_{\pm} - z)^{-1}e_{-1/2}, e_{-1/2} \rightangle.$$  

**Definition**

A Jacobi matrix $J$ is called reflectionless on $E$ if

$$a_0^2 r_+(x + i0) = \frac{1}{r_-(x - i0)} \text{ for almost all } x \in E.$$  

The class $J(E)$ is formed by Jacobi matrices, which are reflectionless on their spectral set $E$. 
Remling’s theorem

Note that for a given $J_+$, since $\{a(n), b(n)\} \in \ell^\infty$, the system of shifts $\{(S_+^*)^n J_+ S_+\}_{n \geq 0}$ forms a relatively compact set in the pointwise convergence topology. Thus, there exists a convergent subsequence.

Theorem (Remling’s theorem)

Let $\{a(n), b(n)\}$ be the Jacobi parameters of a one-sided Jacobi matrix $J_+$ and assume that $\sigma_{a.c.}(J_+) = \sigma_{ess}(J_+) = E$. If

$$\overset{\circ}{a}(n) = \lim_{m_k \to \infty} a_{m_k}, \quad \overset{\circ}{b}(n) = \lim_{m_k \to \infty} b_{m_k},$$

for all $n \in \mathbb{Z}$, then $\overset{\circ}{J} \in J(E)$. 
Theorem

For each system of intervals $E$ there exist differentiable functions $A(\alpha), B(\alpha) : \mathbb{R}^g / \mathbb{Z}^g \to \mathbb{R}$ such that

$$J(E) = \{ J(\alpha) : \alpha \in \mathbb{R}^g / \mathbb{Z}^g \},$$

where $J(\alpha)$ is a two-sided Jacobi matrix with the parameters

$$a(n)^2 = A(\alpha - n\mu), \quad b(n) = B(\alpha - n\mu).$$

Moreover, the map $\alpha \mapsto J(\alpha)$ is one-to-one.

$A, B$ can be given explicitly by means of the $g$-dimensional Theta-function associated with our set $E$. 
Parametrization of the isospectral torus

To each $\alpha \in \mathbb{R}^g / \mathbb{Z}^g$ we can associate $J(\alpha)$ with the parameters $a(n)^2 = A(\alpha - n\mu)$, $b(n) = B(\alpha - n\mu)$. Moreover, we have $J(E) = \{J(\alpha) : \alpha \in \mathbb{R}^g / \mathbb{Z}^g\}$.

- Thus, $J(E)$ can be regarded as a $g$-dimensional real Torus.

$J(E)$ consists of periodic Jacobi matrices if and only if for all $1 \leq k \leq g$, $\mu_i \in \mathbb{Q}$, where $\mu = (\mu_1, \ldots, \mu_g)$. 
Theorem

Let \( E = [b_0, a_0] \setminus \bigcup_{j=1}^{g}(a_j, b_j) \). Then \( E \) is the spectrum of a \( p \)-periodic two-sided Jacobi matrix \( J \) if and only if there exists a polynomial \( T_p(z) \) such that \( T_p^{-1}([-2, 2]) = E \) and

- \( |T'_p(c)| = 0 \) (i.e. \( c \) is a critical point)
Direct spectral problem for periodic $J$

Let $J$ be $p$ periodic and let

$$a(z; a, b) = \begin{bmatrix} 0 & -\frac{1}{a} \\ a & \frac{z-b}{a} \end{bmatrix}.$$  

If we define the transfer matrix

$$\mathcal{A}(z; J) = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} (z) = a(z; a_1, b_0) \ldots a(z; a_p, b_{p-1}),$$

then

$$T_p(z) = \text{tr} \mathcal{A}(z; J).$$

For the transfer matrix we have

$$r_+(z) = \frac{\mathcal{A}_{11}(z)r_+^{(p)}(z) + \mathcal{A}_{11}(z)}{\mathcal{A}_{21}(z)r_+^{(p)}(z) + \mathcal{A}_{22}(z)} = \frac{\mathcal{A}_{11}(z)r_+(z) + \mathcal{A}_{11}(z)}{\mathcal{A}_{21}(z)r_+(z) + \mathcal{A}_{22}(z)}.$$
Theorem (Magic formula)

Let \( E = [b_0, a_0] \setminus \bigcup_{j=1}^{g} (a_j, b_j) \) be the spectrum of a \( p \) periodic Jacobi matrix. Then \( J \in J(E) \) if and only if

\[
T_p(J) = S^{-p} + S^p.
\]

From now on, we assume that \( p = g + 1 \), i.e., we deal with the non-degenerated case.
Main observation:

\[(S_+^*)^p + S_+^p = \begin{bmatrix} 0 & I_{g+1} \\ I_{g+1} & 0 & I_{g+1} \\ & & & \ddots & \ddots & \ddots \end{bmatrix}\]

The idea is, to substitute the condition \(J_+ - \overset{\circ}{J}_+ \in HS\) by 
\(T(J_+) - ((S_+^*)^p + S_+^p) \in HS\).
Matrix version of the Killip-Simon theorem

**Theorem**

Let $G_+$ be a Jacobi block-matrix

$$G_+ = \begin{bmatrix} B(0) & A(1) \\ A(1)^* & B(1) & A(2) \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

where $A(k), B(k)$ are $(g + 1) \times (g + 1)$-matrices, $A(k)$ are lower-triangular, with positive diagonal entries. Let $d\Sigma(y)$ be the corresponding matrix measure. Then the following are equivalent:

**(op)** \[ G_+ - ((S^*_+)^p + S^p_+) \in HS \]

**(sp)** The spectral measure $d\Sigma$ is supported on $Y \cup [-2, 2]$ and

$$\int_{-2}^{2} |\log \det \Sigma'(y)| \sqrt{4 - y^2} dy + \sum_{y_k \in Y} \sqrt{y_k^2 - 4}^3 < \infty.$$
Applying this to $G_+ = T(J_+)$ one can translate the spectral condition on $T(J_+)$ into a spectral condition on $J_+$ and obtains:

**Theorem (Damanik-Killip-Simon theorem)**

The following are equivalent:

**(op)** $T(J_+) - (S_+^p + (S_+^*)^p) \in HS$

**(sp)** The spectral measure $d\sigma$ is supported on $X \cup E$ and

$$\int_E |\log \sigma'(x)| \sqrt{\text{dist}(x, \mathbb{R} \setminus E)} dx + \sum_{x_k \in X} \sqrt{\text{dist}(x_k, E)}^3 < \infty.$$
For $t, \tilde{t} \in \ell^\infty(\mathbb{N})$ and $\eta < 1$ we define

$$\text{dist}^2_{\eta}(t, \tilde{t}) = \sum_{k=0}^{\infty} |t(k) - \tilde{t}(k)|^{2\eta^{2k}},$$

$$\text{dist}^2(J_+, \tilde{J}_+) = \text{dist}^2_{\eta}(a, \tilde{a}) + \text{dist}^2_{\eta}(b, \tilde{b})$$

and finally

$$\text{dist}^2(J_+, J(E)) = \inf \{ \text{dist}^2(J_+, \circ J_+) : \circ J \in J(E) \}.$$

Then

$$T(J_+) - (S^p_+ + (S^*_+)^p) \in HS \iff \sum_{n=0}^{\infty} \text{dist}^2((S^*_+)^n J_+ S^n_+, J(E))$$
Main Result

**Theorem**

Let $E$ be a finite system of intervals and let $d\sigma(x) = \sigma'(x)dx + d\sigma_{\text{sing}}(x)$ be the spectral measure of a one-sided Jacobi matrix $J_+$. Then the following are equivalent:

**(opJ)** There exist $\epsilon_\alpha(n) \in \ell^2_+(\mathbb{R}^g)$ and $\epsilon_a(n) \in \ell^2_+, \epsilon_b(n) \in \ell^2_+$ such that

$$a(n)^2 = A(\alpha(n) - \mu n) + \epsilon_a(n), \quad \alpha(n) = \sum_{k=0}^{n} \epsilon_\alpha(k)$$

$$b(n) = B(\alpha(n) - \mu n) + \epsilon_b(n),$$

**(spJ)** The spectral measure $d\sigma$ is supported on $X \cup E$ and

$$\int_E |\log \sigma'(x)| \sqrt{\text{dist}(x, \mathbb{R} \setminus E)}dx + \sum_{x_k \in X} \sqrt{\text{dist}(x_k, E)}^3 < \infty.$$
Lemma

For a system of intervals $E$ there exists a rational function

$$\Delta(z) = \lambda_0 z + c_0 + \sum_{j=1}^{g} \frac{\lambda_j}{c_j - z}, \quad \lambda_j > 0, c_j \in (a_j, b_j),$$

such that

$$E = [b_0, a_0] \setminus \bigcup_{j=1}^{g} (a_j, b_j) = \Delta^{-1}([-2, 2]).$$

We define certain $\ell^2$ operators $A$ such that the magic formula holds. These operators are called GMP matrices. Let

$$A(E) = \{ \overset{\circ}{A} : \overset{\circ}{A} \text{ is a periodic GMP matrix with } \sigma(\overset{\circ}{A}) = E \}. $$
Let $E$ and $\Delta$ be defined as above. Then

$$A \in A(E) \iff \Delta(A) = S^{g+1} + S^{-(g+1)}.$$

- Assume that $E$ is chosen such that the Magic formula holds for Jacobi matrices and $\deg T = g + 1$.
- $J$ is 3-diagonal,
- but $J^{g+1}$ has $1 + 2(g + 1) = 2g + 3$ diagonals.
- The GMP matrix $\hat{A}$ is already $2g + 3$ diagonal,
- but $(c_j - \hat{A})^{-1}$ are still $2g + 3$ diagonal.
Structure of GMP matrices

Let \( \vec{p}_j = [p_0^{(j)}, \ldots, p_g^{(j)}]^* \), \( \vec{q}_j = [q_0^{(j)}, \ldots, q_g^{(j)}]^* \)

\[
A = \begin{bmatrix}
\vdots & \vdots & \vdots \\
\vdots & \ddots & \vdots \\
A^*(\vec{p}_1) & B(\vec{p}_1, \vec{q}_1) & A(\vec{p}_1) \\
A(\vec{p}_1) & B(\vec{p}_0, \vec{q}_0) & A(\vec{p}_0) \\
A^*(\vec{p}_0) & \ddots & \ddots \\
\ddots & \ddots & \ddots \\
A^*(\vec{p}_0) & \ddots & \ddots \\
B(\vec{p}_0, \vec{q}_0) & A(\vec{p}_0) & A(\vec{p}_1) \\
A^*(\vec{q}_1) & \ddots & \ddots \\
\ddots & \ddots & \ddots \\
A^*(\vec{q}_1) & \ddots & \ddots \\
\ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots \\
\end{bmatrix},
\]

where \( A(\vec{p}) = \delta_g \vec{p}^* \) and

\[
B(\vec{p}, \vec{q}) = \text{diag}[c_1, \ldots, c_g, 0] + (\vec{q}\vec{p}^*)^- + (\vec{p}\vec{q}^*)^+.
\]
By applying the matrix version of the KS-theorem we get

**Theorem (Killip-Simon theorem from GMP-matrices)**

Let $d\sigma(x) = \sigma'(x)dx + d\sigma_{\text{sing}}(x)$ be the spectral measure of a one-sided GMP matrix $A_+$. Then the following are equivalent:

(opA) \[ \Delta(A_+) - (S^p_+ + (S^*_+)^p) \in HS \]

(spA) The spectral measure $d\sigma$ is supported on $X \cup E$ and

\[ \int_E |\log \sigma'(x)| \sqrt{\text{dist}(x, \mathbb{R} \setminus E)} dx + \sum_{x_k \in X} \sqrt{\text{dist}(x_k, E)^3} < \infty. \]
Idea of proof (Main theorem)

**Theorem**

Up to the identification \((p_{m}^{(j)}, q_{m}^{(j)}) \simeq (-p_{m}^{(j)}, -q_{m}^{(j)})\), there exists a one-to-one correspondence between the set of GMP and the set of Jacobi matrices, which preserves the spectral measure.

\[(spJ) \leftrightarrow (spA) \xrightarrow{KS-GMP} (opA) \leftrightarrow ? \rightarrow (opJ)\]
The Jacobi flow is the discrete dynamical system

\[ A_+^{(n+1)} = J A_+^{(n)}, \quad A_+^{(0)} = A_+, \]

where \( J \) is “defined” by the following diagram.
The Jacobi flow on GMP matrices

A(0) \( \vec{p}_{-1}(0), \vec{q}_{-1}(0) \) \( \downarrow \) \( a_0(0), b_0(0) \) \( \downarrow \) \( \vec{p}_0(0), \vec{q}_0(0) \) \( \downarrow \) \( a_1(0), b_1(0) \) \( \downarrow \) \( \vec{p}_1(0), \vec{q}_1(0) \)

A(1) \( \vec{p}_{-1}(1), \vec{q}_{-1}(1) \) \( \downarrow \) \( a_0(1), b_0(1) \) \( \downarrow \) \( \vec{p}_0(1), \vec{q}_0(1) \) \( \downarrow \) \( a_1(1), b_1(1) \) \( \downarrow \) \( \vec{p}_1(1), \vec{q}_1(1) \)

A(2) \( \vec{p}_{-1}(2), \vec{q}_{-1}(2) \) \( \downarrow \) \( \vec{p}_0(2), \vec{q}_0(2) \) \( \downarrow \) \( \vec{p}_1(2), \vec{q}_1(2) \)
The Jacobi flow as an open (input-output) dynamical system

A(0) \rightarrow \bar{p}_0(0), \bar{q}_0(0) \rightarrow \bar{p}_1(0), \bar{q}_1(0)

A(1) \rightarrow \bar{p}_0(1), \bar{q}_0(1) \rightarrow \bar{p}_1(1), \bar{q}_1(1)

A(2) \rightarrow \bar{p}_0(2), \bar{q}_0(2) \rightarrow \bar{p}_1(2), \bar{q}_1(2)
Theorem

The following are equivalent:

(opJ)

\[ a(n)^2 = \mathcal{A} \left( \sum_{k=0}^{n} \epsilon_{\alpha}(k) - \mu n \right) + \epsilon_a(n), \]

\[ b(n) = \mathcal{B} \left( \sum_{k=0}^{n} \epsilon_{\alpha}(k) - \mu n \right) + \epsilon_b(n), \]

(opS)

\[ \int_E |\log \sigma'(x)| \sqrt{\text{dist}(x, \mathbb{R} \setminus E)} \, dx + \sum_{x_k \in X} \sqrt{\text{dist}(x_k, E)}^3 < \infty. \]
Let
\[ a(z; c, p, q) = I - \frac{1}{c - z} \begin{bmatrix} p \\ q \end{bmatrix} \begin{bmatrix} p & q \end{bmatrix} j, \quad j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \]
\[ a(z; \infty, p, q) = \begin{bmatrix} 0 & -p \\ 1 & \frac{z-pq}{p} \end{bmatrix} \]
and let us define the transfermatrix of \( \hat{A} \) by
\[ \mathcal{A}(z, \hat{A}) = a(z, c_1, p_0, q_0), a(z, c_2, p_1, q_1) \ldots a(z, \infty, p_g, q_g). \]
Spectral theory of periodic GMP matrices

**Theorem**

Let $\hat{A} \in A(E)$ and let $\Delta(z) := \text{tr} \mathcal{A}(z; \hat{A})$. Then the spectrum $E$ of $\hat{A}$ is given by

$$E = \Delta^{-1}([-2, 2]) = \{x : \Delta(x) \in [-2, 2]\}.$$ 

Moreover,

$$\Delta(z) = \lambda_0 z + c_0 + \sum_{j=1}^{g} \frac{\lambda_j}{c_j - z},$$

where

$$\lambda_0 p_g = 1, \quad \lambda_0 \sum_{j=0}^{g} p_j q_j + c_0 = 0,$$

and

$$\lambda_k = \Lambda_k(\vec{p}, \vec{q}) := -\text{Res}_{c_k} \text{tr} \mathcal{A}(z; \hat{A}).$$
Theorem

For $A \in \text{GMP}(\mathbb{C})$, let $A(n + 1) = \mathcal{J}A(n)$, $A(0) = A$. Let
\[
\{\tilde{p}_j(n), \tilde{q}_j(n)\}_{j \in \mathbb{N}} \text{ be the forming } A(n) \text{ coefficient sequences. } A \text{ satisfies (opA) if and only if}
\]
\[
\{p_j^{(\pm 1)}(n) - p_j^{(0)}(n)\}_{n \geq 0} \in \ell^2_+, \quad \{q_j^{(\pm 1)}(n) - q_j^{(0)}(n)\}_{n \geq 0} \in \ell^2_+, \\
\{\lambda_0 p_g^{(0)}(n) - 1\}_{n \geq 0} \in \ell^2_+, \quad \{\lambda_0 \langle \tilde{p}_0(n), \tilde{q}_0(n) \rangle + c_0\}_{n \geq 0} \in \ell^2_+, \\
\{\Lambda_k(\tilde{p}_0(n), \tilde{p}_0(n)) - \lambda_k\}_{n \geq 0} \in \ell^2_+
\]

hold for all $j = 0, \ldots, g - 1$ and all $k = 1, \ldots, g$. 