Strong M-bases of reproducing kernels and spectral theory of rank-one perturbations of selfadjoint operators

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joint work with Yurii Belov (St. Petersburg), Alexander Borichev (Marseille) and Dmitry Yakubovich (Madrid)

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Let \( \{x_n\}_{n \in \mathbb{N}} \) be a system of vectors in a separable Hilbert space \( H \) which is both complete (i.e., \( \text{Span}\{x_n\} = H \)) and minimal (i.e., \( \text{Span}\{x_n\}_{n \neq n_0} \neq H \) for any \( n_0 \)). Let \( \{y_n\}_{n \in \mathbb{N}} \) be its (unique) biorthogonal system, \( (x_m, y_n) = \delta_{mn} \).

\[
x \in H \iff \sum_{n \in \mathbb{N}} (x, y_n)x_n.
\]

We are interested in the following "weak reconstruction" property:

**Hereditary completeness or strong M-bases**

\[
x \in \text{Span} \{(x, y_n)x_n\} \quad \text{for any} \quad x \in H.
\]

Then \( \{x_n\}_{n \in \mathbb{N}} \) said to be a hereditarily complete system or a strong M-basis (\( M = \text{Markushevich} \)). Equivalent definition: for any partition \( \mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2, \mathbb{N}_1 \cap \mathbb{N}_2 = \emptyset \), of the index set \( \mathbb{N} \), the mixed system

\[
\{x_n\}_{n \in \mathbb{N}_1} \cup \{y_n\}_{n \in \mathbb{N}_2}
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is complete in \( H \).
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is complete in \( H \).
Let \( e_\lambda(t) = e^{i\lambda t} \). For \( \Lambda = \{\lambda_n\} \subset \mathbb{C} \), consider \( \{e_\lambda\}_{\lambda \in \Lambda} \) in \( L^2(-a,a) \).

- Completeness – Levinson (1940), Beurling and Malliavin (1960-s), Makarov, Poltoratskii (2005).


Exponential systems

Applying the Fourier transform we translate geometric properties of \( \{e_\lambda\}_{\lambda \in \Lambda} \) to the properties of reproducing kernels \( \{k_\lambda\}_{\lambda \in \Lambda} \) in the Paley–Wiener space \( \mathcal{PW}_a = \mathcal{F}(L^2(-a, a)) \):

\[
e_\lambda(t) = e^{i\lambda t} \iff k_\lambda(z) = \frac{\sin a(z - \lambda)}{\pi(z - \lambda)}.
\]

The biorthogonal system for \( \{k_\lambda\}_{\lambda \in \Lambda} \) is then given by

\[
g_\lambda(z) = \frac{G_\Lambda(z)}{G'_\Lambda(\lambda)(z - \lambda)},
\]

where \( G_\Lambda \) is the generating function of \( \Lambda \);

\[
f \in \mathcal{PW}_a \iff \sum_{\lambda \in \Lambda} c_\lambda \frac{\sin a(z - \lambda)}{z - \lambda}, \quad c_\lambda = (f, g_\lambda);
\]

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f \in \mathcal{PW}_a \iff \sum_{\lambda \in \Lambda} f(\lambda) \frac{G_\Lambda(z)}{G'_\Lambda(\lambda)(z - \lambda)}.
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Exponential systems

Let $\{\tilde{e}_\lambda\}$ be the system biorthogonal to a complete and minimal system of exponentials $\{e_\lambda\}$ (the system $\{\tilde{e}_\lambda\}$ is always complete – Young, 1981, Gubreev, Kovalenko, 1981).

Theorem (A.B., Yu. Belov, A. Borichev, 2011)

There exist complete and minimal systems of exponentials which are not hereditarily complete (are not strong $M$-bases).

However, for any partition $\Lambda = \Lambda_1 \cup \Lambda_2$ the corresponding mixed system has the defect at most 1:

$$\dim (\{e_\lambda\}_{\lambda \in \Lambda_1} \cup \{\tilde{e}_\lambda\}_{\lambda \in \Lambda_2})^\perp \leq 1.$$ 

Analogous results hold for reproducing kernels of the Paley–Wiener space: for any complete and minimal system $\{k_\lambda\}_{\lambda \in \Lambda}$,

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but the defect 1 is possible.
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Main problem

We will study strong $M$-basis of reproducing kernels $\{k_\lambda\}_{\lambda \in \Lambda}$ in de Branges spaces of entire functions $\mathcal{H}(E)$.

Why reproducing kernels in de Branges spaces?

- Exponential strong $M$-bases and nonharmonic Fourier series.
- N.K. Nikolski’s question: whether there exist nonhereditarily complete systems of reproducing kernels in model spaces $K_\Theta = H^2 \ominus \Theta H^2$?
- Spectral synthesis for rank one perturbations of selfadjoint operators.
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Spectral synthesis

We say that an operator $T \in \mathcal{L}(H)$ admits the spectral synthesis if for any invariant subspace $E$ of $T$ the restriction $T|_E$ has complete set of eigenvectors (root vectors). Equivalently, the eigenvectors (root vectors) which belong to $E$, span it.

Any normal compact operator admits spectral synthesis (J. Wermer, 1950)

First example of a compact operator without the spectral synthesis – H.L. Hamburger (1951)


Theorem (Markus, 1970)

Let $T$ be a compact operator with eigenvectors (root vectors) $\{x_n\}$ and trivial kernel. Then $T$ admits the spectral synthesis if and only if the system $\{x_n\}$ is hereditarily complete.
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Let $A$ be a compact selfadjoint operator in a Hilbert space $H$. Moreover, let its point spectrum $\sigma_p(A) = \{t_n\}$ be simple and $\text{Ker} A = \{0\}$.

\[
\mathcal{L} = \mathcal{L}(A, a, b) = A + a \otimes b, \quad a, b \in H, \\
\mathcal{L}f = Af + (f, b)a, \quad f \in H.
\]


- $\mathcal{L}(A, a, b) \leftrightarrow T : \mathcal{H}(E) \rightarrow \mathcal{H}(E)$
- Eigenfunctions of $\mathcal{L} \leftrightarrow$ system of reproducing kernels $\{k_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{H}(E)$
- Eigenfunctions of $\mathcal{L}^* \leftrightarrow$ biorthogonal system $\{\tilde{k}_\lambda\}_{\lambda \in \Lambda}$

Thus, the spectral synthesis for rank one perturbation is equivalent to the hereditary completeness of reproducing kernels in de Branges spaces.
A de Branges space is a Hilbert space $\langle \mathcal{H}, (\cdot , \cdot )_\mathcal{H} \rangle$, $\mathcal{H} \neq \{0\}$, with the following properties:

- The elements of $\mathcal{H}$ are entire functions, and for each $w \in \mathbb{C}$ the point evaluation $F \mapsto F(w)$ is a continuous linear functional on $\mathcal{H}$; thus, $\mathcal{H}$ is a reproducing kernel Hilbert space, $f(w) = (f, k_w)_\mathcal{H}$.

- If $F \in \mathcal{H}$, also $F^\#(z) := \overline{F(\bar{z})}$ belongs to $\mathcal{H}$ and $\|F^\#\|_\mathcal{H} = \|F\|_\mathcal{H}$.

- If $w \in \mathbb{C} \setminus \mathbb{R}$ and $F \in \mathcal{H}$, $F(w) = 0$, then

$$\frac{z - \bar{w}}{z - w} F(z) \in \mathcal{H} \quad \text{and} \quad \left\| \frac{z - \bar{w}}{z - w} F(z) \right\|_\mathcal{H} = \|F\|_\mathcal{H}.$$
De Branges spaces via Hermite–Biehler functions

**Definition**

We say that an entire function $E$ belongs to the Hermite–Biehler class $\mathcal{HB}$, if $|E(\bar{z})| < |E(z)|$, $z \in \mathbb{C}^+$. If $E \in \mathcal{HB}$, define

$$\mathcal{H}(E) := \left\{ F \text{ entire} : \frac{F}{E}, \frac{F^\#}{E} \in L^2(\mathbb{C}^+) \right\},$$

$$(F, G)_E := \int_{\mathbb{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2} \, dt, \quad F \in \mathcal{H}(E).$$

**Theorem**

For every function $E \in \mathcal{HB}$, the space $\langle \mathcal{H}(E), (\cdot, \cdot)_E \rangle$ is a de Branges space, and conversely every de Branges space can be obtained in this way.

Example: $E(z) = \exp(-iaz)$, $\mathcal{H}(E) = \mathcal{PW}_a = \mathcal{F}(l^2_{\mathbb{R}}(-a, a))$. 
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Example: $E(z) = \exp(-iaz)$, $\mathcal{H}(E) = \mathcal{P}W_a = \mathcal{F}(L^2(-a, a))$. 
Let $T = \{ t_n \} \subset \mathbb{R}$, $|t_n| \to \infty$, $|n| \to \infty$. Let $\mu = \sum_n \mu_n \delta_{t_n}$, $\sum_n \mu_n (t_n^2 + 1)^{-1} < \infty$. Consider the class of entire functions

$$
\mathcal{H} = \left\{ F : F(z) = A(z) \sum_n \frac{c_n \mu_n^{1/2}}{z - t_n} \right\},
$$

where $A$ is an entire function which is real on $\mathbb{R}$ and vanishes exactly on $T$, and $\{c_n\} \in \ell^2$. Put $\|F\|_{\mathcal{H}} = \|\{c_n\}\|_{\ell^2}$.

**Theorem**

For any spectral data $(T, \mu)$, the space $\mathcal{H}$ is a de Branges space, and conversely every de Branges space can be obtained in this way.

**Example:** $T = \mathbb{Z}$, $\mu_n = 1$, $\mathcal{H} = \mathcal{PW}_\pi$
De Branges spaces via their spectral data

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\[
\mathcal{H} = \left\{ F : F(z) = A(z) \sum_n c_n \mu_n^{1/2} \frac{1}{z - t_n} \right\},
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where \( A \) is an entire function which is real on \( \mathbb{R} \) and vanishes exactly on \( T \), and \( \{ c_n \} \in \ell^2 \). Put \( \| F \|_\mathcal{H} = \| \{ c_n \} \|_{\ell^2} \).

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For any spectral data \((T, \mu)\), the space \( \mathcal{H} \) is a de Branges space, and conversely every de Branges space can be obtained in this way.

Example: \( T = \mathbb{Z}, \mu_n = 1 \), \( \mathcal{H} = \mathcal{P}W_{\pi} \)
Let $E \in \mathcal{H}B$, $E = A - iB$, $A = \frac{E + E^\#}{2}$, $B = \frac{E^\# - E}{2i}$.

The reproducing kernel of $\mathcal{H}(E)$ at the point $w \in \mathbb{C}$:

$$k_w(z) = \frac{\overline{E(w)}E(z) - \overline{E^*(w)}E^*(z)}{2\pi i(\overline{w} - z)} = \frac{A(w)B(z) - B(w)A(z)}{\pi(z - \overline{w})}.$$

**Orthogonal bases of reproducing kernels**

Let $t_{\alpha,n}$ be the zeros of $e^{i\alpha}E - e^{-i\alpha}E^\#$. For any $\alpha \in [0, \pi)$ (except, may be, one), the system of reproducing kernels $k_{t_{\alpha,n}}$ is an orthogonal basis in $\mathcal{H}(E)$.

In particular, $\frac{A(z)}{z - t_n}$ is an orthogonal basis if $A \notin \mathcal{H}(E)$. 
Here we will give a complete description of de Branges spaces where there exist nonhereditarily complete systems of reproducing kernels (\(M\)-bases which are not strong). If, on the contrary, any complete and minimal system of reproducing kernels with the complete biorthogonal system in a de Branges space is a strong \(M\)-basis we say that this de Branges space has strong \(M\)-basis property.

We will say that an increasing sequence \(T = \{t_n\}\) is

- lacunary (or Hadamard lacunary) if

\[
\liminf_{t_n \to \infty} \frac{t_{n+1}}{t_n} > 1, \quad \liminf_{t_n \to -\infty} \frac{|t_n|}{|t_{n+1}|} > 1.
\]

Equivalently, \(d_n := t_{n+1} - t_n \geq \delta |t_n|\) for some \(\delta > 0\).

- power separated if there exist \(c, N > 0\) such that

\[
d_n := t_{n+1} - t_n \geq c |t_n|^{-N}.
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Solution of hereditary completeness problem

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Solution of hereditary completeness problem

Previously (2011) we have shown that if $T$ is power separated and $d_n = o(|t_n|)$, $|n| \to \infty$, then there exist $\mu_n$ such that the corresponding de Branges space $\mathcal{H}(E)$ has no strong $M$-basis property. We now give a complete characterization of such de Branges spaces in terms of their spectral data.

Main theorem (A.B., Yu. Belov, A. Borichev, 2013)

Let $\mathcal{H}(E)$ be a de Branges space with the spectral data $(T, \mu)$. Then $\mathcal{H}(E)$ has the strong $M$-basis property if and only if one of the following conditions holds:

(i) $\sum_n \mu_n < \infty$;

(ii) The sequence $\{t_n\}$ is lacunary and, for some $C > 0$ and any $n$,

$$\sum_{|t_k| \leq |t_n|} \mu_k + t_n^2 \sum_{|t_k| > |t_n|} \frac{\mu_k}{t_k^2} \leq C \mu_n.$$
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Thus, there exist two distinct classes of de Branges spaces with strong $M$-basis property. It seems that there are deep reasons for this property which are essentially different in these two cases:

- For the case $\sum_n \mu_n < \infty$ there exists an operator theory explanation. Passing to the model of rank one perturbations of selfadjoint operators we find ourselves in the case of weak perturbations in the sense of Macaev (e.g., $\mathcal{L} = \mathcal{A}(I + S)$).

- Perturbations of the form (ii) are, on contrary, large, but the spectrum is lacunary. It turns out that in this case de Branges space coincide (as a set with equivalence of norms) with a Fock-type space with area integral norm.
Strong $M$-basis property and Fock-type spaces

Radial Fock-type space (Bargmann–Fock space)

$$
\mathcal{F}_\varphi = \left\{ F \text{ entire} : \|F\|_{\mathcal{F}_\varphi}^2 := \int_{\mathbb{C}} |f(z)|^2 e^{-\varphi(|z|)} dm(z) < \infty \right\},
$$

where $\varphi : [0, \infty) \rightarrow (0, \infty)$ and $m$ stands for the area Lebesgue measure. The classical Fock space corresponds to $\varphi(r) = r^2$.

Borichev–Lyubarskii, 2009: Fock-type spaces with slowly growing weights $\varphi(r) = (\log r)^\gamma$, $\gamma \in (1, 2]$, have Riesz bases of reproducing kernels corresponding to real points and, thus, can be realized as de Branges spaces with equivalence of norms.

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Surprisingly, it turns out that the class of de Branges spaces which can be realized as Fock-type spaces exactly coincides with the class of de Branges spaces (ii) with strong $M$-basis property.

**Theorem (A.B., Yu. Belov, A. Borichev, 2013)**

Let $\mathcal{H}(E)$ be a de Branges space with the spectral data $(T, \mu)$. Then the following conditions are equivalent

1. There exists a Fock-type space $\mathcal{F}_\varphi$ such that $\mathcal{H}(E) = \mathcal{F}_\varphi$;
2. The operator $R_\theta : f(z) \mapsto f(e^{i\theta}z)$ is a bounded invertible operator in $\mathcal{H}(E)$ for all (some) $\theta \in (0, \pi)$;
3. The sequence $T$ is lacunary and

$$\sum_{|t_k| \leq |t_n|} \mu_k + t_n^2 \sum_{|t_k| > |t_n|} \frac{\mu_k}{t_k^2} \leq C \mu_n.$$
Thus, in the space $F_\varphi$ with $\varphi(r) = (\log r)^\gamma$, $\gamma \in (1, 2]$, any complete and minimal system of reproducing kernels is a strong $M$-basis.

And what about the classical Fock space?

$$F = \left\{ F \text{ entire} : \|F\|_F^2 := \int_\mathbb{C} |f(z)|^2 e^{-|z|^2} dm(z) < \infty \right\}.$$

- Yu. Belov, 2013 (a Young-type theorem): for any complete and minimal system of reproducing kernels in $F$, its biorthogonal system is complete.
- A.B., 2014: there exist complete and minimal systems $\{k_\lambda\}_{\lambda \in \Lambda}$ of reproducing kernels in $F$ which are not strong $M$-bases, that is, the mixed system $\{k_\lambda\}_{\lambda \in \Lambda_1} \cup \{g_\lambda\}_{\lambda \in \Lambda_2}$ is not complete in $F$. 
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$$\mathcal{F} = \left\{ F \text{ entire} : \|F\|^2_\mathcal{F} := \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dm(z) < \infty \right\}.$$
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- Yu. Belov, 2013 (a Young-type theorem): for any complete and minimal system of reproducing kernels in $\mathcal{F}$, its biorthogonal system is complete.

- A.B., 2014: there exist complete and minimal systems $\{k_\lambda\}_{\lambda \in \Lambda}$ of reproducing kernels in $\mathcal{F}$ which are not strong $M$-bases, that is, the mixed system

  $$\{k_\lambda\}_{\lambda \in \Lambda_1} \cup \{g_\lambda\}_{\lambda \in \Lambda_2}$$

  is not complete in $\mathcal{F}$. 
Size of the defect

Let \( \{k_\lambda\}_{\lambda \in \Lambda} \) be a complete and minimal system of reproducing kernels with a complete biorthogonal system \( \{g_\lambda\}_{\lambda \in \Lambda} \). For a partition \( \Lambda = \Lambda_1 \cup \Lambda_2 \), define the defect of the corresponding mixed system

\[
def(\Lambda_1, \Lambda_2) := \dim (\{g_\lambda\}_{\lambda \in \Lambda_1} \cup \{k_\lambda\}_{\lambda \in \Lambda_2})^\perp.
\]

We also put

\[
def(\Lambda) = \sup \{\def(\Lambda_1, \Lambda_2) : \Lambda = \Lambda_1 \cup \Lambda_2\},
\]

\[
def(\mathcal{H}(E)) = \sup \{\def(\Lambda) : \{k_\lambda\}_{\lambda \in \Lambda} \text{ is } M\text{-basis}\}.
\]

It turns out that one can construct examples of \( M \)-bases of reproducing kernels with large or even infinite defect.

**Theorem**

Let \( \mathcal{H}(E) \) be a de Branges space with the spectral data \((T, \mu)\), \( \sum_n \mu_n = \infty \). If for some \( N \in \mathbb{N} \) there exists a subsequence \( t_{n_k} \) of \( T \) such that \( \sum_k t_{n_k}^{2N-2} \mu_{n_k} < \infty \), then \( \def(\mathcal{H}(E)) \geq N \).
Let \( \{k_\lambda\}_{\lambda \in \Lambda} \) be a complete and minimal system of reproducing kernels with a complete biorthogonal system \( \{g_\lambda\}_{\lambda \in \Lambda} \). For a partition \( \Lambda = \Lambda_1 \cup \Lambda_2 \), define the defect of the corresponding mixed system

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**Theorem**

Let $T$ be a power separated sequence. Then the following are equivalent:
(i) $\text{def}(\mathcal{H}(E)) = \infty$;
(ii) $\inf_n \mu_n |t_n|^N = 0$ for any $N > 0$.

What about the infinite defect $\text{def}(\Lambda_1, \Lambda_2)$?

**Theorem**

For any increasing sequence $T = \{t_n\}$ with $|t_n| \to \infty$, $|n| \to \infty$, there exists a measure $\mu$ such that in the de Branges space with the spectral data $(T, \mu)$ there exists an $M$-basis of reproducing kernels $\{k_\lambda\}_{\lambda \in \Lambda}$ such that $\text{def}(\Lambda_1, \Lambda_2) = \infty$ for some partition $\Lambda = \Lambda_1 \cup \Lambda_2$. 
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The above results about the possible defects of nonhereditary complete systems of reproducing kernels lead to the following unexpected examples for rank one perturbations of compact selfadjoint operators:

**Theorem (A.B., D. Yakubovich)**

For any compact selfadjoint operator $A$ with simple spectrum there exists its rank one perturbation $L = A + a \otimes b$ such that:

1. $\ker L = \ker L^* = 0$, $L$ is complete, but the eigenvectors of $L^*$ span a subspace with infinite defect;
2. $L$ and $L^*$ are complete, but $L$ admits no spectral synthesis (with infinite defect).

On the other hand, under certain restrictions on the vectors $a$ and $b$ one can show that the spectral synthesis holds for $L$ up to a finite defect.
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On the other hand, under certain restrictions on the vectors $a$ and $b$ one can show that the spectral synthesis holds for $L$ up to a finite defect.
Theorem (A.B., D. Yakubovich)

Let $\mathcal{A}$ be the operator of multiplication by the independent variable in $L^2(\nu)$, $\nu = \sum_n \nu_n \delta_{s_n}$, $s_n \neq 0$. Assume that $\{s_n\}$ is monotonic for $n > 0$ and $n < 0$,

$$|s_{n+1} - s_n| \geq C_1 |s_n|^{N_1}$$

for some $C_1, N_1 > 0$. Let $\mathcal{L} = \mathcal{A} + a \otimes b$, $a, b \notin xL^2(\nu)$ and

$$|a_n|^2 \nu_n \geq C_2 |s_n|^{N_2}.$$ 

If $\mathcal{L}$ is a complete operator with the eigenfunctions $\{f_j\}$, then for any $\mathcal{L}$-invariant subspace $M$,

$$\dim \left( M \ominus \overline{\text{Lin}} \{f_j : f_j \in M\} \right) < \infty,$$

where the upper bound for the dimension depends only $N_1$ and $N_2$. 

Anton Baranov  Saint Petersburg State University  Strong M-bases of reproducing kernels