Strong M-bases of reproducing kernels and spectral theory of rank-one perturbations of selfadjoint operators

Let \( \{x_n\}_{n \in \mathbb{N}} \) be a complete and minimal system in a separable Hilbert space \( H \), and let \( \{y_n\} \) be its biorthogonal system. The system \( \{x_n\} \) is said to be hereditarily complete (or a strong M-basis) if for any \( x \in H \) we have \( x \in \text{Span}\{(x, y_n)x_n\} \). This property may be understood as a very weak form of the reconstruction of a vector \( x \) from its (formal) Fourier series \( \sum_n (x, y_n)x_n \).

In 2013 we solved the spectral synthesis problem for exponential systems in \( L^2(-a, a) \) (equivalently, reproducing kernels of the Paley-Wiener space \( PW_{a}^{2} \)). It turned out that the nonhereditary completeness may occur even in the case of exponential systems, though the defect of incompleteness is always at most one.

In the present talk we discuss the hereditary completeness for the reproducing kernels in Hilbert spaces of entire functions introduced by L. de Branges. One of our motivations is the relation (via a functional model) between this problem and the spectral synthesis for rank one perturbations of compact selfadjoint operators. We give a complete description of de Branges spaces where nonhereditarily complete systems of reproducing kernels exist in terms of their spectral measures. As a corollary, we obtain a series of striking examples of rank one perturbations of compact selfadjoint operators for which the spectral synthesis fails up to finite- or even infinite-dimensional defect.

The talk is based on joint works with Yurii Belov, Alexander Borichev and Dmitry Yakubovich.