Spectral theory of rank one perturbations of selfadjoint operators

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Eigenfunction expansions

Riesz bases of eigenfunctions, bases with brackets, linear summation methods, hereditary completeness, completeness.

Keldyš Theorem, 1951

Suppose $A$ is a selfadjoint Hilbert space operator that belongs to a Schatten ideal $\mathcal{S}_p$, $0 < p < \infty$, and satisfies $\ker A = 0$. Let $L = A(I + S)$, where $S$ is compact and $\ker(I + S) = 0$. Then the operators $L$ and $L^*$ are complete.

Matsaev completeness theorem, 1961 (weak perturbations)

Let $L = A(I + S)$, where $A$, $S$ are compact operators on a Hilbert space, $A$ is selfadjoint, $S \in \mathcal{S}_\omega$ (i.e., $\sum_{n=1}^{\infty} n^{-1} s_n < \infty$) and $\ker(I + S) = 0$. Then $L$ and $L^*$ are complete.

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Let $A$ be a compact selfadjoint operator in a Hilbert space $H$. Moreover, let its point spectrum $\sigma_p(A) = \{t_n\}$ be simple and $\text{Ker} A = \{0\}$.

### Rank one perturbations

$$L = L(A, a, b) = A + a \otimes b, \quad a, b \in H,$$

$$Lf = Af + (f, b)a, \quad f \in H.$$  

### Some questions

- When $L(A, a, b)$ has a complete system of eigenvectors or root vectors ($L$ is complete)?
- When does completeness of $L$ imply completeness of $L^*$?
- When does $L$ admit spectral synthesis?
- For which $A$ there exists a rank one perturbation $L(A, a, b)$ which is a Volterra operator (i.e. $\text{Ker} L = \{0\}$, $\sigma(L) = \{0\}$)?
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Two counterexamples

Completeness of $L$ and $L^*$

Let $L = L(A, a, b)$ be complete. A trivial obstacle for completeness of $L^*$ is that $\text{Ker } L$ is nontrivial, while $\text{Ker } L^*$ is trivial.

First example of a compact operator $L$ such that $\text{Ker } L = \text{Ker } L^* = \{0\}$, $L$ is complete, $L^*$ is not – H.L. Hamburger (1951, Mathematische Nachrichten, Über die Zerlegung des Hilbertschen Raumes durch vollstetige lineare Transformationen)

A simpler example – Deckard, Foias, Pearcy (1979), a Hilbert–Schmidt class operator.

Theorem

For any compact selfadjoint operator $A$ there exists a rank one perturbation $L = A + (\cdot, b)a$ of $A$ such that $\text{Ker } L = \text{Ker } L^* = \{0\}$ and $L$ is complete while $L^*$ is not.
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Spectral synthesis

We say that an operator $T \in \mathcal{L}(H)$ admits the spectral synthesis if for any invariant subspace $E$ of $T$ the restriction $T|_E$ is complete (equivalently, the eigenvectors which belong to $E$, span it).

Any normal compact operator admits spectral synthesis (J. Wermer, 1950)

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Theorem

For any compact selfadjoint operator $A$ there exists a rank one perturbation $L = A + (\cdot, b)a$ of $A$ such that both $L$ and $L^*$ are complete, but $L$ does not admit the spectral synthesis.
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Let $\{x_n\}_{n \in \mathbb{N}}$ be a complete and minimal system in a separable Hilbert space (i.e., $\text{Span}\{x_n\} = H$ and $\text{Span}\{x_n\}_{n \neq n_0} \neq H$ for any $n_0$). Let $\{y_n\}_{n \in \mathbb{N}}$ be its biorthogonal system, $(x_m, y_n) = \delta_{mn}$.

$$x \in H \iff \sum_{n \in \mathbb{N}} (x, y_n)x_n.$$ 

We are interested in the following "weak reconstruction" property:

$$x \in \text{Span}\{(x, y_n)x_n\} \quad \text{for any} \quad x \in H.$$ 

In this case $\{x_n\}_{n \in \mathbb{N}}$ is said to be a hereditarily complete system or a strong $M$-basis ($M =$ Markushevich). Equivalent definition: for any partition $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$, of the index set $N$, the mixed system $\{x_n\}_{n \in N_1} \cup \{y_n\}_{n \in N_2}$ is complete in $H$.

A.S. Marcus (1970): Let $L$ be a compact operator with complete set of eigenvectors $\{x_n\}$ and trivial kernel. Then $L$ admits the spectral synthesis if and only if $\{x_n\}$ is hereditarily complete.
Spectral synthesis and geometry of eigenvectors

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A.S. Marcus (1970): Let \( L \) be a compact operator with complete set of eigenvectors \( \{x_n\} \) and trivial kernel. Then \( L \) admits the spectral synthesis if and only if \( \{x_n\} \) is hereditarily complete.
Let $\mu = \sum_n \mu_n \delta_{t_n}$ be a discrete measure on $\mathbb{R}$ and let $A$ be the (unbounded) operator of multiplication by $x$ in $L^2(\mu)$, $(Af)(x) = xf(x)$, $f \in L^2(\mu)$. Assume that $0 \in \rho(A)$.

Let the triple $(a, b, \kappa)$, where $a = (a_n)$, $b = (b_n)$, satisfy

$$\frac{a}{x}, \frac{b}{x} \in L^2(\mu); \quad \kappa \in \mathbb{C},$$

and

$$\kappa \neq \int_{\mathbb{R}} x^{-1} a(x) \overline{b(x)} \, d\mu(x)$$

in the case when $a \in L^2(\mu)$.

The corresponding singular perturbation $L = L(A, a, b, \kappa)$ of $A$ is defined as follows:

$$\mathcal{D}(L) = \{ y = y_0 + c \cdot A^{-1}a : c \in \mathbb{C}, y_0 \in \mathcal{D}(A), \kappa c + \langle y_0, b \rangle = 0 \};$$

$$Ly = Ay_0, \quad y \in \mathcal{D}(L).$$
Let $E$ be an entire function in the Hermite–Biehler class, that is $E$ has no zeros in $\mathbb{C}_+ \cup \mathbb{R}$, and

$$|E(z)| > |E^*(z)|, \quad z \in \mathbb{C}_+,$$

where $E^*(z) = \overline{E(z)}$. With any such function we associate the de Branges space $\mathcal{H}(E)$ which consists of all entire functions $F$ such that $F/E$ and $F^*/E$ restricted to $\mathbb{C}_+$ belong to the Hardy space $H^2 = H^2(\mathbb{C}_+)$. The inner product in $\mathcal{H}(E)$ is given by

$$(F, G)_E = \int_{\mathbb{R}} \frac{F(t)\overline{G(t)}}{|E(t)|^2} \, dt.$$
Let $\mu = \sum_n \mu_n \delta_{t_n}$, $A$ — multiplication by $x$ in $L^2(\mu)$, $a = (a_n)$, $b = (b_n)$, $\kappa \in \mathbb{C}$.

**Theorem**

Let $L = L(A, a, b, \kappa)$ be a singular rank one perturbation of $A$, where $b$ is a cyclic vector for the resolvent of $A$, i.e., $b_n \neq 0$ for any $n$. Then there exists a de Branges space $\mathcal{H}(E)$ and an entire function $G$ such that

$$G \notin \mathcal{H}(E), \quad \frac{G(z)}{z - z_0} \in \mathcal{H}(E) \quad \text{if} \quad G(z_0) = 0,$$

and $L$ is unitary equivalent to the operator $T = T(E, G)$ which acts on $\mathcal{H}(E)$ by the formulas

$$\mathcal{D}(T) := \{F \in \mathcal{H}(E) : \text{there exists } c = c(F) \in \mathbb{C} : zF - cG \in \mathcal{H}(E)\},$$

$$TF := zF - cG, \quad F \in \mathcal{D}(T).$$
Conversely, any pair \((E, G)\) where \(E\) is an Hermite–Biehler function and the entire function \(G\) satisfies

\[
G \notin \mathcal{H}(E), \quad \frac{G(z)}{z - z_0} \in \mathcal{H}(E) \quad \text{if} \quad G(z_0) = 0,
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corresponds to some singular rank one perturbation \(L = L(A, a, b, \kappa)\) of the multiplication operator \(A\) in some space \(L^2(\mu)\) with \(x^{-1}a(x), x^{-1}b(x) \in L^2(\mu)\).

Functional models: V. Kapustin (rank-one perturbations of unitary operators), S. Naboko, A, Kiselev, V. Ryzhov.
The functions $E$ and $G$ appearing in the model for $L(A, a, b, \kappa)$ are related to the data $(a, b, \kappa)$ by the following formulas. Let $E = A_E - iB_E$, where $A_E = (E + E^*)/2$, $B_E = (E^* - E)/2i$. Then

$$\frac{B_E(z)}{A_E(z)} = \sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n}\right) |b_n|^2 \mu_n, \quad (1)$$

and

$$\frac{G(z)}{A_E(z)} = \kappa + \sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n}\right) a_n \overline{b_n} \mu_n, \quad (2)$$

Note, in particular, that $A_E$ vanishes exactly on the set \{t_n\}. The model essentially uses the expansions with respect to the orthogonal basis \{\frac{A_E}{z - t_n}\} of normalized reproducing kernels of $\mathcal{H}(E)$.
Spectrum of the model operator

\[ TF := zF - cG, \quad F \in \mathcal{D}(T). \]

- \( \sigma(A) = Z(A_E) = \{ t_n \}; \)
- \( \sigma(T) = \sigma_p(T) = Z(G); \)
- The eigenspace of \( T \) corresponding to an eigenvalue \( \lambda \), \( \lambda \in Z(G) \), is spanned by \( g_\lambda \),
  \[ g_\lambda(z) = \frac{G(z)}{z - \lambda}. \]
- Suppose \( T^* \) is correctly defined. Then \( \sigma(T^*) = \overline{Z(G)} \) and \( \ker(T^* - \overline{\lambda}I) \) is spanned by the reproducing kernel \( K_\lambda \).
Positive results about the completeness

**Matsaev completeness theorem, 1961 (weak perturbations)**

Let $L = A(I + S)$, where $A$, $S$ are compact operators on a Hilbert space, $A$ is selfadjoint, $S \in \mathcal{S}_\omega$ (i.e., $\sum_{n=1}^{\infty} n^{-1} s_n < \infty$) and $\ker(I + S) = 0$. Then $L$ and $L^*$ are complete.

**Generalized weak perturbations**

We call $L = L(A, a, b, \kappa)$ a *generalized weak perturbation*, if

$$\sum_n \frac{|a_n b_n| \mu_n}{|t_n|} < \infty, \quad \sum_n \frac{a_n \overline{b_n} \mu_n}{t_n} \neq \kappa.$$  

**Theorem**

If $L = L(A, a, b, \kappa)$ is a *generalized weak perturbations*, then $L$ and $L^*$ are complete.
Positive results about the completeness

Theorem (positive perturbations)

Let $L = L(A, a, b, \kappa)$ be a singular rank one perturbation, such that $a_n \overline{b_n} \geq 0$ for all but possibly a finite number of values of $n$ and $\sum_n |t_n|^{-1}|a_n b_n| \mu_n = \infty$. Then $L^*$ is correctly defined, and $L$ and $L^*$ are complete.
We call \( L = L(A, a, b, \kappa) \) a \textit{generalized weak perturbations}, if

\[
\sum_n \frac{|a_n b_n| \mu_n}{|t_n|} < \infty, \quad \sum_n \frac{a_n \overline{b_n} \mu_n}{t_n} \neq \kappa.
\]

A typical example: \( \sum_n |a_n|^2 t_n^{-2\alpha} \mu_n < \infty, \sum_n |b_n|^2 t_n^{-2+2\alpha} \mu_n < \infty, \alpha \in [0, 1] \).

There exists \( A \in S_p, p > 1/2 \), such that for any \( \alpha_1, \alpha_2 \in (0, 1) \) there exist \( a \in |x|^{\alpha_1} L^2(\mu) \) and \( b \in |x|^{\alpha_2} L^2(\mu) \) such that the spectrum of the perturbed operator is empty.

Thus, there exists a positive compact operator \( A_0 \) such that for any \( \alpha_1, \alpha_2 \in (0, 1) \) and there is a rank one perturbation \( L_0 \) of \( A_0 \) of the form

\[
L_0 = A_0 + A_0^{\alpha_1} S A_0^{\alpha_2},
\]

where \( S \) is a rank one operator and \( \text{Ker} L_0 = \text{Ker} L_0^* = 0 \), such that \( L_0 \) is a Volterra operator.
Completeness of $L$ and of $L^*$

- Eigenfunctions of $L \leftrightarrow$ functions $\frac{G(z)}{z-\lambda}$, $\lambda \in Z(G)$.
- Eigenfunctions of $L^* \leftrightarrow$ reproducing kernels $K_\lambda$, $\lambda \in Z(G)$.


**Theorem**

Assume that $L = L(A, a, b, \kappa)$ and $L^*$ are well defined and $a \notin L^2(\mu)$. Let $L$ be complete. Then its adjoint $L^*$ is also complete if any of the following conditions is fulfilled:

(i) $|a_n|^2 \mu_n \geq C|t_n|^{-N} > 0$ for some $N > 0$;

(ii) $|b_n a_n^{-1}| \leq C|t_n|^N$ for some $N > 0$. 
Theorem

For any cyclic selfadjoint operator $A$ with discrete spectrum $\{t_n\}$ such that $|t_n| \to \infty$, $|n| \to \infty$, there exists a rank one singular perturbation $L$ of $A$ (with real spectrum and trivial kernel), which is complete, while its adjoint $L^*$ is correctly defined and incomplete.

A counterpart for compact operators

For any compact selfadjoint operator $A^\circ$ with simple spectrum and trivial kernel there exists a bounded rank one perturbation $L^\circ$ of $A^\circ$ with real spectrum such that $L^\circ$ is complete and $\ker L^\circ = 0$, while $(L^\circ)^*$ is not complete. Moreover, the orthogonal complement to the eigenvectors of $(L^\circ)^*$ may be infinite-dimensional.
Counterexample

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Perturbations without the spectral synthesis

- Completeness of $L \leftrightarrow$ completeness of a system of reproducing kernels $\{K_\lambda\}$ in $\mathcal{H}(E)$;

- Completeness of $L^* \leftrightarrow$ completeness of the system biorthogonal to the system of reproducing kernels;

- Spectral synthesis for $L \leftrightarrow$ hereditary completeness of $\{K_\lambda\}_{\lambda \in \Lambda}$, i.e., for any partition $\Lambda = \Lambda_1 \cup \Lambda_2$, the system $\{K_\lambda\}_{\lambda \in \Lambda_1} \cup \{g_\lambda\}_{\lambda \in \Lambda_2}$ is complete in $\mathcal{H}(E)$. Recall that $Z(G) = \Lambda$, $g_\lambda = G/(\cdot - \lambda)$.


**Theorem**

For any spectrum $\{t_n\}$ there exists a rank one perturbation $L = A + (\cdot, b)a$ of $A$ such that both $L$ and $L^*$ are complete, but $L$ does not admit the spectral synthesis.
Spectral synthesis for exponential systems

Special case: \( \{t_n\} = \mathbb{Z}, \mu_n \equiv 1, b_n \equiv 1. \) Then \( \mathcal{H}(E) = \mathcal{PW}_\pi. \)

A.B., Yu. Belov, A. Borichev (Adv Math., 2013): there exist nonhereditary complete systems of RK in \( \mathcal{PW}_\pi. \) However, the orthogonal complement to any mixed system \( \{K_\lambda\}_{\lambda \in \Lambda_1} \cup \{g_\lambda\}_{\lambda \in \Lambda_2} \) is always at most one-dimensional.

Systems of reproducing kernels in \( \mathcal{PW}_\pi \) are Fourier images of exponential systems in \( L^2(-\pi, \pi). \) Let \( e_\lambda(t) = e^{i\lambda t}. \) Consider \( \{e_\lambda\}_{\lambda \in \Lambda} \) in \( L^2(-\pi, \pi). \)

- Completeness – Levinson (1940), Beurling and Malliavin (1960-s), Makarov, Poltoratskii (2005).
In the example of a rank one perturbation without synthesis, it may fail with an infinite defect, that is, the orthogonal complement to the root vectors in some invariant subspace $M$ will be infinite-dimensional.

**Theorem**

Let $A_0$ be a compact selfadjoint operator with simple spectrum $\{s_n\}$, $s_n \neq 0$. Assume that $\{s_n\}_{n \in I}$ is ordered so that $s_n > 0$ and $s_n$ decrease for $n \geq 0$, and $s_n < 0$ and increase for $n < 0$ and

$$|s_{n+1} - s_n| \geq C_1 |s_n|^{N_1}$$

for some $C_1, N_1 > 0$. Let $L_0 = A_0 + a \otimes b$ be a bounded rank-one perturbation of $A_0$ such that $a, b \notin xL^2(\mu)$ and $|a_n|^2 \mu_n \geq C_2 |s_n|^{N_2}$ for some $C_2, N_2 > 0$. Assume that operator $L_0$ is complete and that all its eigenvalues are simple and non-zero. Denote by $\{f_j\}_{j \in J}$ the eigenvectors of $L_0$. Then for any $L_0$-invariant subspace $M$ we have

$$\dim \left( M \ominus \overline{\text{Lin} \{f_j : f_j \in M\}} \right) < \infty,$$

where the upper bound for the dimension depends only $N_1$ and $N_2$. 
Volterra rank one perturbations

**Problem**

Let $A_0$ be a compact selfadjoint operator with the (simple) spectrum $\{s_n\}$. When (for which spectra $\{s_n\}$) there exist a rank one perturbation $L_0$ of $A_0$ which is a Volterra operator?

Equivalent problem for inverse operators: when there is a singular rank one perturbation of an unbounded selfadjoint operator $A = A_0^{-1}$ which has empty spectrum?

**Known results:** if $T = A + iB$ is a Volterra operator and $A \in \mathcal{G}_p$, $p > 1$, then $B \in \mathcal{G}_p$ (Matsaev)

**Krein class $\mathcal{K}_1$**

An entire function $F$ is in the Krein class $\mathcal{K}_1$, if it is real on $\mathbb{R}$, has only real simple zeros $t_n$ and may be represented as

$$
\frac{1}{F(z)} = q + \sum_n c_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right), \quad \sum_n t_n^{-2} |c_n| < \infty.
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**Theorem**

Let $t_n \in \mathbb{R}$ and $|t_n| \to \infty$, $|n| \to \infty$. The following are equivalent:

(i) The spectrum $\{t_n\}$ may be removed by a rank one perturbation;

(ii) There exists a function $F \in \mathcal{K}_1$ such that the zero set of $F$ coincides with $\{t_n\}$.

**A counterpart for compact operators**

Let $s_n \in \mathbb{R}$, $s_n \neq 0$, and $|s_n| \to 0$, $|n| \to \infty$, and let $A_0$ be a compact selfadjoint operator with point spectrum $\{s_n\}$. The following are equivalent:

(i) There exists a rank one perturbation $L_0 = A_0 + a_0 \otimes b_0$ such that $L_0$ is a Volterra operator;

(ii) The points $t_n = s_n^{-1}$ form the zero set of some function $F \in \mathcal{K}_1$. 
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Let \( s_n \in \mathbb{R}, s_n \neq 0 \), and \( |s_n| \to 0, |n| \to \infty \), and let \( A_0 \) be a compact selfadjoint operator with point spectrum \( \{s_n\} \). The following are equivalent:

(i) There exists a rank one perturbation \( L_0 = A_0 + a_0 \otimes b_0 \) such that \( L_0 \) is a Volterra operator;
(ii) The points \( t_n = s_n^{-1} \) form the zero set of some function \( F \in \mathcal{K}_1 \).
Volterra rank one perturbations

Theorem

Let $t_n \in \mathbb{R}$ and $|t_n| \to \infty$, $|n| \to \infty$. The following are equivalent:

(i) The spectrum $\{t_n\}$ may be removed by a rank one perturbation;

(ii) There exists a function $F \in \mathcal{K}_1$ such that the zero set of $F$ coincides with $\{t_n\}$.

A rather counterintuitive consequence of this theorem is that adding a finite number of points to the spectrum helps it to become removable, while deleting a finite number of points may make it non-removable.

Examples

Let $t_n = n^\gamma$, $n \in \mathbb{N}$. Then the spectrum $\{t_n\}$ is removable for $\gamma \geq 2$ and non-removable for $\gamma < 2$.

$\gamma = 2$ is the critical exponent. The spectrum $\{n^2\}_{n \geq 1}$ is removable, but $\{n^2\}_{n \geq 2}$ is not.
Theorem (Livshits, 1946)

Let $L^\circ = A^\circ + iB^\circ$ be a Volterra operator (in some Hilbert space $H$) such that both $A^\circ$ and $B^\circ$ are selfadjoint and $B^\circ$ is of rank one, $B^\circ = b^\circ \otimes b^\circ$. Then the spectrum of $A^\circ$ is given by $s_n = c(n + 1/2)^{-1}$, $n \in \mathbb{Z}$, for some $c \in \mathbb{R}$, $c \neq 0$.

From this, one may deduce that $A^\circ$ is unitary equivalent to the integral operator (having the same spectrum)

$$(\tilde{A}f)(x) = i \int_0^{2\pi c} f(t) \text{sign}(x - t) \, dt, \quad f \in L^2(0, 2\pi c),$$

while $L^\circ$ is unitary equivalent to the integration operator

$$(\tilde{L}f)(x) = 2i \int_0^x f(t) \, dt.$$
Theorem (Livshits, 1946)

Let \( L^\circ = A^\circ + iB^\circ \) be a Volterra operator (in some Hilbert space \( H \)) such that both \( A^\circ \) and \( B^\circ \) are selfadjoint and \( B^\circ \) is of rank one, \( B^\circ = b^\circ \otimes b^\circ \). Then the spectrum of \( A^\circ \) is given by

\[ s_n = c(n + 1/2)^{-1}, \quad n \in \mathbb{Z}, \text{ for some } c \in \mathbb{R}, \quad c \neq 0. \]

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Passing to the unbounded inverses we obtain a singular rank one perturbation $L = L(A, a, b, \kappa)$ of the operator $A$ of multiplication by the independent variable in some space $L^2(\mu)$ where $\mu = \sum_n \mu_n \delta_{t_n}$, $t_n = s_n^{-1}$, $\kappa = -1$, $a = ib$.

Let $E = A_E - iB_E$. Then

$$\frac{B_E(z)}{A_E(z)} = \delta + \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b_n|^2 \mu_n,$$

$$\frac{G(z)}{A_E(z)} = -1 + i \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b_n|^2 \mu_n.$$
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\frac{G(z)}{A_E(z)} = -1 + i \sum_n \left( \frac{1}{t_n - z} - \frac{1}{t_n} \right) |b_n|^2 \mu_n,
\]

Since \( L \) (and, thus, the model operator \( T \)) is the inverse to a Volterra operator, \( Z(G) = \emptyset \), and so \( G(z) = \exp(i \pi cz) \) for some real \( c \). Thus,

\[
e^{i \pi cz} = -A_E(z) + i (B_E(z) - \delta A_E(z)),
\]

Hence, \( A_E(z) = \cos \pi cz, \ t_n = c^{-1}(n + 1/2). \)