1. (Exercise 12.5 & 12.6:)

   a) Expand $\varphi(X+H)$ to conclude

   $$\varphi(X+H) = \varphi(X) - 2\,(A^T R, H)_F + \|A\,H\|_F^2 \quad \text{with the residual matrix} \quad R = I \,-\, A\,X$$

   Note that this expansion is, in fact, exact because $\varphi(X)$ is a quadratic functional.

   b) Show that (apart from the dropping step) Alg. 12.4 is indeed the steepest descent algorithm for the problem under consideration. (Cf. the derivation of the SD algorithm in Sec. 7.1, but note the different inner product.)

2. (Exercise 13.3:)

   Define the matrix $I_{N,N/2}$ by

   $$I_{N,N/2} = \begin{pmatrix} \frac12 & & & & \\ 1 & 0 & & & \\ \frac12 & \frac12 & & & \\ 0 & 1 & & & \\ & \frac12 & \frac12 & & \\ & 0 & 1 & 0 & \\ & & \frac12 & \frac12 & \\ & & & \ddots & \ddots \\ & & & & 1 \\ & & & & \frac12 \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N/2-1)}$$

   Let $\hat{u}_{N/2} \in \hat{U}_{N/2} \subset \hat{U}_N$ be associated with the coefficient vector $u_{N/2} \in \mathbb{R}^{N/2-1}$.

   a) Show: The vector $u_N \in \mathbb{R}^{N-1}$ corresponding to $\hat{u}_{N/2}$ (viewed as an element of $\hat{U}_N$!) is then given by $u_N = I_{N,N/2}\,u_{N/2}$. This corresponds to linear interpolation (sketch).

   b) Give an interpretation of the corresponding restriction operator $R_{N/2,N}\colon \hat{U}_N \to \hat{U}_{N/2}$ represented by the matrix

   $$I_{N,N/2}^T \in \mathbb{R}^{(N/2-1)\times(N-1)}$$

   Note that this restriction is not the trivial, pointwise one; it involves local weighting.

   Remark: The row sums in $I_{N,N/2}^T$ are 2, not 1. If we scale the inner products on the two levels in a more natural way by a factor h and $2h$, respectively, then the corresponding adjoint is represented by $\frac12\,I_{N,N/2}^T$.

   c) A trivial restriction $\hat{J}_{N/2,N}\colon \hat{U}_N \to \hat{U}_{N/2}$ is represented by the matrix

   $$J_{N/2,N} = \begin{pmatrix} 0 & 1 & 0 & & & & & & \\ & & 0 & 1 & 0 & & & & \\ & & & & 0 & 1 & 0 & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{(N/2-1)\times(N-1)}$$

   Provide the geometrical interpretation of the trivial restriction operator $\hat{J}_{N/2,N}$ and verify that $J_{N/2,N}$ is the left-inverse of $I_{N,N/2}$.

d) Show that the identities

$$I_{N,N/2}^T\, A_N = A_{N/2}\, J_{N/2,N}$$

and

$$A_{N/2} = A_{N/2}^{Galerkin} := I_{N,N/2}^T\, A_N\, I_{N,N/2}$$

are valid.

e) Show that the A_N - conjugate projector onto the range of the prolongation matrix $I_{N,N/2}$ is given by

$$I_{N,N/2}\, A_{N/2}^{-1}\, I_{N,N/2}^T\, A_N = I_{N,N/2}\, J_{N/2,N} : \quad \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$$

3. (Exercise 13.4:)

Let $A = A_N \in \mathbb{R}^{N\times N}$ be SPD. The iteration matrix (amplification matrix) of a TG scheme with symmetric pre- and post-smoothing has the form

$$G^{TG} = (I - H^T A)(I - CA)(I - HA) =: (I - TA) \tag{1}$$

with a smoother $S = I - HA$ and its adjoint $I - H^T A$, and $C = PA_n^{-1}P^T$, where $P \in \mathbb{R}^{N\times n}$ is a prolongation matrix and $0 < A_n = P^T A\, P \in \mathbb{R}^{n\times n}$ is the Galerkin approximation of A on a coarser level, i.e., in a subspace V_n of dimension $n < N$. (This may result from our context of 'geometric multigrid' (GMG) as introduced above, where we identify vectors with functions. However, it can also be seen in the context of 'algebraic multigrid' (AMG), which directly works with the matrix-vector formulation.) We assume that $P \in \mathbb{R}^{N\times n}$ has full rank n, and $V_n = \text{image}\,(P)$. Then, $PP^T \in \mathbb{R}^{N\times N}$ with $\text{image}(PP^T) = V_n$.

Show:

a) The A - adjoint of $(I - HA)$ is $(I - HA)^A = (I - H^T A)$.

b) $I - CA$ is A - selfadjoint, and the same is true for $G^{TG} = I - TA$.

c) $I - CA$ is *non-expansive* (in the A - norm), i.e., $\rho(I - CA) = \|I - CA\|_A = 1$, hence $\|G^{TG}\|_A = \|I - TA\|_A < 1$, provided $\|I - HA\|_A < 1$.

($\|G^{TG}\|_A$ 'significantly $<$ 1' requires an appropriate smoother H.)

Hint: The subpace- (coarse grid-) correction is a Galerkin approximation. With the error $e = u - u_*$ for given u we have $r = b - A\,u = -A\,e$. The Galerkin subspace (two-grid-) correction is given by $\delta = -PA_n^{-1}P^T\, A\,e$, and this is nothing but the A - best approximation in V_n for the 'exact correction' $-e$. Check once more this fact, i.e., check the Galerkin orthogonality relation $(\delta + e) \perp_A V_n$ by evaluating the inner product $(\delta + e, PP^T y)_A$ for arbitrary vectors $y \in \mathbb{R}^N$. Conclude that $I - GA$ is non-expansive (Pythagoras).

d) Show that the Galerkin approximation operator

$$CA = P\, A_n^{-1}\, P^T A$$

is the A - orthogonal projector onto V_n.

4. (Exercise 13.5:)

Let A be SPD and assume that the TG amplification matrix $G^{TG} = I - TA$ from (1) is contractive, i.e., $\rho(g^{TG}) = \rho(I - TA) < 1$.

Show: The preconditioner T defined in this way is SPD, as required for CG preconditioning.

Hint: Verify that T is symmetric and that $\rho(I - TA) = \|I - TA\|_A < 1$ holds. The desired property $T > 0$ then follows by means of a spectral argument.

5. Consider the 1D Poisson model and use symmetric Gauss-Seidel type pre/post smoothing (without damping) in such a way that you are in the setting of Exercise 3. Use one step of this TG scheme for preconditioning (`pcg`). Compare the performance (residual history) with unpreconditioned cg and cg preconditioned using only symmetric Gauss-Seidel without coarse grid correction.

6. a) (Exercise 13.7:) Assume $\mu = 2$ and $\kappa_l^{TG} \leq \rho \leq \frac{1}{4C}$ on all levels l, where $C \geq 1$ is the constant appearing in (13.39). (Thus, $\rho \leq \frac{1}{4}$ is necessarily assumed.) Show: The W-cycle contraction rate $\kappa_l^{(l)}$ can be uniformly bounded by

$$\kappa_l^{(l)} \leq \frac{1 - \sqrt{1 - 4C\rho}}{2C} \leq 2\rho \leq \frac{1}{2}$$

on all levels $l = 2, 3, 4, \ldots$.

Hint: The sequence $(\kappa_l^{(l)})$ is strictly increasing and majorized by the sequence defined by

$$\xi_2 = \rho, \quad \xi_l = \rho + C\,\xi_{l-1}^2, \quad l = 3, 4, \ldots$$

Consider the latter as a fixed point iteration. Also note that $1 - \sqrt{1-x} \leq x, x \in [0,1]$.

b) Consider again the W-cycle ($\mu = 2$), applied to the 1D ($d = 1$) or 2D ($d = 2$) Poisson problem, with a mesh size h_l proportional to 2^{-l}. The number N_l of unknowns on level l is proportional to the number of mesh points and therefore proportional to 2^{dl}. For the effort E_l of the W-cycle we make the ansatz

$$E_l = \alpha_l\, 2^{dl}\,.$$

We start from the observation that the effort for the smoothing step(s) plus the necessary prolongations and restrictions on level l is bounded by $C\,N_l$ with some constant C. Derive a recursive inequality for α_l and show that the effort N_l for the W-cycle starting on level l is proportional to N_l.