1. MINRES is the variant of GMRES for symmetric (indefinite) systems. Explain what becomes simpler, from the computational point of view, in the symmetric case.

   Hint: What is the shape of R from a QR decomposition of a tridiagonal matrix?

2. (Exercise 11.1:)

   Show that the construction from Sec. 11.1 automatically leads to a full biorthonormal sequence. In particular, we have

   $$0 = (v_{j+1}, w_1) = \ldots = (v_{j+1}, w_{j-1}), \qquad 0 = (w_{j+1}, v_1) = \ldots = (w_{j+1}, v_{j-1})$$

   Hint: Proof by induction, exploiting the structure of the three-term recurrences for the v_j and w_j.

3. Show that Alg. 12.2 is correct.

4. (Exercise 12.3:)

   Assume that the Cholesky decomposition $M = L\,L^T$ of an SPD preconditioner M is available. Consider the split-preconditioned system with $M_L = L$ and $M_R = L^T$, i.e., applying the CG algorithm to the SPD system $L^{-1}A\,L^{-T}u = L^{-1}\,b$.

   Show: The iterates $x_m = L^{-T}u_m$ coincide with those of the left-preconditioned CG method, i.e., Alg. 12.2 with preconditioner M. (Note that the same argument holds for any SPD preconditioner M implicitly specified by a regular matrix C with $M = CC^T$.)

5. Use `pcg` to solve the 2D Poisson problem with left preconditioning by[1] $k \geq 0$ steps of the symmetric (SPD) Gauss-Seidel preconditioner $M = M_{SGS} = (D+L)D^{-1}(D+L^T)$ approximating A. Use two different problem sizes and to different tolerances `TOL` and explore which choice of k leads to the best performance (in terms of number of iterations and in terms of computing time). For specifying the preconditioner, use the `M`- or the `MFUN`-variant – whatever you prefer.

   Hint: For $k = 1$, the action of the preconditioner on a residual r is $M^{-1}r =: \hat{r}$, i.e., the preconditioned residual $\hat{r}$ is the solution of the linear system $M\hat{r} = r$. Verify that this is equivalent to performing one step of the SGS iteration to the linear equation $A\,e = -r$ for the error $e = x - x_*$ starting from the initial value $e_0 = 0$. Then, performing k preconditioning steps means that you perform k SGS steps on this equation starting from $e_0 = 0$.

6. Generally, application of a preconditioner in a particular situation means 'solving a simplified problem'. As an example, we consider the 1D boundary value problem[2]

   $$-\frac{d}{dx}\Big(c(x)\,\frac{du}{dx}\Big) = f(x), \quad u(0) = u(1) = 0,$$

   with a smooth coefficient function $c(x) \geq c_0 > 0$ depending on x.

   We consider an equidistant mesh $(x_0, \ldots, x_n)$ over the interval $[0,1]$, with mesh size $h = 1/n$ and $x_i = i/n$, and define a discrete approximation $(u_0, \ldots, u_n)$ ($u_i \approx u(x_i)$, with $u_0 = u_n = 0$), as the solution of the FD scheme

   $$-\frac{1}{h}\Big(c(x_{i+1/2})\,\frac{u_{i+1}-u_i}{h} - c(x_{i-1/2})\,\frac{u_i-u_{i-1}}{h}\Big) = f(x_i), \quad i = 1\ldots n-1$$

   (with $x_{i\pm1/2} := x_i \pm \frac{h}{2}$).

[1] $k = 0$ means no preconditioing.

[2] This example is not very 'spectacular' and would be more relevant in higher dimension.

a) Show that this gives an SPD linear system for the unknown vector $u = (u_1, \ldots, u_{n-1})$.

b) A simplified problem is $-\bar{c}\,u''(x) = f(x)$, where $\bar{c}$ is the integral mean of $c(x)$ over $[0, 1]$. Implement a stationary iteration, where the FD discretization of the simplified problem is used as an approximation for the original FD discretization.

c) Implement a preconditioned CG iteration, where solution of the simplified problem plays the role of the preconditioner.

d) Choose an example and compare b) with c): How many iterations do we need (starting from $u = 0$) to reduce the norm of the residual by a factor of $1e-5$? Also, compare the corresponding computing times.