Introduction to Matrix Analysis

Richard Bellman

Second Edition

In Applied Mathematics
Introduction to Matrix Analysis

For an elegant probabilistic interpretation of Pueker coordinates, and the analogous multidimensional sets, see Exercises 15 and 16 of Chap. 15, where some results of Karlin and MacGregor are given.

§15. Circulants play an important role in many mathematical-physical theories. See the paper
T. H. Berlin and M. Kac, The Spherical Model of a Ferromagnet, 

for an evaluation of some interesting circulants and further references to work by Onsager, et al., connected with the direct product of matrices and groups.

Treatments of the theory of compound matrices, and further references may be found in
N. G. de Bruijn, Inequalities Concerning Minors and Eigenvalues, 
O. Toeplitz, Das algebraische Analogon zu einem Satze von Fejer, 

13

Stability Theory

1. Introduction. A problem of great importance is that of determining the behavior of a physical system in the neighborhood of an equilibrium state. If the system returns to this state after being subjected to small disturbances, it is called stable; if not, it is called unstable.

Although physical systems can often be tested for this property, in many cases this experimental procedure is both too expensive and too time-consuming. Consequently, when designing a system we would like to have mathematical criteria for stability available.

It was pointed out in Chap. 11 that a linear equation of the form

$$ \frac{dx}{dt} = Ax \quad x(0) = c $$

(1)

can often be used to study the behavior of a system in the vicinity of an equilibrium position, which in this case is $x = 0$.

Consequently, we shall begin by determining a necessary and sufficient condition that the solution of (1) approach zero as $t \to \infty$. The actual economic, engineering, or physical problem is, however, more complicated, since the equation describing the process is not (1), but nonlinear of the form

$$ \frac{dy}{dt} = Ay + g(y) \quad y(0) = c $$

(2)

The question then is whether or not criteria derived for linear systems are of any help in deciding the stability of nonlinear systems. It turns out that under quite reasonable conditions, the two are equivalent. This is the substance of the classical work of Poincaré and Lyapunov. However, we shall not delve into these more recondite matters here, restricting ourselves solely to the consideration of the more tractable linear equations.

2. A Necessary and Sufficient Condition for Stability. Let us begin by demonstrating the fundamental result in these investigations.
Theorem 1. A necessary and sufficient condition that the solution of
\[ \frac{dz}{dt} = Ax \quad z(0) = c \] (1)
regardless of the value of c, approach zero as \( t \to \infty \), is that all the characteristic roots of A have negative real parts.

Proof. If A has distinct characteristic roots, then the representation
\[ e^{zt} = T \begin{bmatrix} 0 & & & \vdots \\ e^{z_1t} & 0 & & \vdots \\ & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{z_Nt} \end{bmatrix} T^{-1} \] (2)
establishes the result. We cannot make an immediate appeal to continuity to obtain the result for general matrices, but we can proceed in the following way.

In place of reducing A to diagonal form, let us transform it into triangular form by means of a similarity transformation, \( T^{-1}AT = B \).

The system of equations in (1) takes the form
\[ \frac{dz}{dt} = Bz \quad z(0) = c' \] (3)
where B is a triangular matrix upon the substitution \( x = Tz \). Written out in terms of the components, we have
\[ \frac{dz_1}{dt} = b_{11}z_1 + b_{12}z_2 + \cdots + b_{1N}z_N \quad z_1(0) = c_1' \\
\frac{dz_2}{dt} = b_{22}z_2 + \cdots + b_{2N}z_N \quad z_2(0) = c_2' \\
\cdots \\
\frac{dz_N}{dt} = b_{NN}z_N \quad z_N(0) = c_N' \] (4)
Since the \( \beta_i \) are the characteristic roots of A, we have, by assumption, \( \text{Re}(\beta_i) < 0 \) for \( i = 1, 2, \ldots, N \).

Solving for \( z_N \),
\[ z_N = e^{\beta_N t}c_N' \] (5)
we see that \( z_N \to 0 \) as \( t \to \infty \).

Stability Theory

In order to show that all \( z_i \to 0 \) as \( t \to \infty \), we proceed inductively based upon the following result.

If \( \xi(t) \to 0 \) as \( t \to \infty \), then \( u(t) \) as determined by
\[ \frac{du}{dt} = bu + \xi(t) \quad u(0) = a_1 \] (6)
approaches zero as \( t \to \infty \), provided that \( \text{Re}(\beta) < 0 \).

Since
\[ u(t) = a_1 e^{\beta t} + \int_0^t e^{\beta(s)}\xi(s) \, ds \] (7)
it is easy to see that the stated result is valid.

Starting with the result for \( z_N \) based upon (5), we obtain successively the corresponding results for \( z_{N-1}, \ldots, z_1 \).

Exercise

1. Prove Theorem 1 using the Jordan canonical form.

2. Stability Matrices. To avoid wearisome repetition, let us introduce a new term.

Definition. A matrix A will be called a stability matrix if all of its characteristic roots have negative real parts.

Exercises

1. Derive a necessary and sufficient condition that a real matrix be a stability matrix in terms of the matrix \( (A^{N+1}) \) (Reus).

2. What is the corresponding condition if A is complex?

4. A Method of Lyapunov. Let us now see how we can use quadratic forms to discuss questions of asymptotic behavior of the solutions of linear differential equations. This method was devised by Lyapunov and is of great importance in the modern study of the stability of solutions of nonlinear functional equations of all types.

Consider the equation
\[ \frac{dz}{dt} = Ax \quad z(0) = c \] (1)
where \( c \) and A are taken to be real, and the quadratic form
\[ u = \langle x, Yx \rangle \] (2)
where \( Y \) is a symmetric constant matrix as yet undetermined. We have
\[ \frac{du}{dt} = \langle x', Yx \rangle + \langle x, Yx' \rangle \]
\[ = \langle Ax, Yx \rangle + \langle x, YAx \rangle \]
\[ = \langle x, (AY + YA)x \rangle \] (3)
Suppose that we can determine $Y$ so that
\[ A^T + YA = -I \quad (4) \]
with the further condition that $Y$ be positive definite. Then the relation in (3) becomes
\[ \frac{du}{dt} = -\langle x, x \rangle \quad (5) \]
which yields
\[ \frac{du}{dt} \leq -\frac{1}{\lambda_N} u \quad (6) \]
where $\lambda_N$ is the largest characteristic root of $Y$. From (6) we have $u \leq u(0)e^{-\lambda_N t}$. Hence $u \to 0$ as $t \to \infty$. It follows from the positive definite property of $Y$ that each component of $x$ must approach zero as $t \to \infty$.

We know, however, from the result of Sec. 13 of Chap. 12 that if $A$ is a stability matrix, we can determine the symmetric matrix $Y$ uniquely from (4). Since $Y$ has the representation
\[ Y = \int_0^\infty e^{\lambda_N t} \varphi_0 \, dt \quad (7) \]
we have
\[ \langle x, Yx \rangle = \int_0^\infty \langle x, e^{\lambda_N t} \varphi_0 \rangle \, dt \]
\[ = \int_0^\infty \langle x, e^{\lambda_N t} \varphi_0 \rangle \, dt \quad (8) \]
It follows that $Y$ is positive definite, since $e^{\lambda_N t}$ is never singular.

**EXERCISES**

1. Consider the equation $dx/dt = Ax + \varphi(x)$, $x(0) = c$, where
   (a) $A$ is a stability matrix,
   (b) $\parallel \varphi(x) \parallel / \parallel x \parallel \to 0$ as $\parallel x \parallel \to 0$,
   (c) $\parallel x \parallel$ is sufficiently small.
   Let $Y$ be the matrix determined above. Prove that if $x$ satisfies the foregoing nonlinear equation and the preceding conditions are satisfied, then
   \[ \frac{d}{dt} \langle x, Yx \rangle \leq -r_i \langle x, Yx \rangle \]
   where $r_i$ is a positive constant. Hence, show that $x \to 0$ as $t \to \infty$.
2. Extend the foregoing argument to treat the case of complex $A$.

**5. Mean-square Deviation.** Suppose that $A$ is a stability matrix and that we wish to calculate
\[ J = \int_0^\infty \langle x, Bx \rangle \, dt \quad (1) \]
where $x$ is a solution of (4.1).

**Stability Theory**

It is interesting to note that $J$ can be calculated as a rational function of the elements of $A$ without the necessity of solving the linear differential equation for $x$. In particular, it is not necessary to calculate the characteristic roots of $A$.

Let us determine a constant matrix $Y$ such that
\[ (x, Bx) = \frac{d}{dt} (x, Yx) \quad (2) \]
We see that
\[ B = A^T + YA \quad (3) \]
With this determination of $Y$, the value of $J$ is given by
\[ J = -\langle c, Yc \rangle \quad (4) \]
Since $A$ is a stability matrix, (3) has a unique solution which can be found using determinants.

**6. Effective Tests for Stability.** The problem of determining when a given matrix is a stability matrix is a formidable one, and at the present time there is no simple solution. What complicates the problem is that we are not so much interested in resolving the problem for a particular matrix $A$ as we are in deriving conditions which enable us to state when various members of a class of matrices, $A(a)$, are stability matrices. Questions of this type arise constantly in the design of control mechanisms, in the field of mathematical economics, and in the study of computational algorithms.

Once the characteristic polynomial of $A$ has been calculated, there are a variety of criteria which can be applied to determine whether or not all the roots have negative real parts. Perhaps the most useful of these are the criteria of Hurwitz.

Consider the equation
\[ \lambda I - A = \lambda^N + a_1 \lambda^{N-1} + \cdots + a_N \lambda + a_N = 0 \quad (1) \]
and the associated infinite array
\[ a_1 \quad 1 \quad 0 \quad 0 \quad 0 \quad \cdots \]
\[ a_2 \quad a_1 \quad 1 \quad 0 \quad 0 \quad \cdots \]
\[ a_3 \quad a_2 \quad a_1 \quad 1 \quad 0 \quad \cdots \]
\[ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots \quad \cdots \]
\[ a_N \quad a_{N-1} \quad a_{N-2} \quad a_{N-3} \quad \cdots \quad 1 \]
\[ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \]
where $a_k$ is taken to be zero for $k > N$. 
A necessary and sufficient condition that all the roots of (1) have negative real parts is that the sequence of determinants

\[
\begin{vmatrix}
    a_1 & 1 \\
    a_2 & 2 \\
\end{vmatrix}
\begin{vmatrix}
    a_1 & 1 & 0 \\
    a_2 & a_3 & a_4 \\
    a_5 & a_6 & a_7 \\
\end{vmatrix}
\]

formed from the preceding array, be positive.

There are no simple direct proofs of this result, although there are a number of elegant proofs. We shall indicate in the following section one line that can be followed, and in Appendix C discuss briefly the chain of ideas, originating in Hermite, giving rise to Hurwitz’s proof. Both of these depend upon quadratic forms. References to other types of proof will be found at the end of the chapter.

The reason why this result is not particularly useful in dealing with stability matrices is that it requires the evaluation of $|A - I|$, something we wish to avoid even if the dimension of $A$ is high.

**EXERCISES**

1. Using the foregoing criteria, show that a necessary and sufficient condition that $\lambda^2 + a_1 \lambda + a_0 = 0$ is a stability polynomial is that $a_1 > 0$, $a_0 > 0$. By this we mean that the roots of the polynomial have negative real parts.

2. For $\lambda^3 + a_1 \lambda^2 + a_0 \lambda + a_0 = 0$ show that corresponding conditions are $a_1 > 0$, $a_0 > a_3 > 0$.

3. For $\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_0 = 0$ show that the conditions are $a_1 > 0$, $a_0 > a_4 > a_3 > a_2 > a_1$.

**7. A Necessary and Sufficient Condition for Stability Matrices.** Let us now show that the results of Sec. 4 yield

**Theorem 2.** Let $Y$ be determined by the relation

\[
AY + YA = -I
\]

Then a necessary and sufficient condition that the real matrix $A$ be a stability matrix is that $Y$ be positive definite.

**Proof.** Referring to Sec. 5, we see that

\[
\int_0^T (x, x) dt = (x(0), x(0)) - (x(T), x(T))
\]

or

\[
(x(T), x(T)) + \int_0^T (x, x) dt = (x(0), x(0))
\]

This is equivalent to the equation

\[
(x'(s), x'(s)) + 4 \int_0^s (x', x') dt + (x(s), x(s)) = c_4
\]

where $c_4 = (a_1, a_2 ^2 + (c_1, c_2 ^2)$.

If $\lambda$ is a root of (2), then (1) has a solution of the form $e^{\lambda t}$. If $\lambda$ is real, $A$ is real. If $\lambda$ is complex, $\lambda = r_1 + ir_2$, then the real part of $e^{\lambda t}$, which has the form $e^{r_1(t \cos r_1 + t \sin r_2)}$, is also a solution. Substituting in (6), we see that

\[
e^{r_1(t \cos r_1 + t \sin r_2)} (b_1, A b_2) + 4 \int_0^s e^{r_1(t \cos r_1 + t \sin r_2)} dt + e^{r_1(t \cos r_1 + t \sin r_2)} c_4 = c_1
\]

where $b_1$ and $b_2$ are constant vectors and $b_1$ is a variable vector given by $(a_1 \cos r_1 + a_2 \sin r_2) cos r_2 + (a_2 \cos r_2 - a_1 \sin r_2) sin r_2$. If $\lambda$ is positive definite, with $B \geq 0$, we see that $r_1 > 0$ leads to a contradiction as $s \to \infty$.

If $A, C \geq 0$, then $B$ positive definite requires that $r_1 \leq 0$. Furthermore, since $a_1 \cos r_1 + a_2 \sin r_2$ is periodic, we see that the integral

\[
\int_0^s (b_1, A b_2) dt
\]

diverges to plus infinity as $s \to \infty$, unless $r_2 = 0$, if $r_1 = 0$.

**Stability Theory**

8. Differential Equations and Characteristic Values. Consider the differential equation

\[
A x'' + 2B x' + C x = 0 \quad x(0) = c_1 \quad x'(0) = c_2
\]

which, if considered to arise from the study of electric circuits possessing capacitances, inductances, and resistances, is such that $A, B$, and $C$ are non-negative definite.

On these grounds, it is intuitively clear that the following result holds.

**Theorem 3.** If $A, B$, and $C$ are non-negative definite, and either $A$ or $C$ positive definite, then

\[
|\lambda A + 2B + C| = 0
\]

has no roots with positive real parts.

If $A$ and $C$ are non-negative definite and $B$ is positive definite, then the only root with zero real part is $\lambda = 0$.

**Proof.** Let us find a proof which makes use of the physical background of the statement. In this case, we shall use energy considerations. Starting with (1), let us write

\[
(x', A x') + 2(x', B x') + (x', C x) = 0
\]

Thus, for any $s > 0$,

\[
\int_0^s [(x', A x') + 2(x', B x') + (x', C x)] dt = 0
\]

or

\[
(x', A x') + 4 \int_0^s (x', B x') dt + (x(s), C x(s)) = 0
\]

This is equivalent to the equation

\[
(x'(s), x'(s)) + 4 \int_0^s (x', B x') dt + (x(s), C x(s)) = c_4
\]