ON ASYMPTOTIC EXPANSIONS OF THE DISCRETIZATION ERROR FOR STIFF EQUATIONS

Winfried Auzinger, Reinhard Frank, Franz Macsek
Vienna, Austria

We discuss asymptotic expansions of the global discretization error for the implicit Euler scheme applied to stiff nonlinear initial value problems:

$$\begin{array}{ll} \text{given problem} & \text{implicit Euler} \\ y'(t) = f(t, y(t)), & \frac{1}{h}(\varsigma_\nu - \varsigma_{\nu-1}) = f(t_\nu, \varsigma_\nu), \quad \nu = 1,2,\ldots, \\ y(0) = z_0, & \varsigma_0 = z_0. \end{array} \tag{1}$$

$z(t)$ denotes the exact solution of the given problem; h is the stepsize used, and $t_\nu = \nu h$.

It can easily be shown (cf. [1]) that the global error of the implicit Euler scheme admits an expansion

$$\varsigma_\nu - z(t_\nu) = he_1(t_\nu) + \ldots + h^q e_q(t_\nu) + R_\nu\,, \tag{2}$$

where the $e_i(t)$ are solutions of the so-called "variational equations":

$$\begin{aligned} e_1'(t) &= f_y(t, z(t))\, e_1(t) + \frac{1}{2} z''(t)\,, \\ e_2'(t) &= f_y(t, z(t))\, e_2(t) - \frac{1}{6} z'''(t) + \frac{1}{2} e_1''(t) + \frac{1}{2} f_{yy}(t, z(t))\, e_1^2(t)\,, \\ &\vdots \end{aligned} \tag{3}$$

The remainder term R_ν satisfies the nonlinear difference equation

$$\frac{1}{h}(R_\nu - R_{\nu-1}) = \hat{J}(R_\nu) \cdot R_\nu \;+\; b_\nu - c_\nu\,, \tag{4}$$

where

$$\hat{J}(R_\nu) := \int_0^1 f_y\big(t_\nu,\, z(t_\nu) + he_1(t_\nu) + \ldots + h^q e_q(t_\nu) + \sigma R_\nu\big)\, d\sigma\,; \tag{5}$$

the inhomogeneity of $b_\nu - c_\nu$ of (4) is defined as follows:

$$c_\nu := -\frac{1}{h} I_{\nu,0} - I_{\nu,1} - hI_{\nu,2} - \ldots - h^{q-1} I_{\nu,q}\,, \tag{6}$$

where the $I_{\nu,i}$ are the remainder terms of the Taylor expansions

$$\begin{aligned} z(t_\nu - h) &= z(t_\nu) - hz'(t_\nu) + \ldots + (-1)^{q+1} \frac{h^{q+1}}{(q+1)!} z^{(q+1)}(t_\nu) + I_{\nu,0}\,, \\ e_1(t_\nu - h) &= e_1(t_\nu) - he_1'(t_\nu) + \ldots + (-1)^q \frac{h^q}{q!} e_1^{(q)}(t_\nu) + I_{\nu,1}\,, \\ &\vdots \\ e_q(t_\nu - h) &= e_q(t_\nu) - he_q'(t_\nu) + I_{\nu,q}\,, \end{aligned} \tag{7}$$

and b_ν is defined as the collection of all terms of an order $\geq q+1$ of the Taylor expansion of $f(t_\nu, z(t_\nu) + he_1(t_\nu) + \ldots + h^q e_q(t_\nu))$ around $(t_\nu, z(t_\nu))$, i.e.,

$$b_\nu := h^{q+1} f_{yy}(t_\nu, z(t_\nu))\, e_1(t_\nu) e_q(t_\nu) + \ldots . \tag{8}$$

For $\nu = 0$, (2) reads

$$0 = \varsigma_0 - z(0) = he_1(0) + \ldots + h^q e_q(0) + R_0 ; \tag{9}$$

hence a natural choice for the starting values would be $e_1(0) = e_2(0) = \ldots = e_q(0) = R_0 = 0$. However, another choice of starting values satisfying (9) will be appropriate in the stiff case. Further, having nonequidistant grids in mind, we consider an integration interval $[0, T]$ which is to be interpreted as a *subinterval* (with constant stepsize h) of the whole integration interval. Then, in contrast to (9), $\varsigma_0 - z(0)$ is not 0 but is the accumulated error from the preceding subintervals.

The essential question is whether the remainder term R_ν satisfies

$$R_\nu = O(h^{q+1}) . \tag{10}$$

In the "classical" (non-stiff) case this can easily be shown under suitable smoothness assumptions (i.e., all occuring derivatives are to be assumed to exist and to be of moderate size): Then, $b_\nu - c_\nu = O(h^{q+1})$, and (10) follows immediatetely by a conventional stability estimate for (4).

For stiff problems, however, where any Lipschitz bound L (w.r.t. y) for $f(t, y)$ (cf. (1)) is large, the classical argumentation is useless and does not describe the situation in a realistic way. One might expect that a refined "B-convergence" estimate for the difference equation (4), based on a one-sided Lipschitz bound m (instead of L), leads again to (10) (with a O-constant which does only depend on moderately sized quantities): Due to

$$\left\| \left(I - h\hat{J}(R_\nu)\right)^{-1} \right\| \leq \frac{1}{1 - hm} , \tag{11}$$

R_ν can be estimated by

$$\|R_\nu\| \leq \begin{cases} \|R_0\| \; + \; t_\nu \cdot \max\limits_{\mu=1,\ldots,\nu} \|b_\mu - c_\mu\|, & m = 0 , \\ \left(\frac{1}{1-hm}\right)^\nu \|R_0\| \; + \; \frac{1}{m} \left[\left(\frac{1}{1-hm}\right)^\nu - 1 \right] \cdot \max\limits_{\mu=1,\ldots,\nu} \|b_\mu - c_\mu\|, & m \neq 0 . \end{cases} \tag{12}$$

But from (12) the desired estimate $R_\nu = O(h^{q+1})$ (in the B-sense) can only be concluded if $R_0 = O(h^{q+1})$ and $b_\nu - c_\nu = O(h^{q+1})$ for all ν. Unfortunately, the latter cannot be expected in general: The variational equations (3) are obviously stiff (with the same stiff spectrum as for the given problem (1)), and, except for special starting values $e_i(0)$, their solutions $e_i(t)$ are not smooth. In particular, the $e_i(t)$ will contain transient terms of the type $e^{\lambda t}$, where $\lambda \ll 0$ denotes a stiff eigenvalue. Recall that c_ν is a linear combination of terms of the form

$$h^{i-1} I_{\nu,i} = \frac{h^{q+1}}{(q+1-i)!} \int_0^1 \frac{d^{q+2-i}}{dt^{q+2-i}} e_i(t_\nu - \sigma h)(1-\sigma)^{q+1-i} \, d\sigma ; \tag{13}$$

since $\frac{d^k}{dt^k} e^{\lambda t} = \lambda^k e^{\lambda t}$, c_ν will be influenced by powers of λ and therefore by powers of L. Thus, $b_\nu - c_\nu = O(h^{q+1})$ (with a moderate O-constant) cannot be expected in general, and the desired result cannot be concluded immediately from (12).

A closer look at simple stiff models (with a stiff eigenvalue λ) shows that, for these models, $\|b_\nu - c_\nu\| \leq const \cdot h^{q+1}$ is satisfied for all ν (with some moderate constant) if $0 < \frac{1}{|\lambda|} \ll h$ ("strongly stiff case"). If, however, $\frac{1}{|\lambda|} \ll h$ is not satisfied ("mildly stiff case"), we only have $\|b_\nu - c_\nu\| \leq const \cdot h$ at the first grid points. The consequence of the latter is that the order of the remainder term R_ν itself breaks down at the first grid points in the mildly stiff case. Therefore these models show that full asymptotic expansions cannot be expected to exist in general for stiff problems.

In the scalar case, where the one-sided Lipschitz constant m satisfies $m \approx \lambda \ll 0$, it is immediately clear that order reduction effects at the begin are *quickly damped away* by the factor $(\frac{1}{1-hm})^\nu$ (cf. (12)). *However*, it is important to notice that for non-scalar problems such a damping effect cannot be concluded from (12): In the vector case there are usually non-stiff eigenvalues of moderate size besides the stiff eigenvalues. Since m is influenced by these non-stiff eigenvalues (note that $m \geq \max Re(\lambda_i)$, $\lambda_i \ldots$ eigenvalues of f_y), the $(\frac{1}{1-hm})^\nu$-factors have no damping power: in many cases, $m > 0$ and then $(\frac{1}{1-hm})^\nu$ increases with ν; but also if $m < 0$ (but not $m \ll 0$), $h|m|$ is small and $(\frac{1}{1-hm})^\nu$ decreases very slowly.

Studying certain non-scalar models we observed that, however, similar damping properties hold as in the scalar case. Therefore our main concern was to develop a general theory and to show that these damping effects do indeed hold for a large class of stiff problems. This is by no means trivial: Conventional B-convergence estimates provide no insight into damping effects; our theory is based on a singular perturbation analysis of the remainder term equation (4). (In the context of stiff equations, singular perturbation concepts were first extensively used by Van Veldhuizen, cf. for instance [4].)

Order reductions within $b_\nu - c_\nu$ can be avoided if one chooses special starting values $e_i(0)$ for the variational equations (3) such that the $e_i(t)$ are *smooth solutions* of these stiff equations. This means that any terms of the type $e^{\lambda t}$ within the $e_i(t)$ must contain an additional factor λ^{-k} – with $k = q + 2 - i$ (cf. (13)) – such that all derivatives up to the order k are at $O(1)$-level. But then another problem arises: From (9),

$$R_0 = -he_1(0) - \ldots - h^q e_q(0)\,, \tag{14}$$

and thus it must be expected that only $R_0 = O(h)$ holds for these special starting values $e_i(0)$ (which do usually not vanish). Therefore, again, an order reduction within R_ν at the begin is unavoidable, and the damping of R_ν has to be studied. It turned out that the damping of an order reduction caused by $R_0 = O(h)$ can better be understood than the effects of $b_\nu - c_\nu = O(h)$ (in the case of non-smooth solutions). The idea to work with smooth solutions of the variational equations is essentially due to Dahlquist and Lindberg [2].

We now present a short outline of our theory, the details of which can be found in [1]. The class of stiff problems under consideration is the following: We assume that

$$f_y(t, z(t)) = T(t)\,\Lambda(t)\,T^{-1}(t) \tag{15}$$

with a smooth, well-conditioned transformation $T(t)$, and

$$\Lambda(t) = \text{diag}\left(c_1(t),\ -\frac{c_2(t)}{\varepsilon}\right), \tag{16}$$

where the $c_k(t)$ are smooth and $Re(c_2(t)) \geq \kappa > 0$. $\varepsilon > 0$ is a small real parameter characterizing the stiffness. Our analysis refers to the 2-dimensional case (i.e., $c_1(t)$ and $c_2(t)$ are scalar functions). However, all considerations may also be understood in the sense that the $c_k(t)$ are vector-valued. The higher derivatives of f are assumed to be smooth and bounded, i.e.,

$$f_{yy}(t,y) = O(\varepsilon^0), \qquad f_{yyy}(t,y) = O(\varepsilon^0), \qquad \ldots \tag{17}$$

To enable the application of singular perturbation techniques we now transform (3) and (4) according to

$$\bar{e}_i(t) := T^{-1}(t)\, e_i(t), \qquad \rho_\nu := T^{-1}(t_\nu) R_\nu \equiv T_\nu^{-1} R_\nu\,. \tag{18}$$

This leads to transformed variational equations of the type

$$\bar{e}_i'(t) = \Lambda(t)\, \bar{e}_i(t) + A(t)\, \bar{e}_i(t) + \bar{g}_i(t)\,, \tag{19}$$

where $A(t) := -T^{-1}(t)\, T'(t)$ and where $\bar{g}_i(t)$ denotes the transformed inhomogeneity (which depends recursively on all $\bar{e}_j(t),\ j < i$). The transformed remainder term equation reads

$$\frac{1}{h}(\rho_\nu - \rho_{\nu-1}) = \Lambda_\nu\, \rho_\nu + \Theta_\nu\, \rho_{\nu-1} + \Gamma_\nu(\rho_\nu) + \delta_\nu\,, \tag{20}$$

where $\Lambda_\nu \equiv \Lambda(t_\nu)$, $\Theta_\nu \equiv \Theta(t_\nu) := -T_\nu^{-1} \frac{1}{h}(T_\nu - T_{\nu-1})$, $\delta_\nu := T_\nu^{-1}(b_\nu - c_\nu)$, and where Γ_ν is a certain smooth nonlinear function (cf. [1,(5.8c)]). Using the denotations

$$\bar{e}_i(t) = \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix}, \quad \bar{g}_i(t) = \begin{pmatrix} r_i(t) \\ s_i(t) \end{pmatrix}, \qquad \rho_\nu = \begin{pmatrix} x_\nu \\ y_\nu \end{pmatrix}, \quad \delta_\nu = \begin{pmatrix} \delta_{\nu;1} \\ \delta_{\nu;2} \end{pmatrix}, \tag{21}$$

and multiplying the second components by ε, we obtain from (19),

$$\begin{aligned} x_i'(t) &= c_1(t)x_i(t) + a_{1,1}(t)x_i(t) + a_{1,2}(t)y_i(t) + r_i(t)\,, \\ \varepsilon\, y_i'(t) &= -c_2(t)y_i(t) + \varepsilon\, a_{2,1}(t)x_i(t) + \varepsilon\, a_{2,2}(t)y_i(t) + \varepsilon\, s_i(t)\,, \end{aligned} \tag{22}$$

and from (20),

$$\begin{aligned} \frac{1}{h}(x_\nu - x_{\nu-1}) &= c_1(t_\nu)x_\nu + \vartheta_{1,1}(t_\nu)x_{\nu-1} + \vartheta_{1,2}(t_\nu)y_{\nu-1} + \gamma_1(t_\nu, x_\nu, y_\nu) + \delta_{\nu;1}\,, \\ \frac{\varepsilon}{h}(y_\nu - y_{\nu-1}) &= -c_2(t_\nu)y_\nu + \varepsilon\,\vartheta_{2,1}(t_\nu)x_{\nu-1} + \varepsilon\,\vartheta_{2,2}(t_\nu)y_{\nu-1} + \varepsilon\,\gamma_2(t_\nu, x_\nu, y_\nu) + \varepsilon\,\delta_{\nu;2}\,. \end{aligned} \tag{23}$$

(Here, $a_{k,l}(t)$, $\vartheta_{k,l}(t_\nu)$ and $\gamma_k(t_\nu, x_\nu, y_\nu)$ denote the respective components of $A(t)$, Θ_ν and $\Gamma_\nu(\rho_\nu)$.)

The structure of solutions $\bar{e}_i(t)$ of the transformed variational equations (22) can be described by means of singular perturbation techniques (cf. for instance O'Malley [3]). We introduce the "stretched time variable" $\tau := \frac{t}{\varepsilon}$, and make the ansatz

$$\bar{e}_i(t) = \begin{pmatrix} x_i(t) \\ \\ y_i(t) \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{N} \varepsilon^k X_{k,i}(t) + \varepsilon \sum_{k=0}^{N-1} \varepsilon^k m_{k,i}(\tau) + x_{rem,i}(t,\varepsilon) \\ \sum_{k=0}^{N} \varepsilon^k Y_{k,i}(t) + \sum_{k=0}^{N} \varepsilon^k n_{k,i}(\tau) + y_{rem,i}(t,\varepsilon) \end{pmatrix}. \tag{24}$$

After inserting into (22) and equating coefficients of $\varepsilon^k, k = 0, 1, \dots$, it turns out that the $X_{k,i}(t)$ and $Y_{k,i}(t)$ are smooth functions of t and that $Y_{0,i}(t) \equiv 0$. Further the $m_{k,i}(\tau)$ and $n_{k,i}(\tau)$ are (essentially) of the form $p(\tau)e^{-c_2(0)\tau}$ with certain polynomials $p(\tau)$. Moreover, $x_{rem,i}(t,\varepsilon) = O(\varepsilon^{N+1})$, $y_{rem,i}(t,\varepsilon) = O(\varepsilon^{N+1})$.

Derivatives of the $m_{k,i}(\tau)$ and $n_{k,i}(\tau)$ w.r.t. t are not bounded uniformly for $\varepsilon \to 0$. A solution $\bar{e}_i(t)$ of (22) is called *smooth* if $m_{k,i}(\tau) \equiv n_{k,i}(\tau) \equiv 0$ up to the ε^k-level for $k = q+2-i$ (cf. (13)). In this case, $\frac{d^k}{dt^k}\bar{e}_i(t) = O(1)$, and therefore $b_\nu - c_\nu = O(h^{q+1})$, as required. To obtain smooth solutions $\bar{e}_i(t)$ in the above sense it is necessary to choose *special starting values* $\bar{e}_i(0)$. The considerations in [1, Section 3] show that the first components $x_i(0)$ of these special starting values can be chosen arbitrarily at $O(1)$-level whereas the second components $y_i(0)$ are then fixed.

To discuss the behaviour of the solution ρ_ν of the transformed remainder term equation (23) we assume $\varepsilon \le Ch$ with some moderate constant C. (The case ε small but $\varepsilon \le Ch$ not satisfied will be discussed below.) We make a discrete singular perturbation ansatz:

$$\rho_\nu = \begin{pmatrix} x_\nu \\ y_\nu \end{pmatrix} = \begin{pmatrix} h\sum_{k=0}^{q-1} h^k X_\nu^{(k)} + h^2 \sum_{k=0}^{q-2} h^k \xi_\nu^{(k)} + x_\nu^{(rem)} \\ h\sum_{k=0}^{q-1} h^k Y_\nu^{(k)} + h \sum_{k=0}^{q-1} h^k \eta_\nu^{(k)} + y_\nu^{(rem)} \end{pmatrix}. \tag{25}$$

In contrast to (24) we use an ansatz in powers of the stepsize h (and not of ε). This is quite natural now because we are concerned with the study of order reduction effects (i.e., reduced powers of h) within ρ_ν, and, moreover, because any power ε can be written in the form $\varepsilon^l = (\frac{\varepsilon}{h})^l h^l$ where $\frac{\varepsilon}{h}$ is a moderate quantity due to assumption $\varepsilon \le Ch$.

Consider any starting value $\rho_0 = O(h)$. By inserting (25) into (23) and equating coefficients of h^k, $k = 1, 2, \dots$, it can be shown that the $\xi_\nu^{(k)}$ and $\eta_\nu^{(k)}$ are (essentially) of the type

$$\frac{\varepsilon}{h} p(\nu) Q^{-\nu}, \quad \text{where} \quad Q := 1 + \frac{c_2(0)h}{\varepsilon}, \tag{26}$$

with certain polynomials $p(\nu)$. (These terms are discrete analoga of the $m_{k,i}(\tau)$ and $n_{k,i}(\tau)$ (cf. (24)) which are of the type $p(\tau)e^{-c_2(0)\tau}$.) For $\varepsilon \le Ch$ we have $Q \ge 1 + \frac{c_2(0)}{C}$ and therefore $p(\nu)Q^{-\nu}$ is bounded and rapidly decaying with increasing ν. Our goal is now to ensure that – once that the ξ_ν- and η_ν-terms of (25) are damped away – ρ_ν is at h^{q+1}-level. The latter is true if

$$X_\nu^{(k)} \equiv Y_\nu^{(k)} \equiv 0,\ k = 0, \dots, q-1, \quad \text{and} \quad x_\nu^{(rem)} = O(h^{q+1}),\ y_\nu^{(rem)} = O(h^{q+1}). \tag{27}$$

Whether this is the case or not depends on the particular starting value $\rho_0 = T^{-1}(0)R_0$. Due to (14), ρ_0 depends on the starting values $\bar{e}_i(0) = T^{-1}(0)e_i(0)$. In [1, Section 5] it is shown how to use the degree of freedom within the first components of the starting values for smooth $\bar{e}_i(t)$ such that (27) is satisfied. The details of this argumentation are cumbersome and somewhat technical. A discussion of all these technical details is outside the scope of the present paper. The interested reader is referred to [1].

We now report the results of our analysis. To keep the presentation transparant we here restrict ourselves to the special case of an equidistant grid and a starting value $\varsigma_0 = z(0)$ on a smooth solution $z(t)$ of the original problem (cf. (1)). (In [1] the general case of non-equidistant grids, including a transient phase, is discussed extensively.)

Theorem 1. (*The Strongly Stiff Case, Theorem* 4.1 *of* [1].)

Assume that (1) *satisfies the smoothness assumptions* (15) – (17) *and assume* ε *so small that*

$$\varepsilon \leq C\, h^q \tag{28}$$

with some moderate constant C. *Then the discretization error of the implicit Euler scheme admits a full asymptotic expansion*

$$\varsigma_\nu - z(t_\nu) = he_1(t_\nu) + \ldots + h^q e_q(t_\nu) + R_\nu \tag{29}$$

with smooth, h-independent functions $e_i(t)$ *which are solutions of the variational equations* (3), *and*

$$\|R_\nu\| \leq C\, h^{q+1} \tag{30}$$

(*with some moderate constant* C) *at all grid points* t_ν.

Theorem 2. (*The Mildly Stiff Case, Theorem* 5.1 *of* [1].)

Assume that (1) *satisfies the smoothness assumptions* (15) – (17). *Assume further* $q = 4$ *and*

$$\varepsilon \leq C\, h \tag{31}$$

with some moderate constant C. *Then the discretization error of the implicit Euler scheme admits an asymptotic expansion*

$$\varsigma_\nu - z(t_\nu) = he_1(t_\nu) + \ldots + h^4 e_4(t_\nu) + R_\nu \tag{32}$$

with smooth, h-independent functions $e_i(t)$ *which are solutions of the variational equations* (3), *and* $R_\nu = T_\nu\, \rho_\nu$ (*cf.* (18)) *with*

$$\rho_\nu = \begin{pmatrix} h^3 \xi_\nu^{(1)} + h^4 \xi_\nu^{(2)} + x_\nu^{(rem)} \\ h^2 \eta_\nu^{(1)} + h^3 \eta_\nu^{(2)} + h^4 \eta_\nu^{(3)} + y_\nu^{(rem)} \end{pmatrix}. \tag{33}$$

The terms $\xi_\nu^{(j)}$, $\eta_\nu^{(j)}$ *are bounded by*

$$\frac{\varepsilon}{h}\, C(\varepsilon, h)\, p(\nu) |Q|^{-\nu} \tag{34}$$

(*with* Q *from* (26)), *which is rapidly decaying with increasing* ν. $C(\varepsilon, h)$ *denotes some quantity which is of moderate size for* $\varepsilon \leq C\, h$, *and* $p(\nu)$ *denotes some polynomial of low degree. Furthermore,*

$$\left\| \begin{pmatrix} x_\nu^{(rem)} \\ y_\nu^{(rem)} \end{pmatrix} \right\| \leq C\, h^5 \tag{35}$$

with some moderate constant C.

The argumentation which is used for the proof of Theorem 2 cannot be extended in all points to the general case. Consequently, we only have a slightly weaker formulation of our results for $q \geq 5$:

Theorem 3. (*The Mildly Stiff Case, Theorem* 5.6 *of* [1].)

Assume that (1) *satisfies the smoothness assumptions* (15) – (17). *Let* q *be arbitrary and assume*

$$\varepsilon \leq C\,h \tag{36}$$

with some moderate constant C. *Then the discretization error of the implicit Euler scheme admits an asymptotic expansion*

$$\zeta_\nu - z(t_\nu) = h e_1(t_\nu; h) + \ldots + h^q e_q(t_\nu; h) + R_\nu \tag{37}$$

with smooth functions $e_i(t;h)$ *which are solutions of the variational equations* (3) *and depend in a moderate way on* h *for* $\varepsilon \leq C\,h$. *Further,* $R_\nu = T_\nu\, \rho_\nu$ *where*

$$\rho_\nu = \begin{pmatrix} h^3\,\xi_\nu^{(1)} + \ldots + h^q\,\xi_\nu^{(q-2)} + x_\nu^{(rem)} \\ h^2\eta_\nu^{(1)} + h^3\eta_\nu^{(2)} + \ldots + h^q\eta_\nu^{(q-1)} + y_\nu^{(rem)} \end{pmatrix}. \tag{38}$$

The terms $\xi_\nu^{(j)}$, $\eta_\nu^{(j)}$ *are rapidly decaying and bounded by* (34). *Moreover,*

$$\left\| \begin{pmatrix} x_\nu^{(rem)} \\ y_\nu^{(rem)} \end{pmatrix} \right\| \leq C\,h^{q+1} \tag{39}$$

with some moderate constant C.

The question arises whether asymptotic expansions in the sense of (37) with h-dependent functions $e_i(t) = e_i(t;h)$ are reasonable and useful for practical applications. The analysis of acceleration techniques like Defect Correction Methods, which strongly relies upon asymptotic error expansions, shows that an assertion like (37) is completely sufficient: It is not required that the $e_i(t)$ are independent of h (as in the assertion of Theorem 2 for $q = 4$); only the *smoothness* of the $e_i(t)$ and the structure of the variational equations (from which bounds for the derivatives of the $e_i(t) = e_i(t;h)$ can be concluded) are essential. In the mildly stiff case the ***really*** essential point is the presence of the inner solution terms which cause order reductions at the first grid points: The high convergence order of Defect Correction methods does indeed break down at the first grid points after a change of the stepsize, but it reappears by and by at the subsequent grid points.

Up to now we have assumed (Theorems 2 and 3) that a relation $\varepsilon \leq C\,h$ holds, which is a realistic assumption for many practical applications. However, $\varepsilon \leq C\,h$ is violated in the following situations:

i) in a transient phase, where the solution $z(t)$ is not smooth and where very small stepsizes h are necessary to compensate the large derivatives of $z(t)$ (ensuring small local and global errors),

ii) after a transient phase, where $z(t)$ is smooth, if the accuracy requirements are so strong that a stepsize h satisfying $h \ll \varepsilon$ is necessary.

In case i) the problem may be considered non-stiff and the classical theory applies, ensuring full asymptotic expansions for $h \to 0$. In case ii), however, the situation is quite different: In the spirit of the B-theory it is natural to strive for an estimate $\|R_\nu\| \leq Ch^{q+1}$ (with some moderate constant C independent of the stiffness, i.e., independent of ε) which is not only satisfied for $\varepsilon \leq Ch$ (Theorems 1 - 3) but also for extremely small stepsizes $h \ll \varepsilon$. Similarly as for $\varepsilon \leq Ch$, such an estimate can neither be concluded from classical stability estimates (based on $L = O(\frac{1}{\varepsilon})$) nor from the B-stability estimate (12). Indeed, as can be seen from simple models, we have again to reckon with order reductions within R_ν at the first grid points after a change of the stepsize. The following theorem again ensures that these order reduction effects are damped away.

Theorem 4. (*The Mildly Stiff Case, Theorem* 5.7 *of* [1].)

Assume that (1) *satisfies the smoothness assumptions* (15) - (17). *Let* q *be arbitrary and assume*

$$h \leq C\,\varepsilon \tag{40}$$

with some moderate constant C. *Then the discretization error of the implicit Euler scheme admits an asymptotic expansion*

$$\varsigma_\nu - z(t_\nu) = he_1(t_\nu;h) + \ldots + h^q e_q(t_\nu;h) + R_\nu \tag{41}$$

with smooth functions $e_i(t;h)$ *which are solutions of the variational equations* (3) *and depend in a moderate way on* h *for* $h \leq C\,\varepsilon$. *Further,* $R_\nu = T_\nu\, \rho_\nu$ *where*

$$\rho_\nu = \begin{pmatrix} \varepsilon^3\,\xi_\nu^{(1)} + \ldots + \varepsilon^q\,\xi_\nu^{(q-2)} + x_\nu^{(rem)} \\ \varepsilon^2\eta_\nu^{(1)} + \varepsilon^3\eta_\nu^{(2)} + \ldots + \varepsilon^q\eta_\nu^{(q-1)} + y_\nu^{(rem)} \end{pmatrix}. \tag{42}$$

The terms $\xi_\nu^{(j)}$, $\eta_\nu^{(j)}$ *are bounded by*

$$\frac{h}{\varepsilon}\,C(\varepsilon,h)\,p(\tau_\nu)\left|1+\frac{\kappa h}{\varepsilon}\right|^{-\nu} \tag{43}$$

(*where* $Re(c_2(t)) \geq \kappa > 0$), *which is rapidly decaying with increasing* ν. $C(\varepsilon,h)$ *denotes some quantity which is of moderate size for* $h \leq C\,\varepsilon$; $p(\tau_\nu)$ *denotes some polynomial in* $\tau_\nu = \nu\frac{h}{\varepsilon}$. *Moreover,*

$$\left\|\begin{pmatrix} x_\nu^{(rem)} \\ y_\nu^{(rem)} \end{pmatrix}\right\| \leq \varepsilon^{q+1}\frac{h}{\varepsilon}\,C(\varepsilon,h)\,p(\tau_\nu)\left|1+\frac{\kappa h}{\varepsilon}\right|^{-\nu} + C\,h^{q+1}\,. \tag{44}$$

Remark. In contrast to Theorems 1 - 3, h/ε is now a moderate quantity (due to assumption $h \leq C\,\varepsilon$).

References

[1] W.Auzinger, R.Frank and F.Macsek, *Asymptotic error expansions for stiff equations: The implicit Euler scheme*, Report Nr. 72/87 - Inst. f. Angew. u. Numer. Math., TU Wien. (Submitted to SIAM J. Numer. Anal.)

[2] G.Dahlquist and B.Lindberg, *On some implicit one-step methods for stiff differential equations*, Dept. of Computer Sciences, Royal Institute of Technology, Rep. TRITA-NA-7302, 1973.

[3] R.E.O'Malley Jr., *Introduction to singular perturbations*, Academic Press, New York - London - Toronto - Sydney - San Francisco, 1974.

[4] R.Van Veldhuizen, *D-stability*, SIAM J. Numer. Anal. 18 (1981), pp. 45-64.