A Study of Numerical Elimination for the Solution of Multivariate Polynomial Systems

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Abstract

In an earlier paper we had motivated and described am algorithm for the computation of the zeros of multivariate polynomial systems by means of multiplication tables and solution of an associated eigenvalue problem. In this paper we give a more transparent description of this approach, including degenerate situations. Both the theoretical results and the algorithmic procedures are applied to examples, and the results are displayed and discussed.

1 Introduction

We consider systems of multivariate polynomial equations:

$$F(x) := \begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ \ldots \\ f_n(x_1, \ldots, x_n) = 0 \end{cases} \tag{1.1}$$

Here, $f_\nu(x) := \sum_j a_j^{(\nu)} x^j := \sum_j a^{(\nu)}_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_n^{j_n}, \quad |j| = \sum_{\nu=1}^{n} j_\nu \le m_\nu, \quad a_j^\nu \in \mathbb{C}.$

The *zeros* of (1.1) are denoted by $\xi_\mu := (\xi_{\mu 1}, \ldots, \xi_{\mu n}) \in \mathbb{C}^n$.

The n polynomials f_ν, $\nu = 1(1)n$, form the *polynomial ideal* $\mathcal{F}$. $g(x) \equiv h(x) \bmod \mathcal{F}$ will denote the fact that $g(x) - h(x) \in \mathcal{F}$.

We will also consider a further linear "dummy" polynomial

$$f_0(x) := a_1^{(0)} x_1 + \ldots + a_n^{(0)} x_n + a_0^{(0)}$$

with *indeterminate* coefficients $a_\nu^{(0)}$. The ideal of the $n+1$ polynomials f_ν, $\nu = 0(1)n$, will be called $\mathcal{F}_0$.

It is useful to consider also the natural embedding of (1.1) into the projective n-space $\mathbb{C}\mathrm{P}^n$ (homogenization):

$$\begin{aligned} \tilde{f}_1(x_0, \ldots, x_n) &:= x_0^{m_1} f_1\Big(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\Big) = 0 \\ &\ldots \\ \tilde{f}_n(x_0, \ldots, x_n) &:= x_0^{m_n} f_n\Big(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\Big) = 0 \end{aligned} \tag{1.2}$$

Each zero of (1.1) is a zero of (1.2); but (1.2) may have solutions with $\xi_{\mu 0} = 0$ (and some other $\xi_{\mu\nu} \neq 0$); these are called *zeros at infinity* of (1.1).

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The *multiplicity* of an isolated zero of (1.2) is defined by the factorization of the *resultant* $R(a_0^{(0)}, \ldots, a_n^{(0)})$ of the polynomial system $f_0, f_1, \ldots, f_n$; cf. any text of classical algebra. $R(a_\nu^{(0)})$ is a homogeneous polynomial of degree $m = \prod_{\nu=1}^{m} m_\nu$ (the Bézout number of (1.1)) in the dummy coefficients $a_\nu^{(0)}$ and factors into

$$R(a_\nu^{(0)}) = \prod_{\mu=1}^{m} \left(a_0^{(0)} \xi_{\mu 0} + a_1^{(0)} \xi_{\mu 1} + \ldots + a_n^{(0)} \xi_{\mu n} \right) \tag{1.3}$$

where $(\xi_{\mu 0}, \xi_{\mu 1}, \ldots, \xi_{\mu n}) \in \mathbb{C}\mathrm{P}^n$ are the isolated zeros of (1.2). For the finite zeros of (1.2), the multiplicity carries over to the corresponding zero of (1.1).

Unfortunately, $R(a_\nu^{(0)})$ vanishes identically when (1.2) has solution manifolds. If (1.2) has no solution manifolds, it has precisely m isolated zeros counting multiplicities (cf. (1.3)) and (1.1) has at most m isolated zeros.

In [1], the following method for the numerical determination of all zeros of a multivariate polynomial system (1.1) has been derived:

Consider a set $\bar{Z}_0$ of $\bar{m}$ *power products* (PPs) of the unknowns x_ν:

$$x^j := x_1^{j_1} \cdots x_n^{j_n}, \qquad j \in \bar{\mathcal{J}}_0 \subset \mathbb{N}_0^n, \qquad |\bar{\mathcal{J}}_0| = \bar{m}, \quad 0 \in \bar{\mathcal{J}}_0 \,,$$

and denote them as *basis power products* (BPPs). Let $\bar{Z}_0$ also denote the formal *vector* of theses BPPs in some agreed order, with $x^0 = 1$ as last component.

Assume that we are able to find, for $\nu = 1(1)n$, $\bar{m}$-vectors of multivariate polynomials $X_\lambda^{(\nu)}(x) = \left(\chi_{\lambda\mu}^{(\nu)}(x) \right)$, $\lambda = 1(1)n$, $\mu = 1(1)\bar{m}$, such that the associated polynomial combinations of the f_λ reduce to

$$\sum_{\lambda=1}^{n} X_\lambda^{(\nu)}(x)\, f_\lambda(x) = \left(\bar{B}^{(\nu)} - x_\nu I \right) \bar{Z}_0, \qquad \nu = 1(1)n. \tag{1.4}$$

(The $\bar{B}^{(\nu)}$ are $\bar{m}\times\bar{m}$-matrices with real resp. complex elements for real resp. complex f_λ.) Then we say that the $\bar{B}^{(\nu)}$ determine a *multiplication table* mod $\mathcal{F}$ w.r.t. the PP basis $\bar{Z}_0$.

Consider a zero $\xi_\mu = (\xi_{\mu 1}, \ldots, \xi_{\mu n}) \in \mathbb{C}^n$ of (1.1) and denote the *numerical* vector which arises after substitution of ξ_μ into $\bar{Z}_0$ by $\bar{z}_{0\mu} \in \mathbb{C}^{\bar{m}}$. By (1.4), $\bar{z}_{0\mu}$ has to satisfy

$$\left(\bar{B}^{(\nu)} - \xi_{\mu\nu} I \right) \bar{z}_{0\mu} = 0, \qquad \text{for} \quad \nu = 1(1)n, \tag{1.5}$$

i.e., $\bar{z}_{0\mu}$ has to be a *joint eigenvector* of the $\bar{B}^{(\nu)}$, with resp. eigenvalues $\xi_{\mu\nu}$. Thus, the joint eigenvectors of the $\bar{B}^{(\nu)}$ are the only candidates for BPP vectors $\bar{z}_{0\mu}$ associated with a zero of (1.1) ("*zero BPP vectors*").

Actually, it turns out that (1.5) is a full characterization of the zero BPP vectors of the system (1.1). For a suitable basis $\bar{Z}_0$ of PPs, *each* joint eigenvector of the $\bar{B}^{(\nu)}$ of (1.4) defines a zero of (1.1) through its eigenvalues $\xi_{\mu\nu}$ as well as through its components corresponding to a BPP x_ν. Thus, the determination of all (finite) zeros of (1.1) is reduced to the generation of the $\bar{B}^{(\nu)}$ for an appropriate $\bar{Z}_0$ basis and to the computation of the $\bar{m}$ joint eigenvectors of the $\bar{B}^{(\nu)}$ (they commute), and the computation of the zeros proceeds entirely in the context of Linear Algebra.

In [1], we had motivated and described the generation of multiplication tables (1.4) and the computation of the zeros from associated eigenvalue problems only for the "non-degenerate case" where all zeros of (1.1) are finite. (We will use the terms "non-deficient" resp. "deficient" in this paper.) Even for this case, a more refined analysis of the situation had not been given.

In the present paper, we will give a more transparent presentation of the relation between the module $\mathcal{F}$ of the polynomials in (1.1) and the multiplication tables mod $\mathcal{F}$ w.r.t. a PP basis, for the non-deficient case as well as for the deficient case. Furthermore, we will describe the algorithmic generation of the PP basis $\bar{Z}_0$ and the multiplication table matrices $\bar{B}^{(\nu)}$ in the general situation. Both the theoretical results and the algorithmic procedures will be applied to examples, and the results will be displayed and discussed.

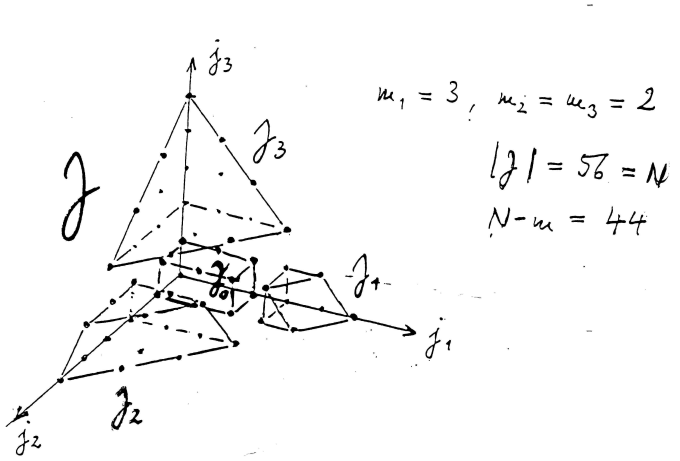

Figure 2.1: Partitioning of $\mathcal{J}$

2 Preparations

Naturally, the PP basis $\bar{Z}_0$ introduced in Section 1 must be a basis for the *residue class ring* $\bar{\mathcal{F}}$ of the polynomial ideal $\mathcal{F}$ if it is to characterize the zeros of (1.1) fully. However, we have not been able to find a constructive approach to the generation of bases for $\bar{\mathcal{F}}$ in the algebraic literature. Therefore, we have based our algorithmic approach to the generation of the PP basis $\bar{Z}_0$ on the resultant $R(a_\nu^{(0)})$ about which many constructive results may be found in the older algebraic literature.

By (1.3), the knowledge of the resultant $R(a_\nu^{(0)})$ of $\{f_0, f_1, \ldots, f_n\}$, which is also called the n-resultant of (1.1) in the algebraic literature, opens the way to the zeros of (1.1), at least theoretically. According to classical results, $R(a_\nu^{(0)})$ permits a representation (not unique)

$$R(a_\nu^{(0)}) = \sum_{\nu=0}^{n} \psi_\nu(x)\, f_\nu(x); \tag{2.1}$$

this makes it natural to look for multivariate polynomials ψ_ν which make all PPs except x^0 cancel out of the polynomial combination (2.1).

The following ansatz[1] for the ψ_ν is also classical, cf. e.g. [2]: Consider the set

$$\mathcal{J} := \Big\{ j \in \mathbb{N}_0^n \colon \sum_{\nu=1}^{n} j_\nu \le \sum_{\nu=1}^{n} m_\nu \; - \; n + 1 \Big\}, \qquad \text{with} \quad |\mathcal{J}| =: N := \binom{\sum_{\nu=1}^{n} m_\nu \; + \; 1}{n}. \tag{2.2}$$

$\mathcal{J}$ may be partitioned into *disjoint* sets (cf. Fig. 2.1)

$$\begin{aligned} \mathcal{J}_0 &:= \Big\{ j \in \mathcal{J} \colon \; j_\lambda < m_\lambda, \; \lambda = 1(1)n \Big\} \\ \mathcal{J}_\nu &:= \Big\{ j \in \mathcal{J} \colon \; j_\nu \ge m_\nu, \; j_\lambda < m_\lambda, \; \lambda = \nu{+}1(1)n \Big\}, \qquad \nu = 1(1)n. \end{aligned} \tag{2.3}$$

Note that this partition depends on the order of the polynomials in (1.1).

Now we compose the ψ_ν in (2.1) from the PPs in the sets

$$\mathcal{J}_0 \qquad \text{resp.} \qquad \check{\mathcal{J}}_\nu := \mathcal{J}_\nu - (0, \ldots, 0, \underbrace{m_\nu}_{\nu\text{–th position}}, 0, \ldots, 0), \quad \nu = 1(1)n, \tag{2.4}$$

[1] We represent sets of PPs in n variables by the sets of their exponents in $\mathbb{N}_0^n$.

(i.e., the $\check{\mathcal{J}}_\nu$ are defined by "shifting" the $\mathcal{J}_\nu$ towards the origin). Then $\mathcal{J}$ represents the set of all PPs which may occur in the right hand side of (2.1); this makes (2.1) a linear homogeneous system for the coefficients $c_j^{(\nu)}$ of the ψ_ν and the unknown left-hand side R, with an $N\times N$-matrix $A(a_\nu^{(0)})$ described below.

The irreducibility and normalization required for the resultant imply that

$$\bar{R}(a_\nu^{(0)}) := \det\Big(A(a_\nu^{(0)}\Big) = \rho \cdot R(a_\nu^{(0)}) \tag{2.5}$$

where ρ is a polynomial in the coefficients of the f_ν of (1.1) only and does not contain the dummy coefficients $a_\nu^{(0)}$. Actually, ρ is the determinant of a certain submatrix of A according to [2]. By (1.3) and (2.5), the factorization of $\bar{R}(a_\nu^{(0)})$ fully determines the zeros of (1.1), except when $\bar{R}$ and ρ vanish while $R(a_\nu^{(0)})$ is not identically zero.

In this case one may perturb (1.1) into a non-deficient system for which ρ will not vanish, and then let the perturbation parameter δ approach zero:

$$R(a_\nu^{(0)}) = \lim_{\delta\to 0} \frac{\bar{R}(a_\nu^{(0)};\delta)}{\rho(\delta)}. \tag{2.6}$$

As both ρ and $\bar{R}$ are polynomials in δ, this is a purely algebraic process on the elements in the matrix $A(a_\nu^{(0)})$. Thus, this matrix determines the resultant - and hence its factorization and the zeros of (1.1) - in the deficient case as well.

Only if (1.1) possesses solution manifolds and $R(a_\nu^{(0)})$ vanishes identically, the matrix A of (2.1)/(2.4) is not a suitable starting-point for the computation of the isolated zeros of (1.1).

To define A formally, we denote by $Z^{(\nu)}$ the vector of the PPs in $\check{\mathcal{J}}_\nu,\ \nu = 0(1)n$. The order of the PPs within these vectors is irrelevant, except that we assume $x^0 = 1$ to be the last element of $Z_0 := Z^{(0)}$. The set resp. vector Z of all PPs in $\mathcal{J}$ will be segmented into

$$Z = \left(\begin{array}{c} Z_1 \\ \hline Z_0 \end{array}\right) \begin{array}{l} \updownarrow\ N-m \\ \\ \updownarrow\ m \end{array} \tag{2.7}$$

with a fixed but arbitrary order of the PPs in Z_1 (the union of the $Z^{(\nu)},\ \nu = 1(1)n$).

Now, the polynomial combination (cf. (2.1) and (2.4))

$$\left[\left(\begin{array}{c} Z^{(1)} \\ \hline 0 \\ \vdots \\ \vdots \\ 0 \end{array}\right) f_1(x) + \ldots + \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline Z^{(n)} \\ \hline 0 \end{array}\right) f_n(x) + \left(\begin{array}{c} 0 \\ \vdots \\ \vdots \\ 0 \\ \hline Z_0 \end{array}\right) f_0(a_\nu^{(0)}, x)\right] =: \left(\begin{array}{c|c} A_{11} & A_{10} \\ \hline A_{01}(a_\nu^{(0)}) & A_{00}(a_\nu^{(0)}) \end{array}\right)\left(\begin{array}{c} Z_1 \\ \hline Z_0 \end{array}\right)$$

$$=:\ A(a_\nu^{(0)}) \cdot Z \tag{2.8}$$

defines the matrix $A(a_\nu^{(0)})$ and its segmentations corresponding to (2.7). Obviously, each row of A contains the coefficients of a particular f_ν as non-zero elements; their position depends on the PP within $Z^{(\nu)}$ with which f_ν has been multiplied and on the order of the PPs in Z.

The dummy coefficients $a_\nu^{(0)}$ of f_0 occur on A_{01} and A_{00} only; due to the linearity of f_0,

$$A_{0i}(a_\nu^{(0)}) =: \sum_{\nu=0}^{n} a_\nu^{(0)} A_{0i}^{(\nu)}, \qquad i = 0, 1, \tag{2.9}$$

and each matrix $A_0^{(\nu)} := \left(A_{01}^{(\nu)} \,|\, A_{00}^{(\nu)}\right)$, $\nu = 0(1)n$, has one element 1 in each row as the only non-zero element. By (2.8), the $A_0^{(\nu)}$ satisfy

$$x_\nu \cdot Z_0 = A_0^{(\nu)}\, Z, \qquad \nu = 1(1)n, \tag{2.10}$$

or

$$\left(\begin{array}{c|c} A_{01}^{(\nu)} & A_{00}^{(\nu)} - x_\nu I \end{array}\right) = 0. \tag{2.11}$$

The system (2.1) for the coefficients $c_j^{(\nu)}$ of the ψ_ν now takes the form

$$\left(\begin{array}{c|c|c|c|c} c_j^{(1)} & c_j^{(2)} & \cdots & c_j^{(n)} & c_j^{(0)} \end{array}\right) A = \left(0 \cdots 0 \; R\right) \tag{2.12}$$

which explains once more (2.5). However, (2.12) will not play a role in our further development. Also, the introduction of the dummy polynomial $f_0(a_\nu^{(0)}, x)$ served only to relate our approach to the classical algebraic literature. For our purposes, we could as well have *defined* $A_0^{(\nu)}$ and $A_{01}^{(\nu)}$, $A_{00}^{(\nu)}$ by (2.10)/(2.11).

3 Multiplication Tables mod $\mathcal{F}$; case I: A_{11} regular

We will now explain the generation of the multiplication tables introduced in Section 1 and establish their properties. For simplicity, we treat the "non-degenerate" case at first, i.e. we assume at first that A_{11} in (2.8) is regular. In this case Z_0 will take the role of the PP basis $\bar{Z}_0$ of Section 1.

According to (2.8) and (2.11) we have, for $\nu = 1(1)n$,

$$\begin{aligned} -A_{01}^{(\nu)}\, A_{11}^{-1} \left[\left(\begin{array}{c} Z^{(1)} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right) f_1(x) + \ldots + \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline Z^{(n)} \end{array}\right) f_n(x) \right] &= \left(\begin{array}{c|c} 0 & B^{(\nu)} - x_\nu I \end{array}\right) \left(\begin{array}{c} Z_1 \\ \hline Z_0 \end{array}\right) \\ &= \left(B^{(\nu)} - x_\nu I\right) Z_0, \end{aligned} \tag{3.1}$$

with

$$B^{(\nu)} := A_{00}^{(\nu)} - A_{01}^{(\nu)}\, A_{11}^{-1}\, A_{10} \in \mathbb{R}^{m\times m} \quad \text{or} \quad \mathbb{C}^{m\times m} \quad \text{resp.}^2 \tag{3.2}$$

Note that we have formed a polynomial combination of the f_ν in $\mathcal{F}$ only; cf. the end of the previous section.

(3.1) establishes the $B^{(\nu)}$ of (3.2) as the matrices determining a multiplication table mod $\mathcal{F}$ w.r.t. the PP basis Z_0:

$$x_\nu \cdot Z_0 \equiv B^{(\nu)}\, Z_0 \mod \mathcal{F}, \qquad \nu = 1(1)n, \tag{3.3}$$

and we have (cf. Section 1) for "zero BPP vector")

Proposition 3.1: *For each (finite) zero $\xi_\mu = (\xi_{\mu 1}, \ldots, \xi_{\mu n}) \in \mathbb{C}^n$ of (1.1), the associated BPP vector $z_{0\mu} \in \mathbb{C}^n$ (w.r.t. Z_0) satisfies*

$$B^{(\nu)}\, z_{0\mu} = \xi_{\mu\nu}\, z_{0\mu}, \qquad \nu = 1(1)n, \tag{3.4}$$

i.e. the joint eigenvectors of the $B^{(\nu)}$ are the only candidates for the zero BPP vectors of (1.1).

The fact that the $B^{(\nu)}$ have common eigenvectors suggests that they commute:

[2] For simplicity, we will assume that the f_ν have *real* coefficients. There is no change in the formal development if the coefficients $a_j^{(\nu)}$ are complex numbers.

Proposition 3.2: *The $B^{(\nu)}$ of* (3.2) *commute.*

Proof: By (3.3), $B^{(\nu)}B^{(\lambda)}Z_0 \equiv x_\nu x_\lambda Z_0 \equiv x_\nu x_\lambda Z_0 \equiv B^{(\lambda)}B^{(\nu)}Z_0 \bmod \mathcal{F}$. The existence of a non-vanishing row c^T in $B^{(\nu)}B^{(\lambda)} - B^{(\lambda)}B^{(\nu)}$ would thus imply the existence of a nontrivial linear dependence $c^T Z_0 \equiv 0 \bmod \mathcal{F}$ within the BPPs of Z_0. However, according to [2], the matrix $\left(A_{11} \,|\, A_{10}\right)$ contains all information about the ideal $\mathcal{F}$, and the regularity of A_{11} contradicts the possibility of an elimination of Z_1 from $\left(A_{11} \,|\, A_{10}\right) \begin{pmatrix} Z_1 \\ Z_0 \end{pmatrix} \equiv 0 \bmod \mathcal{F}$. $\square$

On the basis of Proposition 3.2, we may define, for all $j \in \mathbb{N}_0^n$:

$$B^j := (B^{(1)})^{j_1} \cdots (B^{(n)})^{j_n} \in \mathbb{R}^{m \times m} \tag{3.5}$$

$$b_j^T := (0 \cdots 0\, 1)\, B^j \in \mathbb{R}^m \tag{3.6}$$

The b_j^T furnish the *basis representation* of all PPs mod $\mathcal{F}$:

Proposition 3.3: *For all $j \in \mathbb{N}_0^n$,*

$$x^j \equiv b_j^T\, Z_0 \mod \mathcal{F}. \tag{3.7}$$

Proof: $x^j = (0 \cdots 0\, 1)\, x^j \cdot Z_0 \equiv (0 \cdots 0\, 1)\, B^j\, Z_0 = b_j^T\, Z_0 \mod \mathcal{F}$ by (3.3), (3.5), (3.6). $\square$

As an immediate consequence of (3.7), we have

$$b_j^T = (0 \cdots 1 \cdots 0) \qquad \text{for} \quad j \in \mathcal{J}_0, \tag{3.8}$$

where the position of the only non-vanishing entry 1 is determined by the position of x^j in Z_0.

Proposition 3.4: *For any joint eigenvector $z_0 \in \mathbb{C}^m$ of the $B^{(\nu)}$, the last component is $\neq 0$:*

$$(0 \cdots 0\, 1)\, z_0 \neq 0. \tag{3.9}$$

Proof: Assume that the last component of z_0 vanishes. Then, for each $j \in \mathcal{J}_0$,

$$(z_0)_j = (0 \cdots \overset{\overset{j}{\downarrow}}{1} \cdots 0)\, z_0 \overset{(3.8)}{=} b_j^T z_0 \overset{(3.6)}{=} (0 \cdots 0\, 1)\, B^j\, z_0 \overset{\text{hyp.}}{=} x_0^j\, (0 \cdots 0\, 1)\, z_0 \overset{\text{ass.}}{=} 0,$$

where $x_0^j = \xi_1^{j_1} \cdots \xi_n^{j_n}$ and ξ_ν are the eigenvalues of z_0 for $B^{(\nu)}$. Hence our assumption would imply $z_0 = 0$. $\square$

Thus we may normalize each joint eigenvector of the $B^{(\nu)}$ by setting its last component equal to 1.

The following relations link the polynomials f_ν directly to the multiplication tables:

Proposition 3.5: *For $\nu = 1(1)n$,*

$$\sum_j a_j^{(\nu)}\, b_j^T = 0. \tag{3.10}$$

Proof: $0 \equiv f_\nu(x) = \sum_j a_j^{(\nu)} x^j \equiv \sum_j a_j^{(\nu)}\, b_j^T\, Z_0 \mod \mathcal{F}$ by (3.7).

But a nontrivial linear dependence mod $\mathcal{F}$ of the BPPs in Z_0 cannot exist; cf. the proof of Proposition 3.2. $\square$

Theorem 3.6: *Let $z_{0\mu} \in \mathbb{C}^m$ be a joint eigenvector of the $B^{(\nu)}$, with eigenvalues $\xi_{\mu\nu}$, $\nu = 1(1)n$, resp. Then $\xi_\mu = (\xi_{\mu 1}, \ldots, \xi_{\mu n}) \in \mathbb{C}^n$ is a zero of* (1.1).

Proof: $f_\nu(\xi_\mu) = \sum_j a_j^{(\nu)}\, \xi_\mu^j = \sum_j a_j^{(\nu)}\, \xi_{\mu 1}^{j_1} \cdots \xi_{\mu n}^{j_n} \underbrace{(0 \cdots 0\, 1)\, z_{\mu 0}}_{1} \overset{\text{hyp.}}{=} \sum_j a_j^{(\nu)}\, (0 \cdots 0\, 1)\, B^j\, z_{\mu 0} \overset{(3.6)}{=}$

$\overset{(3.6)}{=} \sum_j a_j^{(\nu)}\, b_j^T\, z_{\mu 0} \overset{(3.10)}{=} 0.$ □

For regular A_{11}, Theorem 3.6 and Proposition 3.1 establish a complete equivalence of a multiplication table (3.3) and the system (1.1) in the following sense:

Proposition 3.7: *If A_{11} is regular,* (1.1) *has no zero at infinity and hence exactly m isolated zeros (counting multiplicities).*

Proof: From the definition of A in (2.8) it follows that

$$\begin{aligned} \det\Big(A(a_\nu^{(0)})\Big) &= \det(A_{11}) \cdot \det\Big(A_{00}(a_\nu^{(0)}) - A_{01}(a_\nu^{(0)})\, A_{11}^{-1}\, A_{10}\Big) \\ &= \det(A_{11}) \cdot \det\Big(\sum_{\nu=0}^{n} a_\nu^{(0)}\, B^{(\nu)}\Big) \qquad \text{by (2.9) and (3.2),} \end{aligned} \tag{3.11}$$

with $B^{(0)} = I$. Hence $\bar{R}(a_\nu^{(0)}) = \det(A(a_\nu^{(0)}))$ has the full degree m in $a_0^{(0)}$ so that the factorization (1.3) of the resultant R must have a term $a_0^{(0)}\, \xi_{\mu 0}$ in each factor, which implies the assertion. □

Proposition 3.8: *If the m zeros of* (1.1) *are simple, the multiplication table matrices $B^{(\nu)}$ have exactly m joint eigenvectors.*

Proof: A trivial consequence of Proposition 3.1. □

Remark 3.9: If simple zeros of (1.1) have some coinciding components (say $\xi_{\mu_1 1} = \xi_{\mu_2 1}$) then the corresponding $B^{(\nu)}$ will have multiple eigenvalues with eigenspaces of dimension greater 1 (here $B^{(\nu)}$ will have a double eigenvalue with an eigenspace of dimension 2). The fact that a zero BPP vector has to be a joint eigenvector of *all* $B^{(\nu)}$ serves to distinguish the zero BPP vectors within the eigenspace.

Proposition 3.10: *If* (1.1) *has a zero ξ_μ of multiplicity k, then each $B^{(\nu)}$ has an eigenvalue $\xi_{\mu\nu}$ with algebraic multiplicity k but geometric multiplicity* 1*, and the associated eigenvector is the same for all $B^{(\nu)}$ (viz. the zero BPP vector for ξ_μ).*

Proof: By (3.11), (2.5) and (1.3), we have for $\nu = 1(1)n$:

$$\begin{aligned} \det\Big(B^{(\nu)} - \lambda I\Big) &= \Big[\det(A_{11})\Big]^{-1} \cdot \bar{R}\,(-\lambda,\, 0 \cdots \overset{\nu}{\underset{\downarrow}{1}} \cdots 0) \\ &= \rho\,\Big[\det(A_{11})\Big]^{-1} \prod_{\mu=1}^{m} (-\lambda\, \xi_{\mu 0} + \xi_{\mu\nu}) \end{aligned} \tag{3.12}$$

so that the algebraic multiplicities of the zeros of (1.1) and of the corresponding eigenvalues of the $B^{(\nu)}$ are the same.

If we consider a perturbation of (1.1) which takes the k-fold zero ξ_μ into k simple zeros and then regard the limit process back to (1.1), the corresponding procedure for the multiplication tables will at first produce k (separate but close) joint eigenvectors for the $B^{(k)}$ which must merge into one eigenvector. □

Naturally, the situations of Remark 3.9 and Proposition 3.10 may occur in combination, e.g. zeros of a multiplicity $k > 1$ may coincide in some components, with obvious consequences for the $B^{(\nu)}$ and their eigenvalues and eigenvectors.

The preceding results may be combined into

Theorem 3.11: *If a particular multiplication table matrix $B^{(\nu)}$ has only eigenspaces of dimension 1, then the zeros of* (1.1) *may be completely determined by an eigenanalysis of this $B^{(\nu)}$. If $B^{(\nu)}$ has an eigenspace of a dimension greater than* 1*, then the selection of the zero BPP vectors must use further $B^{(\nu)}s$.*

Remark 3.12: Note that Z_0 contains all linear powers $x_\nu,\ \nu = 1(1)n$, if $m_\nu \geq 2$ for all ν, cf. (2.3). In this case, the $\xi_{\mu\nu}$ occur explicitly in the normalized eigenvectors of *each* $B^{(\nu)}$. If there are linear equations in (1.1), they must be linearly independent to exclude solution manifolds; hence the $\xi_{\mu\nu}$ missing in the eigenvectors may be computed from the linear part of (1.1). Of course, it should be advantageous to reduce the number of unknowns in (1.1) by means of the linear equations before a solution of (1.1) is considered.

4 Multiplication Tables mod $\mathcal{F}$; case II: A_{11} singular

For the matrix $A_1 := \left(A_{11} \,|\, A_{10}\right)$ we define

$$\bar{N} := \operatorname{rank}(A_1) \leq N - m. \tag{4.1}$$

Note that the case $\bar{N} = N - m$ is perfectly possible for singular A_{11}.

The following assumption appears to be necessary and sufficient for a transfer of the results of Section 3 to the more general situation.

Assumption M: The N PPs in Z (cf. (2.7)) may be ordered and segmented into

$$Z = \left(\begin{array}{c} \hat{Z}_1 \\ \hline \bar{Z}_1 \\ \hline \bar{Z}_0 \end{array}\right) \begin{array}{l} \updownarrow \hat{N} \\ \updownarrow \bar{N} \\ \updownarrow \bar{m} \end{array}, \qquad \hat{N} + \bar{N} + \bar{m} = N, \tag{4.2}$$

such that the following conditions hold:

(i) The ordering and segmentation of A_1 induced by (4.2) (cf. (2.8))

$$\left[\left(\begin{array}{c} Z^{(1)} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right) f_1(x) + \ldots + \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline Z^{(n)} \end{array}\right) f_n(x)\right] = \left(\begin{array}{c|c|c} \hat{A}_{11} & \bar{A}_{11} & \bar{A}_{10} \end{array}\right) \left(\begin{array}{c} \hat{Z}_1 \\ \bar{Z}_1 \\ \bar{Z}_0 \end{array}\right) \tag{4.3}$$

generates a matrix $\bar{A}_{11}$ of rank $\bar{N}$ (i.e. the $\bar{N}$ columns of $\bar{A}_{11}$ are linearly independent).

(ii) No x_ν-neighbor of $\bar{Z}_0$ is in $\hat{Z}_1$, i.e. (cf. (2.10))

$$x_\nu \cdot \bar{Z}_0 =: \bar{A}_0^{(\nu)} Z =: \left(\begin{array}{c|c|c} 0 & A_{01}^{(\nu)} & A_{00}^{(\nu)} \end{array}\right) \left(\begin{array}{c} \hat{Z}_1 \\ \bar{Z}_1 \\ \bar{Z}_0 \end{array}\right) \qquad \text{for} \quad \nu = 1(1)n. \tag{4.4}$$

(iii) For a left pseudoinverse $\bar{A}_{11}^{+}$ of $\bar{A}_{11}$ (i.e. $\bar{A}_{11}^{+}\,\bar{A}_{11} = I \in \mathbb{R}^{\bar{N}\times\bar{N}}$),

$$\bar{A}_{01}^{(\nu)}\,\bar{A}_{11}^{+}\,\hat{A}_{11} = 0 \qquad \text{for} \quad \nu = 1(1)n. \qquad \square \tag{4.5}$$

Under this assumption which we suppose to be true in the following, we may form multiplication tables mod $\mathcal{F}$ w.r.t. the PP basis $\bar{Z}_0$, cf. (1.4) and (3.1)–(3.3):

$$-\bar{A}_{01}^{(\nu)}\,\bar{A}_{11}^{+}\left[\begin{pmatrix} Z^{(1)} \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} f_1(x) + \ldots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \hline Z^{(n)} \end{pmatrix} f_n(x)\right] = \left(\begin{array}{c|c|c} 0 & 0 & \bar{B}^{(\nu)} - x_\nu I \end{array}\right) \begin{pmatrix} \hat{Z}_1 \\ \bar{Z}_1 \\ \bar{Z}_0 \end{pmatrix} \tag{4.6}$$

$$= \left(\bar{B}^{(\nu)} - x_\nu I\right) \bar{Z}_0$$

with

$$\bar{B}^{(\nu)} := \bar{A}_{00}^{(\nu)} - \bar{A}_{01}^{(\nu)}\,\bar{A}_{11}^{+}\,\bar{A}_{10} \in \mathbb{R}^{\bar{m}\times\bar{m}}, \qquad \nu = 1(1)n, \tag{4.7}$$

so that

$$x_\nu \cdot \bar{Z}_0 \equiv \bar{B}^{(\nu)}\,\bar{Z}_0 \mod \mathcal{F}, \qquad \nu = 1(1)n. \tag{4.8}$$

This shows that - under Assumption M - we are able to achieve the situation described in Section 1.

Proposition 4.1: *For a given (admissible) segmentation* (4.2)*, the* $\bar{B}^{(\nu)}$ *are uniquely defined (i.e. independent of the choice of the pseudoinverse* $\bar{A}_{11}^{+}$*).*

Proof: Since $\bar{N}$ is the rank of A_1, the columns of $\bar{A}_{10}$ must be linear combinations of the $\bar{N}$ columns of $\bar{A}_{11}$. Hence $\bar{A}_{11}^{+}\,A_{10}$ is independent of the particular choice of $\bar{A}_{11}^{+}$ since $\bar{A}_{11}^{+}\,\bar{A}_{11} = I$. $\square$

As a consequence of (4.6), Proposition 3.1 remains valid, with m, $z_{0\mu}$, Z_0, $B^{(\nu)}$ replaced by $\bar{m}$, $\bar{z}_{0\mu}$, $\bar{Z}_0$, $\bar{B}^{(\nu)}$. For the validity of the analogously adapted Propositions 3.2–3.5, we need

Proposition 4.2: *There exists no nontrivial linear dependence*

$$c^T\,\bar{Z}_0 \equiv 0 \mod \mathcal{F}, \tag{4.9}$$

i.e. $\bar{Z}_0$ *is a basis of the residue class ring* $\bar{\mathcal{F}}$*.*

Proof: We distinguish two cases:

a) $\bar{N} = N - m$: Then $\bar{A}_{11} \in \mathbb{R}^{\bar{N}\times\bar{N}}$ is regular and we may argue as in the proof of Proposition 3.2.

b) $\bar{N} = \operatorname{rank}(A_1) = \operatorname{rank}(\bar{A}_{11}) < N - m$: Assume that (4.9) holds. The row $(0\,|\,0\,|\,c^T) \in \mathbb{R}^n$ is obviously independent of any $\bar{N}$ linearly independent rows in A_1 for which the segments in $\bar{A}_{11}$ are also linearly independent. But at the same time, (4.9) must be formable as a linear combination of rows in (4.3) according to (2.6), which constitutes a contradiction to $\bar{N} = \operatorname{rank}(A_1)$. $\square$

With Proposition 4.2, Propositions 3.2–3.5 may be immediately adapted and we have (as analogy to Proposition 3.1 and Theorem 3.6)

Theorem 4.3: *For each (finite) zero* $\xi_\mu = (\xi_{\mu 1}, \ldots, \xi_{\mu n}) \in \mathbb{C}^n$ *of* (1.1)*, the associated zero BPP vector* $\bar{z}_{0\mu} \in \mathbb{C}^{\bar{m}}$ *w.r.t.* $\bar{Z}_0$ *satisfies*

$$\bar{B}^{(\nu)}\,\bar{z}_{0\mu} = \xi_{\mu\nu}\,\bar{z}_{0\mu}, \qquad \nu = 1(1)n, \tag{4.10}$$

i.e. it is a joint eigenvector of the $\bar{B}^{(\nu)}$*. Conversely, if* $\bar{z}_{0\mu} \in \mathbb{C}^{\bar{m}}$ *is a joint eigenvector of the* $\bar{B}^{(\nu)}$*, with eigenvalues* $\xi_{\mu\nu}$*,* $\nu = 1(1)n$*, then* $\xi_\mu := (\xi_{\mu 1}, \ldots, \xi_{\mu n})$ *is a zero of* (1.1)*.*

Furthermore, due to the mutual commutativity of the $\bar{m} \times \bar{m}$-matrices $B^{(\nu)}$ under Assumption M, we must have a set of $\bar{m}$ joint eigenvectors counting multiplicities in the sense of Propositions 3.8 and 3.10 and Remark 3.9; thus there are precisely $\bar{m}$ (finite) zeros for system (1.1) counting multiplicities.

Let us now discuss the requirements of Assumption M:

Condition (i) merely serves to distinguish a set of $\bar{N}$ linearly independent columns in the matrix A_1; because of $\bar{N} = \text{rank}\,(A_1)$, such a set must exist and there cannot be more than $\bar{N}$ linearly independent columns.

Algorithmically, one must select the columns of $\bar{A}_{11}$ such that their linear independence is numerically well established, i.e. with column pivoting; see Section 5. The determination of the (numerical) rank of A_1 is then a byproduct of the determination of $\bar{A}_{11}$.

Condition (ii) is a control criterion for the separation of the remaining columns of A_1 into $\hat{A}_{11}$ and $\bar{A}_{10}$ or, equivalently, of the remaining PPs in Z into $\hat{Z}_1$ and $\bar{Z}_0$ (cf. (4.3)): Each PP in $\bar{Z}_0$ must have all its x_ν-neighbors in $\bar{Z}_0$ or in $\bar{Z}_1$.

Since $x^0 = 1$ must be in $\bar{Z}_0$, this requires that each increasing path in $\mathcal{J}$ which issues from 0 must reach a point in $\bar{Z}_1$; the PPs for all previous points must be in $\bar{Z}_0$. Geometrically, this means that the points in $\mathcal{J}$ associated with $\bar{Z}_1$ must separate the origin from the "outer" part of $\mathbb{N}^n$.

Condition (iii) is a genuine hypothesis; it can only be tested after the separation (4.3) has been effected.

If there are no (finite) solution manifolds for the system (1.1), the residue class ring $\bar{\mathcal{F}}$ has a finite dimension and thus there exists a finite PP basis for it (containing the element 1). On the other hand, no finite bases for $\bar{\mathcal{F}}$ exist when there are finite solution manifolds of a positive dimension.

Thus it is trivial that Assumption M *cannot* be satisfied if there are finite solution manifolds, and any attempt to effect a separation (4.2) satisfying (4.3)–(4.5) must fail. The critical question is the reverse one: May a reasonable algorithm for the selection of the PPs in $\bar{Z}_1$ fail *although* there exists a finite PP basis for $\bar{\mathcal{F}}$.

Such a failure could be due to three reasons:

a) Any PP basis for $\bar{\mathcal{F}}$ contains PPs not in Z (cf. (2.7)). This would seem to contradict the constructive theory for the resultant $R(a_\nu^{(0)})$ except (possibly) in the case where $R(a_\nu^{(0)})$ vanishes identically due to a zero manifold at *infinity*.

b) The selection process for independent columns in A_1 leads to a set $\bar{Z}_1$ which does not separate the PP 1 from the "farfield" PPs although appropriate sets $\bar{Z}_1$ would exist.

c) The selection process leads to a separation (4.2) of Z which satisfies (4.3) and (4.4) but not (4.5).

Conjecture 4.4: *If* (1.1) *has no (finite) zero manifolds, the algorithm explained in Section 5 (and any other reasonable algorithm for the separation* (4.2)*) will succeed, possibly after a suitable reduction of* A_1 *(by an omission of appropriate rows) in the case that* (1.1) *has zero manifolds at infinity.*

Remark 4.5: In this conjecture, "success" refers to an execution of the algorithm in precise arithmetic (i.e. in $\mathbb{R}$ or $\mathbb{C}$ resp.). For a floating-point algorithm it is clear that it may also fail "numerically " if (1.1) is very close to a system with zero manifolds.

Remark 4.6: It appears that the ideas of T. Y. Li (e.g. [3]) may be used for an appropriate reduction of the matrix A_1 in the case of zero manifolds at infinity.

5 Algorithmic Realization

Our algorithm may indeed be called an *elimination algorithm* since the multiplication tables $B^{(\nu)}$ resp. $\bar{B}^{(\nu)}$ are obtained by row elimination and back-substitution within the matrix $A_1 = \Big(A_{11} \,|\, A_{10}\Big)$. In Sections 3 resp. 4 above the "non-deficient" (A_{11} non-singular) and the "deficient" case (A_{11} singular) were treated separately; however, for an algorithmic realization such a distinction is not necessary as will become clear below.

(1) Preparations.

For the purpose of internal data representation a suitable order has to be chosen for the exponents (multiindices) $j = (j_1, \ldots, j_n)$ within the "hypertetraeder" $\mathcal{J}$ (cf. (2.2) and Fig. 2.1) associated with the power products $x^j = x_1^{j_1} \cdots x_n^{j_n} \in Z$. Lexicographic ordering, for instance, is a convenient choice since it is easy to manage algorithmically. By the "address" of a PP x^j we understand the position of its exponent $j \in \mathcal{J}$ w.r.t. the prescribed order (e.g., its lexicographic position within $\mathcal{J}$). Note that we do not a priori mark a set of basis power products (except x^0 which must be a BPP).

The auxiliary functions required for the management of the underlying (lexicographic) data structure include (among others)

- `GETNPP` (j): returns the (lexicographic) successor of a given multiindex j;
- `ADDRPP` (j, ℓ): returns the (lexicographic) address ℓ of a given multiindex j;
- `PPADDR` (ℓ, j): returns the multiindex j whose (lexicographic) address is given by ℓ (inverse of `ADDRPP`).

`GETNPP` is mainly used within the generation of the matrix A_1 (step (2) below); a modified version of `GETNPP` is also required to generate the sets $\check{\mathcal{J}}_\nu$ (cf. (2.4) and step (2) below). `ADDRPP` and `PPADDR` perform combinatorical tasks providing the necessary link between the columns of A_1 and their corresponding multiindices $j \in \mathcal{J}$ (steps (3)–(6) below).

These auxiliary functions are used in a "black box" manner; thus the algorithm can easily be adjusted (by substituting `GETNPP`, ...) to another ordering or even to a reduced ansatz for the efficient treatment of special problem types (cf. Remark 4.6).

(2) Generation of the matrix A_1.

Now we compose the $N\times(N+m)$-matrix A_1 according to the rules described in Section 2: Each column of A_1 is associated (via the chosen ordering[3]) with a PP x^j, $j \in \mathcal{J}$; each row of A_1 represents a "sparse" linear relation (mod f_ν) between these PPs,

$$x^{\check{\jmath}} f_\nu(x) = \sum_j a_j^{(\nu)} x^{j+\check{\jmath}} = 0, \tag{5.1}$$

which is a "shifted version" of one of the given polynomial equations $f_\nu(x) = 0$, $\nu = 1, \ldots, n$ (cf. for instance (2.8)). The various "shift factors" $x^{\check{\jmath}}$ for f_ν are taken from the set of PPs with exponents from $\check{\mathcal{J}}_\nu$ (cf. (2.3), (2.4)) and are processed in some convenient order (e.g., lexicographically).[4] Now, according to (5.1), the polynomial coefficient $a_j^{(\nu)}$ becomes the ℓ-th entry in the corresponding row of A_1 where ℓ is the address of $j + \check{\jmath}$. Thus the generation of A_1 is easy to perform algorithmically.

(3) Elimination.

Next we apply row eliminations in conjunction with a special column pivoting strategy (involving column permutations) to determine the (numerical) rank $\bar{N} \le N$ of the sparse rectangular matrix

[3] In Section 2, a different arrangement of columns, $A_1 = \Big(A_{11} \,|\, A_{10}\Big)$, was used for theoretical purposes.

[4] Recall that, by construction, any $j+\check{\jmath}$ ever occuring in (5.1) is indeed contained in $\mathcal{J}$, cf. Section 2.

A_1 and to select a suitable set of $\bar{N}$ linearly independent columns.[5] Basically, we use conventional column pivoting with the aim of optimal numerical stability.[6] However, we restrict the pivot search by distinguishing different "types" of columns ($\equiv$ PPs $x^j \in Z$ $\equiv$ exponents $j \in \mathcal{J}$):

(δ) First a maximal linearly independent set is selected among the PPs x^j where j is in

$$\partial\mathcal{J} := \Big\{ j \in \mathbb{N}_0^n \colon \sum_{\nu=1}^{n} j_\nu = \sum_{\nu=1}^{n} m_\nu \,-\, n+1 \Big\} \tag{5.2}$$

(i.e. the exterior boundary of $\mathcal{J}$). These are considered first because none of the $x^j,\ j \in \partial\mathcal{J}$, can be a BPP (no x_ν-neighbor is in $\mathcal{J}$ and so Assumption M, (ii) would be violated).

(γ) Next, all other PPs in our ansatz, except those in Z_0 (cf. (β)), are considered as "pivot candidates".

(β) The PPs $x^j \in Z_0$ (cf. (2.3)), which form a PP basis in the non-deficient case (cf. Section 3), are pivot candidates of lowest priority (except x^0, cf. (α)): They are only considered as "pivot PPs" if it has not been possible to find a full set of N linearly independent columns of the type (δ) and (γ).

(α) The PP x^0 must be a BPP; so it is not admitted as a pivot candidate.

This procedure leads to an assortment of $\bar{N} \le N$ pivot PPs of the type (δ), (γ) (and, possibly, (β)) which form the set $\bar{Z}_1$ in the sense of Section 4. The remaining non-pivot PPs of the type (α), (β) and (γ) (not: (δ)) constitute the set of "basis candidates" $\subset Z \setminus \bar{Z}_1$, from which a definitive PP basis will be selected in step (5).

(4) Back-substitution.

By back-substitution within the $\bar{N}$ linearly independent rows of the triangularized matrix A_1 arising from step (3) we obtain, for each of the $\bar{N}$ pivot PPs $x^k \in \bar{Z}_1$, a representation

$$x^k \equiv \sum_{j\colon\, x^j \in Z\setminus\bar{Z}_1} b_{k,j}\, x^j \mod \mathcal{F} \tag{5.3}$$

in terms of non-pivot PPs $x^j \in Z \setminus \bar{Z}_1$.

(5) Determination of a PP basis.

Now we attempt to extract a PP basis $\bar{Z}_0$ from the set of basis candidates (cf. (3)) which satisfies Assumption M. To this end we choose an initial (minimal) "test basis" $\tilde{Z}_0$ simply consisting of the element x^0 and apply the following recursive testing & extension procedure:

Consider all x_ν-neighbors ($\nu = 1, \ldots, n$) of the current test basis $\tilde{Z}_0$ which are not contained in $\tilde{Z}_0$. Let x^k denote the x_ν-neighbor under consideration. We distinguish several cases:

- If x^k is a basis candidate, it is added to $\tilde{Z}_0$. Continue with the new test basis.
- If x^k is a pivot PP (cf. (3)), the search path has reached its endpoint w.r.t. the x_ν-direction and x^k permits a representation (5.3) in terms of basis candidates. Three cases are possible:
  - If x^k is representable by PPs from $\tilde{Z}_0$ solely such that (5.3) reduces to

$$x^k \equiv \sum_{j\colon\, x^j \in \tilde{Z}_0} b_{k,j}\, x^j \mod \mathcal{F}, \tag{5.4}$$

there is no need to extend $\tilde{Z}_0$. *Passed.* Continue with the current test basis.

[5] Cf. Remark (ii) at the end of this section.

[6] Cf. Remark (iii) at the end of this section.

– If in (5.3) there occur non-vanishing[7] coefficients $b_{k,j}$ for which x^j is not contained in $\tilde{Z}_0$ but is a basis candidate, the respective PPs x^j are added to $\tilde{Z}_0$. Continue with the new test basis.

– If in (5.3) there occurs at least one non-vanishing[7]coefficient $b_{k,j}$ for which x^j is of the type (δ), the algorithm has *failed* since x^j is not admissible as a basis PP (cf. step (3)).

- Otherwise x^k is of the type (δ) and is neither a basis candidate nor a pivot PP. This means that the search path has met the exterior boundary $\partial\mathcal{J}$ and the algorithm has *failed.*

Once all PPs in the current test basis $\tilde{Z}_0$ *pass* the above test, a minimal PP basis $\tilde{Z}_0 =: \bar{Z}_0$ (of dimension $\bar{m} \le m$) has been found which, by construction, satisfies Assumption M, and the system (1.1) has $\bar{m}$ finite isolated zeros. In the case of failure, the fact has been established that (1.1) has zero manifolds (cf. Conjecture 4.4).

(6) Multiplication table(s). Solution of eigenproblem(s), extraction of the zeros.

If step (5) has been successful we form, for some $\nu \in \{1, \ldots, n\}$, the multiplication table mod $\mathcal{F}$ w.r.t. the chosen PP basis: For each $x^k \in \bar{Z}_0$,[8]

$$x_\nu \cdot x^k = x^{k+i_\nu} \equiv \sum_{j:\ x^j \in \bar{Z}_0} b_{k+i_\nu,j}\, x^j \mod \mathcal{F} \tag{5.5}$$

with

$$\begin{aligned} &b_{k+i_\nu,j} \quad \text{from (5.3)} && \text{if} \quad x^{k+i_\nu} \notin \bar{Z}_0, \\ &b_{k+i_\nu,j} := \delta_{k+i_\nu,j} && \text{if} \quad x^{k+i_\nu} \in \bar{Z}_0, \end{aligned} \tag{5.6}$$

and the $\bar{m}\times\bar{m}$ coefficients $b_{k+i_\nu,j}$ in (5.6) define the multiplication table $\bar{B}^{(\nu)}$ in the sense of Section 4. Now we solve the eigenproblem $\bar{B}^{(\nu)}\,\bar{z}_0 = \bar{\lambda}\,\bar{z}_0$ by means of a standard library procedure and obtain $\bar{m}$ eigenvalues $\bar{\lambda}_\mu$ and their associated eigenvectors $\bar{z}_{0\mu}$, which we normalize such that the x^0-component is 1.

If there are simple eigenvalues only, we can immediately extract the components of the $\bar{m}$ isolated zeros ξ_μ: For $\mu = 1, \ldots, \bar{m}$,

$$\xi_{\mu\ell} := \begin{cases} \bar{\lambda}_\mu, & \ell = \nu, \\ \textit{component of } \bar{z}_{0\mu} \textit{ corresponding to the} \\ \textit{position of the linear PP } x_\ell \textit{ in } \bar{Z}_0, & \ell \ne \nu. \end{cases} \tag{5.7}$$

It may occur that a linear PP x_ℓ $(\ell \ne \nu)$ is *not* a member of $\bar{Z}_0$, e.g. if the ℓ-th equation in (1.1) is linear. In this case the solution component $\xi_{\mu\ell}$ may be obtained as a scalar product on the basis of the representation (5.5)[9] for x_ℓ (with the numerical vector $\bar{z}_{0\mu}$ substituted for $\bar{Z}_0$).

If there occur multiple eigenvalues, we solve the eigenproblem(s) for further multiplication tables $\bar{B}^{(\nu)}$ and "merge" the results if necessary; cf. the end of Section 3 (Remark 3.9 ff.); we omit the details.

Remarks.

(i) In the non-deficient case our algorithm automatically selects the "generic PP basis" Z_0 (cf. Sections 2 & 3) due to the strategy used in steps (3) and (4) above.

[7] Naturally, "non-vanishing" has to be understood in a numerical sense.

[8] i_ν denotes the ν-th unit exponent $(0, \ldots, \overset{\nu}{\underset{\downarrow}{1}}, \ldots, 0)$.

[9] If x_ℓ is not a BPP then it permits a representation (5.5) because 1 is a BPP and Assumption M is satisfied.

(ii) Recall that the PP x^0 is not admitted as a pivot candidate in step (3) above. Thus, for $\bar{N} < N$ it is possible that the addition of the x^0-column of A_1 to the pivot columns selected in step (3) increases the rank by 1. This has to be checked after step (3) has been completed. If so, there exists a linear combination of shifted equations (5.1) which results in $1 = 0$, and the fact has been established that the given system (1.1) is *inconsistent*.

(iii) The number N of equations represented by the matrix A_1 strongly increases with increasing m_ν and n (cf. (2.2)), but A_1 has a very sparse structure. (In special cases, N can be chosen much smaller than in (2.2), cf. Remark 4.6.) The use of *sparse matrix techniques* is therefore indispensable for an efficient implementation, and it will be favourable to combine the column selection procedure in step (3) with a fill-in minimization strategy.

(iv) For the treatment of cases where the determination of rank (A_1) is numerically "critical" one may also use a suitably adapted QR decomposition in our elimination procedure.

(v) Recall that our algorithm fails if there are (finite) zero manifolds (cf. Section 6, Example 6.4). We have, however, been successful in computing all finite isolated zeros for a number of systems with a zero manifold at infinity (cf. Section 6, Example 6.6). Obviously, the information about these manifolds is "filtered out" in steps (3) and (4).

6 Numerical Examples; Discussion

We have implemented a (not yet completely optimized) version of our algorithm in FORTRAN 77 and have successfully solved a number of non-deficient as well as deficient systems. In the following we present some simple examples featuring different "types of deficiency" and briefly comment on the results.

Example 6.1: $n = 2,\ m_1 = m_2 = 2;\ m = 4,\ N = 6.$ *Non-deficient case.*

$$\begin{aligned} x_1^2 + 4x_2^2 - 4 &= 0 \\ (x_2 - x_1)(4x_2 + x_1) &= 0 \end{aligned} \tag{6.1}$$

(6.1) has a full set of 4 finite isolated zeros.

The 6×6-matrix A_{11} (in the sense of Section 2) is nonsingular, i.e. the 6 columns of the type (δ) and (γ) (cf. Section 5, step (3)) are linearly independent. Thus our algorithm selects the generic PP basis

$$\bar{Z}_0 = Z_0 = \left\{x^j :\ j = (0,0),\ (0,1),\ (1,0),\ (1,1)\right\}$$

and delivers the isolated (real) zeros $(2a, 2a)$, $(-2a, -2a)$, $(4a, -a)$ and $(-4a, a)$ $(a = 1/\sqrt{5})$ on the basis of the multiplication table for $\nu = 1$ or $\nu = 2$.

Example 6.2: $n = 2,\ m_1 = m_2 = 2;\ m = 4,\ N = 6.$ *One zero at infinity.*

$$\begin{aligned} x_1 x_2 - x_1 &= 0 \\ x_1^2 - x_2 &= 0 \end{aligned} \tag{6.2}$$

Due to the first equation in (6.2), the PPs $x_1 \in Z_0$ and $x_1 x_2 \in Z_0$ are not independent; so this is a deficient situation. Indeed, (6.2) has only 3 finite isolated zeros and one isolated zero at infinity.

Let us have a closer look on the matrix A_1 for this example (each column is labelled by its corresponding PP exponent):

$$A_1 = \left(\begin{array}{cccccccccc} (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (2,0) & (2,1) & (3,0) \\ \hline & & & & -1 & 1 & & & & \\ & & & & & -1 & 1 & & & \\ & & & & & & & -1 & 1 & \\ & -1 & & & & & & 1 & & \\ & & -1 & & & & & & 1 & \\ & & & & & -1 & & & & 1 \end{array}\right)$$

A_1 has the full rank $\bar{N} = N = 6$. But $(0,3)$ is a zero column; in step (3) our algorithm chooses the PP x_1x_2 as a pivot PP instead of x_2^3 and selects the reduced PP basis

$$\bar{Z}_0 = \left\{x^j : \ j = (0,0),\ (0,1),\ (1,0)\right\}$$

of the correct dimension $\bar{m} = 3$; the zero column $(0,3)$ is simply ignored. The isolated zeros $(0.,0.)$, $(1.,1.)$ and $(-1.,1.)$ are obtained on the basis of the multiplication table for $\nu = 1$. (Note that for $\nu = 2$ there occurs the 2-fold eigenvalue $x_2 = 1$.)

Example 6.3: $n = 2,\ m_1 = m_2 = 2;\ m = 4,\ N = 6.$ *Deficient case with a full set of finite isolated zeros.*

$$\begin{aligned} x_1x_2 - x_1 - 1 &= 0 \\ x_1^2 + x_2^2 - x_2 &= 0 \end{aligned} \tag{6.3}$$

(6.3) has a full set of 4 finite isolated zeros. However, similarly as in (6.2), the first equation in (6.3) relates the PPs $1,\ x_1$ and $x_1x_2 \in Z_0$ such that Z_0 cannot be a PP basis.

A_1 has the full rank $\bar{N} = N = 6$. Here no zero columns occur; a linearly independent set of columns is obtained by choosing x_1x_2 as a pivot PP instead of x_1^2. This leads to the PP basis

$$\bar{Z}_0 = \left\{x^j : \ j = (0,0),\ (0,1),\ (1,0),\ (2,0)\right\}$$

of the correct dimension $\bar{m} = m = 4$, and our algorithm delivers two pairs of complex conjugate zeros on the basis of the multiplication table for $\nu = 1$ or $\nu = 2$.

Example 6.4: $n = 3,\ m_1 = m_2 = 2,\ m_3 = 3;\ m = 12,\ N = 44.$ *Several zeros at infinity.*

$$\begin{aligned} 16x_1^2 + 4x_1x_2 - 4x_3 + 1 &= 0 \\ 2x_2x_3 + 4x_1 + 1 &= 0 \\ 2x_1^2x_3 - x_1 - 2x_2 &= 0 \end{aligned} \tag{6.4}$$

(6.4) has 7 finite isolated zeros and 5 isolated zeros at infinity.

It turns out that the matrix A_1 has the rank $\bar{N} = 39$; thus 5 equations in A_1 are redundant. In step (4) it turns out that 10 columns can be ignored (they correspond to the PP set $\hat{Z}_1$ of Section 4); our algorithm selects the PP basis

$$\bar{Z}_0 = \left\{x^j : \ j = (0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,2),\ (1,0,0),\ (1,0,1),\ (1,1,1)\right\}$$

of the correct dimension $\bar{m} = 7$ and delivers the isolated zeros (three real zeros and two complex conjugate pairs) on the basis of any of the multiplication tables.

Example 6.5: $n = 2,\ m_1 = m_2 = 2;\ m = 4,\ N = 6.$ *Finite zero manifold.*

$$\begin{aligned} x_1(x_1 - x_2) &= 0 \\ (x_2 - 1)(x_1 - x_2) &= 0 \end{aligned} \tag{6.5}$$

(6.5) has a finite zero manifold: $x_1 = x_2$, and our algorithm fails in step (4): It turns out that the PP $x_1 x_2^2$ does not permit a representation (5.3), which would be necessary to construct a valid PP basis.

Example 6.6: $n = 3,\ m_1 = m_2 = m_3 = 2;\ m = 8,\ N = 27.$ *Zero manifold at infinity.*

$$\begin{aligned} (x_1 + 1)(x_1 + x_2 + x_3 + 2) &= 0 \\ (x_1 + 5)(x_2 - 3) &= 0 \\ (x_1 - 2)(x_3 + 2) &= 0 \end{aligned} \tag{6.6}$$

(6.6) has a zero manifold at infinity besides 4 finite isolated zeros.

Here the matrix A_1 has the rank $\bar{N} = 24$; thus 3 equations in A_1 are redundant. In step (4) it turns out that 7 columns can be ignored (they correspond to the PP set $\hat{Z}_1$ of Section 4); our algorithm selects the PP basis

$$\bar{Z}_0 = \left\{ x^j :\ j = (0,0,0),\ (0,0,1),\ (0,1,0),\ (1,0,0) \right\}$$

of the correct dimension $\bar{m} = 4$ and delivers the isolated zeros $(-1., 3., -2.)$, $(-5., 5., -2.)$, $(2., 3., -7.)$ and $(-3., 3., -2.)$ on the basis of any of the multiplication table for $\nu = 1$. The occurrence of a zero manifold at infinity does not hurt us. (Note, however, that for Example 6.6 a suitably reduced ansatz would lead to a more efficient algorithm.)

References

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