An Elimination Algorithm for the Computation of All Zeros of a System of Multivariate Polynomial Equations

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Abstract

A direct numerical method is proposed for the determination of all isolated zeros of a system of multivariate polynomial equations. By "polynomial combination", the system is reduced to a special form which may be interpreted as a multiplication table for power products modulo the system. The zeros are then formed from an ordinary eigenvalue problem for the matrix of the multiplication table. Degenerate situations may be handled by perturbing them into general form and reaching the zeros of the unperturbed system via a homotopy method.

1 Introduction

This paper is concerned with the numerical computation of all isolated zeros $\xi_\mu := (\xi_{\mu 1}, \ldots, \xi_{\mu n})$, $\mu = 1, \ldots$ of a system of multivariate polynomial equations:

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= 0 \\ &\ldots \\ f_n(x_1, \ldots, x_n) &= 0 \end{aligned} \tag{1.1}$$

where

$$f_\nu(x_1, \ldots, x_n) = \sum_j a^{(\nu)}_{j_1 \ldots j_n} x_1^{j_1} \cdots x_n^{j_n}.$$

Our approach is similar to that of B. Buchberger who reduces (1.1) to a so-called Gröbner basis; see eg. [1] or [2]. This leads to an equivalent system of multivariate polynomials of a very special form which normally consists of one equation in one variable and further equations expressing the other variables in terms of the distinguished one. The fact that Buchberger assumes precise operations (e.g. in a computer algebraic system) is of a secondary importance although the sensitivity of the algorithm to round-off has not yet been established.

Like Buchberger's algorithm, our algorithm is essentially an elimination algorithm. However, we attempt to keep the problem size fixed by starting from the classical theory of resultants for the solution of (1.1). In the final interpretation of our approach the essential idea becomes the (numerical) construction of a multiplication table for power products modulo (1.1) and the numerical solution of an eigenvalue problem associated with the multiplication table which yields *all* components of the ξ_μ as a subset of the components of the eigenvectors.

An approach which is quite similar to ours has been proposed by Lazard ([3]). Although his starting point is in algebraic geometry, his further developments are parallel to ours in several aspects; also the use of an eigenvalue problem for the numerical computation of *one* solution component has been considered. However, Lazard has not paid sufficient attention to the construction of what will be the matrix A in section 4; thus his elimination problem will be singular except in trivial situations, which prohibits a numerical treatment in floating-point arithmetic. Nevertheless, we gladly acknowledge that this work has helped us to clarify some issues although it came to our attention at a late state in our own research.

As in most approaches to the solution of (1.1), the presence of solution manifolds is detrimental. Also a number of degeneracies of (1.1) cannot be treated by our algorithm in its present form. Therefore we suggest to perturb (1.1) slightly in these cases, and to use Newton steps (or a homotopy method) to recover the solutions of the original from those of the perturbed system.

In sections 2 and 3 we present the necessary background from classical algebra which is nearly a hundred years old. In section 4, we show how a multiple of the resultant of the polynomials in (1.1) plus one special polynomial may be numerically computed by an elimination algorithm. The factorization problem for this resultant is reduced to a standard eigenvalue problem in section 5; its solution contains the desired zeros ξ_μ. Zeros at infinity may be treated by homogenization of the problem.

In section 6, we reinterpret our approach as the construction of a multiplication table for power products $x_1^{j_1} \cdots x_n^{j_n}$ modulo (1.1); this view permits the modification of the elimination in a number of degenerate situations. In section 7, we indicate the use of systematic perturbations of (1.1) for the generation of approximate zeros which may be used as excellent starting points for homotopy methods. In this fashion, one may even reach points on solution manifolds; cf. e.g. [4].

Details of the algorithmic construction of the resultant and other technical aspects of the algorithm will be presented in a separate report [5]. This report will also contain some nontrivial numerical examples.

2 Some facts from classical algebra

The vanishing of the *resultant* of two polynomials f_1 and f_2 in one variable x is the classical criterion for the existence of a common linear factor of f_1 and f_2, or of a common zero of the associated system

$$f_1(x) = f_2(x) = 0.$$

With

$$f_\nu(x) = \sum_{j=0}^{m_\nu} a_j^{(\nu)} x^j, \quad \nu = 1, 2,$$

the resultant of f_1 and f_2 is defined as

$$R\begin{pmatrix} f_1 & f_2 \\ x & 1 \end{pmatrix} = \det \begin{pmatrix} a_{m_1}^{(1)} & \cdots & \cdots & a_0^{(1)} & & \\ & \ddots & & & \ddots & \\ & & a_{m_1}^{(1)} & \cdots & \cdots & a_0^{(1)} \\ a_{m_2}^{(2)} & \cdots & \cdots & a_0^{(2)} & & \\ & \ddots & & & \ddots & \\ & & a_{m_2}^{(2)} & \cdots & \cdots & a_0^{(2)} \end{pmatrix} \begin{matrix} \uparrow \\ m_2 \\ \downarrow \\ \uparrow \\ m_1 \\ \downarrow \end{matrix} ; \tag{2.1}$$

it is a *homogeneous polynomial* in the coefficients of the two polynomials, of degree m_2 in the coefficients $a_j^{(1)}$ and of degree m_1 in the $a_j^{(2)}$. When we define the *weight* of an $a_j^{(\nu)}$ as $m_\nu - j$, and the weight of a product as the sum of the weights, then each term of R has weight $m = m_1 m_2$ as is easily seen from (2.1).

The use of this device in the solution of two polynomial equations in two variables

$$f_1(x_1, x_2) = f_2(x_1, x_2) = 0 \tag{2.2}$$

is a time-honored approach. Consider the polynomials of total degree m_ν ($\nu = 1, 2$) as polynomials in x_1 only:

$$\begin{aligned} f_\nu(x_1, x_2) &= \sum_{j_1 + j_2 \le m_\nu} a_{j_1 j_2}^{(\nu)} x_1^{j_1} x_2^{j_2} = \\ &= \sum_{j_1=0}^{m_\nu} \left(\sum_{j_2=0}^{m_\nu - j_1} a_{j_1 j_2}^{(\nu)} x_2^{j_2} \right) x_1^{j_1} = \sum_{j_1=0}^{m_\nu} A_{j_1}^{(\nu)}(x_2)\, x_1^{j_1}; \end{aligned} \tag{2.3}$$

then the resultant criterion may be applied to (2.3) to check for the existence of a common "zero" x_1. But the introduction of the $A_{j_1}^{(\nu)}$ into (2.1) turns R into a polynomial in x_2; thus the zeros of this polynomial are the only candidates for the x_2-component of zeros $(\xi_{\mu 1}, \xi_{\mu 2})$ of (2.2).

This polynomial $R(x_2)$ is of degree $m_1 m_2$ since the weight of the $A_{j_1}^{(\nu)}$ as coefficients of $f_\nu(x_1)$ is equal to their polynomial degree in x_2; see (2.3). The $m_1 m_2$ values $\xi_{\mu 2}$ which satisfy $R(x_2) = 0$ and their associated $\xi_{\mu 1}$ constitute the full solution set of (2.2), except in the case where f_1 and f_2 have a common polynomial factor of positive degree and $R(x_2)$ vanishes identically.

Naturally, the transition from (2.2) to $R(x_2) = 0$ is simply an *elimination* of x_1 from (2.2): Let us try to determine polynomials $\psi_1(x_1)$ and $\psi_2(x_1)$ such that x_1 disappears from

$$\psi_1(x_1) f_1(x_1, x_2) + \psi_2(x_1) f_2(x_1, x_2). \tag{2.4}$$

With a ψ_1 of degree m_2 and ψ_2 of degree m_1, the linear homogeneous system for the coefficients of the ψ_ν has a matrix which is essentially the transpose of the matrix in (2.1), with the $A_j^{(\nu)}$ in place of the $a_j^{(\nu)}$, and the vanishing of its determinant is a criterion for the existence of nontrivial polynomials in (2.4). Thus the resultant approach and elimination by "polynomial combination" are closely related concepts.

The success of this approach for two polynomials in two variables makes one look for generalizations which would help to attack the general multivariate case (1.1).

The generalization of resultants to multivariate polynomials is less widely known though it dates back to the end of the 19th century; but their study disappeared from algebra textbooks by the middle of this century together with a host of other computationally relevant material. For the following facts we have relied on Perron's Algebra, vol. 1, 2nd edition of 1931 ([6]); but supposedly any serious algebra textbook from this time would do as well.

We consider polynomials f_ν of total degree m_ν in n variables $x_1, x_2, \ldots, x_n$, and we use multi-indices and multi-exponents whereever feasible:

$$a_j\, x^j := a_{j_1 j_2 \cdots j_n}\, x_1^{j_1} \cdots x_n^{j_n}, \tag{2.5}$$

$$f_\nu(x) := \sum_{|j|=0}^{m_\nu} a_j^{(\nu)} x^j, \qquad |j| := \sum_{\mu=1}^{n} j_\mu. \tag{2.6}$$

For multivariate polynomials there is the concept of (algebraic) *dependence*: $n+1$ polynomials f_ν, $\nu = 0(1)n$, in n variables are always dependent, i.e. there exists a polynomial $F \not\equiv 0$ in $n+1$ variables such that

$$F\Big(f_0(x), f_1(x), \ldots, f_n(x)\Big) \equiv 0. \tag{2.7}$$

$F(y)$ contains only terms $C_j\, y^j$ with

$$\sum_{\nu=0}^{n} m_\nu j_\nu \le \prod_{\nu=0}^{n} m_\nu =: m, \tag{2.8}$$

and the C_j are rational functions in the $a_j^{(\nu)}$.

If each set of n polynomials from $\{f_0, f_1, \ldots, f_n\}$ is independent, then F is uniquely determined except for a factor which may be chosen such that the C_j are *polynomials* in the $a_j^{(\nu)}$ and that C_0 contains the "principal term"

$$\Big(a^{(1)}_{m_1 0 \cdots 0}\Big)^{\ell_1} \Big(a^{(2)}_{0 m_2 0 \cdots 0}\Big)^{\ell_2} \cdots \Big(a^{(n)}_{0 \cdots 0 m_n}\Big)^{\ell_n} \Big(a^{(0)}_{0 \cdots 0}\Big)^{\ell_0} \tag{2.9}$$

with the coefficient 1; here

$$\ell_\nu = m/m_\nu, \quad \nu = 0(1)n. \tag{2.10}$$

With this normalization of F, the quantity C_0 defines the *resultant* of the $(n+1)$ polynomials $f_0, f_1, \ldots, f_n$ in the n variables $x_1, \ldots, x_n$; it is denoted by

$$R \begin{pmatrix} f_1 & \ldots & f_n & f_0 \\ x_1 & \ldots & x_n & 1 \end{pmatrix}. \tag{2.11}$$

This resultant is an irreducible homogeneous polynomial in all the coefficients $a_j^{(\nu)}$, with *degree* ℓ_μ *of homogeneity* with respect to the $a_j^{(\mu)}$, $\mu = 0(1)n$. If the *weight* of an $a_j^{(\nu)}$ is now defined as $m_\nu - |j|$, then each term in the resultant has the same weight m ((2.9) is an example of such a term). For

$n = 1$ and f_0 renamed into f_2, we arrive at our familiar situation (2.1). However, for $n > 1$, the resultant may no longer be written as one determinant.

The resultant (2.11) permits a multitude of "representations"

$$R\begin{pmatrix} f_1 & \dots & f_n & f_0 \\ x_1 & \dots & x_n & 1 \end{pmatrix} \equiv \sum_{\nu=0}^{n} \varphi_\nu(x) f_\nu(x), \tag{2.12}$$

where the φ_ν are polynomials in the x_j whose coefficients are polynomials in the $a_j^{(\nu)}$. Note that the left hand side R in (2.12) no longer contains any of the x_j; thus (2.12) represents a polynomial combination of the f_ν which eliminates all power products x^j in the right hand side of (2.12) and R is the constant term of this polynomial combination. Moreover, it is known that any polynomial in the $a_j^{(\nu)}$ which permits a representation

$$\bar{R}(a) = \sum_{\nu=0}^{n} \psi_\nu(a, x) f_\nu(a, x) \tag{2.13}$$

must be divisible by R ; i.e. (2.13) implies

$$\bar{R}(a) = \rho(a) \cdot R\begin{pmatrix} f_1 & \dots & f_n & f_0 \\ x_1 & \dots & x_n & 1 \end{pmatrix} \tag{2.14}$$

with some polynomial ρ in the $a_j^{(\nu)}$.

These last equations show that the close relation between resultants and elimination by polynomial combination is retained for the multivariate case, which makes resultants an interesting object for our enterprise.

Finally, it is clear from the definition of R as the constant term C_0 of the polynomial F in (2.7) that the set of $n + 1$ polynomials f_ν, $\nu = 0(1)n$, in n variables can have common zeros only if their resultant vanishes.

3 The resultant and the zeros

The fact that

$$R\begin{pmatrix} f_1 & \dots & f_n & f_0 \\ x_1 & \dots & x_n & 1 \end{pmatrix} = 0 \tag{3.1}$$

is a necessary condition for the existence of a common zero of $f_0, f_1, \dots, f_n$ or of a solution of

$$f_0(x) = f_1(x) = \dots = f_n(x) = 0 \tag{3.2}$$

may be used in two ways towards the actual determination of the zeros of a system (1.1).

A) In a straightforward generalization of the procedure at the beginning of section 2, we replace n by $(n-1)$ in (3.1) and identify f_n with f_0; the coefficients of these polynomials thus remain polynomials in x_n of a degree specified by their weight and (3.1) becomes a polynomial equation of degree m in x_n. The (scalar) solutions $\xi_{\mu n}$ of this equation are the only candidates for the n-th components of zeros $\xi_\mu \in \mathbb{R}^n$ of (1.1). (Solutions at infinity and solution manifolds are not considered at this time.)

To obtain the remaining $n-1$ components, one may construct polynomials $P_0(x_n), \dots, P_{n-1}(x_n)$, with coefficients which are polynomials in the $a_j^{(\nu)}$ such that the

$$\xi_{\mu\nu} = P_\nu(\xi_{\mu n})/P_0(\xi_{\mu n}), \quad \nu = 1(1)\, n-1, \tag{3.3}$$

supplement $\xi_{\mu n}$ to a full zero vector ξ_μ of (1.1). However, P_0 will vanish at a multiple zero and also if there is a zero at infinity with $x_n = \xi_{\mu n}$. We will not further pursue this approach at this time.

B) The other approach which also appears in the old algebra books (e.g. [6]) seems unnecessarily complicated at first; but it is more general and more powerful in dealing with exceptional situations.

Let us, at first, consider one polynomial in one variable and complicate the simple assertion of the fundamental theorem of algebra by supplementing the given polynomial $f(x)$ of degree m by the *linear* polynomial

$$f_0(x) = a_1^{(0)} x + a_0^{(0)},$$

with indeterminate coefficients $a_j^{(0)}$. For a common zero of f and f_0, we must have

$$\begin{aligned} R\begin{pmatrix} f & f_0 \\ x & 1 \end{pmatrix} &= \det \begin{pmatrix} a_m & a_{m-1} & \cdots & \cdots & a_0 \\ a_1^{(0)} & a_0^{(0)} & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & a_1^{(0)} & a_0^{(0)} \end{pmatrix} = \\ &= a_m\big(a_0^{(0)}\big)^m - a_{m-1}\big(a_0^{(0)}\big)^{m-1} a_1^{(0)} + \dots + (-1)^m a_0\big(a_1^{(0)}\big)^m = 0. \end{aligned} \tag{3.4}$$

In a suitable extension field, this homogeneous polynomial in $a_1^{(0)}$ and $a_0^{(0)}$ factors:

$$R\begin{pmatrix} f & f_0 \\ x & 1 \end{pmatrix} = \prod_{\mu=1}^{m} \big(\xi_{\mu 0}\, a_0^{(0)} + \xi_{\mu 1}\, a_1^{(0)}\big). \tag{3.5}$$

Consider one of the linear factors, with coefficients $\xi_{\mu 0}$, $\xi_{\mu 1}$; by (3.5), f and f_0 will have a common zero whenever the coefficients of f_0 satisfy

$$\xi_{\mu 0}\, a_0^{(0)} + \xi_{\mu 1}\, a_1^{(0)} = 0 \tag{3.6}$$

which characterizes the set of all linear functions - or the bundle of all straight lines

$$f_0(x) = a_1^{(0)} x + a_0^{(0)} = a_1^{(0)} x - \frac{\xi_{\mu 1}}{\xi_{\mu 0}} a_1^{(0)}$$

with the common zero $\xi_{\mu 1}/\xi_{\mu 0}$. Hence, $\xi_\mu = \xi_{\mu 1}/\xi_{\mu 0}$ must be a zero of f, which is, of course, immediate from (3.4) and (3.6).

The generalization to n polynomial equations in n variables is now straightforward, at least in principle: Supplement the polynomials of (1.1) with the linear polynomial

$$f_0(x) = \sum_{\nu=1}^{n} a_\nu^{(0)} x_\nu + a_0^{(0)} \tag{3.7}$$

with indeterminate coefficients. The resultant of (3.1) is now a homogeneous polynomial in these indeterminates of degree m where $m = \prod_{\nu=1}^{n} m_\nu$ because of $m_0 = 1$. Classical algebra tells us that - in an appropriate extension field - this polynomial will factor:

$$R\begin{pmatrix} f_1 & \dots & f_n & f_0 \\ x_1 & \dots & x_n & 1 \end{pmatrix} = \prod_{\mu=1}^{m} \left(\xi_{\mu 0}\, a_0^{(0)} + \xi_{\mu 1}\, a_1^{(0)} + \ldots + \xi_{\mu n}\, a_n^{(0)} \right). \tag{3.8}$$

Each one of the linear factors characterizes a *hyperplane bundle* through $(\xi_{\mu 1}/\xi_{\mu 0}, \ldots, \xi_{\mu n}/\xi_{\mu 0})$. Since each hyperplane of a fixed bundle satisfies (3.1) and should have a common zero with $f_1, \ldots, f_n$, the intersection point of the bundle is the only candidate for that common zero. Thus the coefficients $\xi_{\mu\nu}$ of the factorization (3.8) determine the isolated zeros of (1.1) completely since one can also show that any isolated zero of (1.1) corresponds to a factor of (3.8); see e.g. [6].

In this approach, zeros at infinity present no problem; they correspond to linear factors with $\xi_{\mu 0} = 0$. Also the multiplicity of a zero of our original system (1.1) is simply the multiplicity with which the associated linear factor occurs in (3.8). This leads to a concise form of Bézout's Theorem; see [6], section 57. Manifolds of zeros cannot be dealt with; but they manifest themselves through $R \equiv 0$; see [6].

While this approach is quite elegant from the algebraic point of view - particularly if homogeneous coordinates are used throughout so that the special role of the $\xi_{\mu 0}$ disappears - it is still far from being constructive in an algorithmic and numerical sense: So far we cannot deal with resultants constructively for $n > 1$, and clearly there is no way of finding the factorization (3.8) numerically for a given homogeneous polynomial of degree m (m large!) in $n + 1$ indeterminates.

We will deal with these problems in the following sections.

4 Numerical computation of the resultant

Remember that the resultant (2.11) has been defined as the constant term C_0 of the polynomial F of (2.7) which expresses the dependence of $f_0, f_1, \ldots, f_n$; thus we could obtain R by determining F. In [6], there is a constructive proof for the existence of F which may even be turned into an algorithm; but it does not appear effective for our purpose, at least not in the general case. For strongly degenerate situations, as they may frequently occur in practice, the construction of R through F may be a feasible approach; this will have to be clarified further.

At present, we assume that the principal term (2.9) of the resultant does not vanish. This may always be achieved by a suitable linear transformation of the variables $x_1, \ldots, x_n$ ($a^{(0)}_{0\ldots0}$ is an indeterminate). But such transformations may introduce many additional terms into the f_ν and should therefore be avoided; cf. the end of this section.

Our approach is the following: We use a representation (2.13) of the resultant, i.e. we determine polynomials

$$\psi_\nu(c, x) := \sum_j c^{(\nu)}_j x^j, \quad \nu = 0(1)n, \tag{4.1}$$

such that the $c^{(\nu)}_j$ are polynomials in the $a^{(\nu)}_j$ and that all power products x^j cancel out of the polynomial combination

$$\sum_{\nu=0}^{n} \psi_\nu(c(a), x)\, f_\nu(a, x) =: \bar{R}(a). \tag{4.2}$$

This implies that (2.14) holds: $\bar{R}(a) = \rho(a)\, R(a)$.

Furthermore, we will see to it that the $a^{(0)}_\nu$ of (3.7) appear in $\bar{R}$ with the same degree of homogeneity as in R, viz. $\ell_0 = m/m_0 = m$, cf. (2.10). Then $\rho(a)$ does not contain the $a^{(0)}_\nu$; for given numerical values of the $a^{(\nu)}_j$, $\nu = 1(1)n$, it is simply a numerical factor. Hence the factorization (3.8) of $\bar{R}$ is equivalent to that of R and displays the isolated zeros of (1.1); cf. section 3.

For $\psi_0(c, x)$ we use an ansatz with precisely $l_0 = m$ terms:

$$\psi_0(x) = \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_n=0}^{m_n-1} c^{(0)}_{j_1\ldots j_n}\, x_1^{j_1} \cdots x_n^{j_n}. \tag{4.3}$$

The structure of the other ψ_ν is chosen subject to the following conditions:

(i) each ψ_ν must have no less than $\ell_\nu = m/m_\nu$ terms;

(ii) the total number of coefficients $c^{(\nu)}_j$ must equal the number of elimination conditions represented by (4.2);

(iii) the total number of coefficients $c^{(\nu)}_j$ should be kept small, i.e. close to $\sum_{\nu=0}^{n} \ell_\nu$.

The power product structure of the individual ψ_ν, $\nu \neq 0$, is not uniquely determined by these side conditions. We have conceived a conforming power product pattern for the various ψ_ν which may easily be generated algorithmically for any choice of n and the m_ν. Its principal idea is that the power products occuring in $x_\nu^{m_\nu}\psi_\nu(x)$ should be disjoint for $\nu = 1(1)m$. This construction, and a good number of other algorithmic and technical details of our appraoch will be described in a forthcoming report [5].

Note that (4.2) is a system of linear equations in the $c_j^{(\nu)}$ if we equate the coefficients of the power products x^j, $j \neq 0$, to zero. For reasons which will become apparent later, we write this system in transposed form:

$$\Big(c_j^{(1)} \dots c_j^{(n)}\, c_j^{(0)}\Big) \begin{pmatrix} \textit{coefficients of } f_1 \\ - - - - - - - - - - \\ \dots \\ - - - - - - - - - - \\ \textit{coefficients of } f_n \\ - - - - - - - - - - \\ a_\nu^{(0)} \textit{ of } f_0 \end{pmatrix} = \Big(0 \dots 0\, \bar{R}\Big). \tag{4.4}$$

The elements of the sparse matrix A of (4.4) - which is quadratic due to condition (ii) above - are individual coefficients of the f_ν as is easily seen. Each column of A corresponds to a particular power product x^j on the left-hand side of (4.2); some fixed order for these power products - which corresponds to an order in the set of the exponent vectors - is assumed from now on. At this point, we require only that the power products x^j, $j_\nu < m_\nu$, which occur in ψ_0 (cf. (4.3)), are at the low end of our order, with x^0 at the bottom. If we arrange the $c_j^{(0)}$ in the same order, the right lower $m \times m$ corner of A looks like

$$\begin{pmatrix} a_0^{(0)} & & 0 \\ \textit{other} & \ddots & \\ a_\nu^{(0)} & & a_0^{(0)} \end{pmatrix} =: A_{00}. \tag{4.5}$$

Naturally, $\bar{R}$ is not known in (4.4) but to be determined. It must be chosen as a polynomial in the $a_j^{(\nu)}$ such that the solutions $c_j^{(\nu)}$ of (4.4) become polynomials in the $a_j^{(\nu)}$, i.e. in the elements of the matrix A; at the same time it should not introduce unnecessary terms into $\rho(a)$, cf. (2.14). Hence

$$\bar{R}(a) := \det(A) \tag{4.6}$$

is the natural choice. Due to the ansatz (4.3) for ψ_0, this $\bar{R}$ has also the correct degree m of homogeneity in the $a_\nu^{(0)}$ (cf. also (4.5)) so that

$$\bar{R}(a) = \rho\Big(a_j^{(1)}, \dots, a_j^{(n)}\Big) \cdot R \begin{pmatrix} f_1 & \dots & f_n & f_0 \\ x_1 & \dots & x_n & 1 \end{pmatrix}. \tag{4.7}$$

If the coefficients $a_j^{(\nu)}$ of the f_ν, $\nu = 1(1)n$, are also considered as *indeterminates* then $\rho(a_j^{(\nu)})$ may easily be shown not to vanish because the diagonal of A contains no zeros for a reasonable order of the power products. Thus our approach is feasible in the general case (without degeneracies).

Of course, we are really interested in $\bar{R}$ as a *polynomial in the* $a_j^{(0)}$, for specified *numerical* values of the $a_j^{(\nu)}$, $\nu \neq 0$. We wish to use elimination in the upper rows of A to find the coefficients of the products of the $a_j^{(0)}$ numerically. (Note that this is indeed a numerical - but not a symbolic - process because the indeterminates $a_j^{(0)}$ are not involved.) Here, degeneracies of A may hurt us; in particular, they may make $\rho(a_j^{(\nu)})$ zero (or very nearly zero) although R is well-defined.

If $\rho(a_j^{(\nu)}) = 0$ for the specified $f_1, \dots, f_n$ of (1.1), the structure of the ψ_ν has to be modified or the whole approach has to be altered; some special situations of this kind will be discussed in [5]. Within the presentation of this paper, we will assume that we may *perturb* the f_ν in order to avoid this difficulty as will be explained in section 7.

If $\rho(a_j^{(\nu)}) \not\approx 0$, difficulties in the elimination in the upper rows of A may be relieved by algorithmic techniques, e.g. by a transformation of the vector x, possibly at considerable computational cost. This situation will also be studied in more detail in [5].

In order not to confuse the general exposition, we will assume in the following that the quadratic upper left submatrix of A (supplementary to A_{00}) is numerically regular as it is when no degeneracies are present. We subdivide the matrix A into

$$A = \left(\begin{array}{c|c} A_{11} & A_{10} \\ \hline A_{01} & A_{00} \end{array}\right), \tag{4.8}$$

where A_{11} and A_{10} contain only elements $a_j^{(\nu)}$, $\nu = 1(1)n$, and A_{01} only $a_\nu^{(0)}$, $\nu = 1(1)n$, while A_{00} is given by (4.5).

Due to our regularity assumption for A_{11}, we may compute its triangularization and represent A as

$$A = \left(\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & I \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ \hline A_{01} & I \end{array}\right) \left(\begin{array}{c|c} I & A_{11}^{-1}A_{10} \\ \hline 0 & \bar{A}_{00} \end{array}\right) \quad \text{with} \tag{4.9}$$

$$\bar{A}_{00} := A_{00} - A_{01}A_{11}^{-1}A_{10}, \tag{4.10}$$

so that

$$\det(A) = \det(A_{11}) \cdot \det(\bar{A}_{00}). \tag{4.11}$$

Obviously, the $a_\nu^{(0)}$, $\nu = 0(1)n$, are contained only in $\bar{A}_{00}$. Thus, by (4.6), (4.7) and (4.11),

$$\det(\bar{A}_{00}) = \bar{\rho}\big(a_j^{(1)}, \ldots, a_j^{(n)}\big) \cdot R \begin{pmatrix} f_1 & \ldots & f_n & f_0 \\ x_1 & \ldots & x_n & 1 \end{pmatrix}. \tag{4.12}$$

As we have stated in the paragraph below (4.2), the polynomial $\det(\bar{A}_{00})$ in the $a_\nu^{(0)}$ is fully equivalent to R for our purpose of determining the zeros of (1.1).

5 Computation of the zeros from the resultant

The numerical factorization of a homogeneous polynomial of a degree > 1 in more than two variables is not feasible except in trivial cases. But if we set $a_1^{(0)} = a_2^{(0)} = \ldots = a_{n-1}^{(0)} = 0$ in (3.8), then

$$R\Big(a_0^{(0)}, 0, \ldots, 0, a_n^{(0)}\Big) = \prod_{\mu=1}^{m} \Big(\xi_{\mu 0}\, a_0^{(0)} + \xi_{\mu n}\, a_n^{(0)}\Big), \tag{5.1}$$

with the same $\xi_{\mu 0}$, $\xi_{\mu n}$ as previously, and the determination of the $\xi_{\mu n}/\xi_{\mu 0}$ requires merely the computation of the zeros of a polynomial in one variable; cf. (3.5) and (3.4).

In (4.12), this specialization of the $a_\nu^{(0)}$ leads to a further simplification: All elements of $\bar{A}_{00}$ are *linearly* homogeneous in the $a_\nu^{(0)}$ and, due to (4.5) and (4.10), the $a_0^{(0)}$ occur only on the main diagonal of $\bar{A}_{00}$. Hence

$$\bar{A}_{00}\Big(a_0^{(0)}, a_1^{(0)}, \ldots, a_n^{(0)}\Big) = a_0^{(0)} I + a_1^{(0)} B^{(1)} + \ldots + a_n^{(0)} B^{(n)}, \tag{5.2}$$

with the $m \times m$ - matrices $B^{(\nu)}$ determined by (4.10).

(5.2) tells us that

$$\bar{A}_{00}\Big(a_0^{(0)}, 0, \ldots, 0, a_n^{(0)}\Big) = a_0^{(0)} \cdot I + a_n^{(0)} \cdot B^{(n)}$$

so that $\det(\bar{A}_{00}(a_0^{(0)}, 0, \ldots, 0, a_n^{(0)}))$ can only vanish if

$$1 \cdot a_0^{(0)} + \lambda_\mu^{(n)} \cdot a_n^{(0)} = 0, \quad \mu = 1(1)m, \tag{5.3}$$

where the $\lambda_\mu^{(n)}$ are the eigenvalues of $B^{(n)}$. Thus the n-th components $\xi_{\mu n}/\xi_{\mu 0}$ of the zeros ξ_μ of (1.1) are given by the eigenvalues of $B^{(n)}$.

Again, we disregard the potential degenerations of the eigenvalue problem for $B^{(n)}$ and assume that there are m distinct (complex) eigenvalues $\lambda_\mu^{(n)}$. We must now complement the n-th components of our zeros ξ_μ of (1.1) by their companion components. To compute the eigenvalues of the analogously defined matrices $B^{(\nu)}$, $\nu = 1(1)\, n-1$, will not help except if we are willing to sort out the correct combinations of components by substituting *all* combinations into (1.1).

Instead, we observe that for $\det(\bar{A}_{00}) = \det(A) = 0$ the eigenvector $z_{\mu 0} \neq 0$ of B associated with the eigenvalue λ_μ (we drop the superscript n at the moment) satisfies

$$\bar{A}_{00}\, z_{\mu 0} = 0 \tag{5.4}$$

due to (5.3); also, with

$$z_{\mu 1} := -A_{11}^{-1} A_{10}\, z_{\mu 0}, \tag{5.5}$$

we have

$$A \begin{pmatrix} z_{\mu 1} \\ z_{\mu 0} \end{pmatrix} = 0. \tag{5.6}$$

We normalize $z_{\mu 0}$ and $z_{\mu 1}$ by setting the last component equal to 1. Then the components of $z_{\mu 0}$ and $z_{\mu 1}$ correspond to the power products x^j which occur on the left hand side of (4.2); cf. (4.4) and the paragraph below it.

On the other hand, the linear factor $(a_0^{(0)} + \lambda_\mu a_n^{(0)})$ of $\det(\bar{A}_{00}(a_0^{(0)}, 0, \ldots, 0, a_n^{(0)}))$ is associated with a linear factor $(a_0^{(0)} + \xi_{\mu 1} a_1^{(0)} + \ldots + \xi_{\mu, n-1} a_{n-1}^{(0)} + \lambda_\mu a_n^{(0)})$ of $\det(\bar{A}_{00}(a_0^{(0)}, a_1^{(0)}, \ldots, a_n^{(0)}))$ which determines the zero $\xi_\mu = (\xi_{\mu 1}, \ldots, \xi_{\mu, n-1}, \lambda_\mu)$ of (1.1). If we form the power products of the components of ξ_μ and arrange them in the order used in (4.4), they must also satisfy (5.6).

Since they have the same normalization, the eigenvector $z_{\mu 0}$ of B for the eigenvalue λ_μ and the vector composed of the values of those power products for ξ_μ which occur in (4.3) must be identical! Thus those components of $z_{\mu 0}$ which correspond to the linear monomials $x_1, x_2, \ldots, x_{n-1}$ are the remaining components of the solution vector ξ_μ of (1.1) with n-th component λ_μ. (Actually, x_n will also occur among the power products; this component will reproduce the eigenvalue λ_μ because of the structure of B.)

Thus the eigenvalue problem for $B^{(n)}$ from (5.2) gives us *all* components of *all* zeros of (1.1) as long as no degeneracies occur. The linear monomial x_k would only be missing in (4.3) if $m_k = 1$, i.e. if the k-th equation of (1.1) is linear; then $\xi_{\mu k}$ may be recovered from that linear equation in terms of the other $\xi_{\mu\nu}$. (This trivially generalizes to several linear equations in (1.1).)

Naturally, any $k \in \{1, \ldots, n\}$ may be chosen in place of n in (5.1). This liberty may be used to avoid certain numerical difficulties in the eigenvalue problem. E.g., there may be several zeros of (1.1) with the same value of the n-th component $\xi_{\mu n}$ while all k-th components may be different; then $B^{(k)}$ has only simple eigenvalues and well-defined eigenvectors while $B^{(n)}$ has multiple eigenvalues and the eigenproblem code may not give us the proper eigenvectors from the associated eigenspace.

Furthermore, it is obvious that $B^{(k)}$, $k = 1(1)n$, is found as

$$B^{(k)} := A_{00}^{(k)} - A_{01}^{(k)} A_{11}^{-1} A_{10} \tag{5.7}$$

where $A_{01}^{(k)}$, $A_{00}^{(k)}$ are the m lower rows of A which arise for $f_0(x) = x_k$; cf. (5.2). Thus $\left(\begin{array}{c|c} A_{01}^{(k)} & A_{00}^{(k)} \end{array} \right)$ has one entry 1 per row; otherwise it contains only zeros, and the formation of $A_{11}^{-1} A_{10}$ is the only computational work necessary.

Therefore, we have the following algorithm for the numerical computation of all zeros ξ_μ of a multivariate polynomial system of equations (1.1) with isolated zeros:

Algorithm (regular case):

(i) From the coefficients $a_j^{(\nu)}$ of the f_ν, form $\left(\begin{array}{c|c} A_{11} & A_{10} \end{array} \right)$ according to the algorithmic procedure indicated in section 3 and specified in [5].

(ii) Select k from $\{1, \ldots, n\}$ (cf. [5] for some guidelines) and form $\left(\begin{array}{c|c} A_{01}^{(k)} & A_{00}^{(k)} \end{array} \right)$.

(iii) Compute $A_{11}^{-1} A_{10}$ and form $B^{(k)}$ by (5.7).

(iv) Compute all eigenvalues $\lambda_\mu^{(k)}$ and associated eigenvectors $z_{\mu 0}^{(k)}$, $\mu = 1(1)m$, of $B^{(k)}$.

(v) Select the components corresponding to the powers x_ν, $\nu = 1(1)n$, from the $z_{\mu 0}^{(k)}$ to obtain the $\xi_{\mu\nu}$.

6 Basis representation for power products

Let us denote by Z_1 and Z_0 the vectors of the power products (PPs) x^j which have appeared in (5.4)–(5.6) but with indeterminate x_ν. Then the matrices $A_{01}^{(k)}$, $A_{00}^{(k)}$ in (5.7), which contain only elements 0 or 1, may be defined by the identity

$$x_k \cdot Z_0 \equiv \left(\begin{array}{c|c} A_{01}^{(k)} & A_{00}^{(k)} \end{array} \right) \begin{pmatrix} Z_1 \\ Z_0 \end{pmatrix}. \tag{6.1}$$

On the other hand, (5.5) holds only for vectors of PPs of the components $\xi_{\mu\nu}$ of a zero ξ_μ of (1.1). Considering the generation of the matrix A via (4.4), we may also say that (5.5) is a PP identity *modulo* $\{f_1, \ldots, f_n\}$, i.e. for PPs satisfying (1.1):

$$Z_1 \equiv -A_{11}^{-1} A_{10}\, Z_0 \qquad \mod \{f_\nu\}. \tag{6.2}$$

If we substitute (6.2) into (6.1) and consider (5.7) we obtain

$$x_k \cdot Z_0 \equiv B^{(k)}\, Z_0 \qquad \mod \{f_\nu\}. \tag{6.3}$$

Note that the identity (6.3) has been obtained from (1.1) by polynomial combination and elimination of all PPs not in $x_k Z_0$ and by "solving" for those PPs in $x_k Z_0$ which are not in the "basis" Z_0; it is an immediate consequence of (6.1) that all components of (6.3) with an $x^j \in Z_0$ on the left hand side are trivial, e.g. $x_k \cdot x_1^2 x_2 x_k = x_1^2 x_2 x_k^2$.

Obviously, the matrices $B^{(k)}$, $k = 1(1)n$, define the *multiplication table* which holds for PPs *modulo* $\{f_\nu\}$. This multiplication table permits the reduction of an arbitrary multivariate polynomial (in the x_ν) to a polynomial composed only of *basis* PPs in Z_0, modulo $\{f_\nu\}$. At the same time, each multiplication table (6.3) (for any fixed value of k) defines the zeros of (1.1) by the interpretation of (6.3) as an *eigenvalue problem*; cf. section 5.

In the regular case, with m isolated zeros of (1.1), the basis Z_0 must contain m PPs and our generic assumption

$$x^j \in Z_0 \quad \Longleftrightarrow \quad j_\nu \le m_\nu - 1 \tag{6.4}$$

which was introduced through the ansatz (4.3) is appropriate. However, our changed interpretation of the matrices $B^{(k)}$ now permits us to attack degenerate situations:

If we can - by polynomial combination of the equations in (1.1) and elimination of PPs - obtain a multiplication table (6.3) for *some* set $\bar{Z}_0$ of PPs, then we have established the identity

$$\Big(B^{(k)} - x_k\, I\Big)\bar{Z}_0 \equiv \sum_{\nu=1}^{n} \chi_\nu(x)\, f_\nu(x) \tag{6.5}$$

and the eigenvectors $\bar{z}_{\mu 0}$ of $B^{(k)}$ are the only candidates for vectors of PPs of solution components of (1.1). Under suitable assumptions about Z_0 and the reduction procedure one can also show that the "linear" components of the $\bar{z}_{\mu 0}$ generate solutions of (1.1) indeed.

The algorithmic construction of adapted multiplication tables (6.3) in degenerate situations will be analyzed in a separate paper. Note that it is the occurrence of isolated solutions at infinity - a very common phenomenon in multivariate polynomial equations - which leads to a reduction of the generic PP basis (6.4). This case may also be treated by avoiding the distinction of $a_0^{(0)}$ in (5.2) which derives from our insistence on designating 1 as the "lowest" PP in our order, cf. (4.5). In a homogenization of the problem (1.1), 1 becomes simply the first power of the homogenizing variable x_0, and any other variable x_ℓ can take the role of x_0 as lowest variable. No reduction in the size of Z_0 will then appear if none of the zeros of (1.1) has a vanishing ℓ-th component.

If (1.1) has solution manifolds of a positive dimension it is clear that an identity of type (6.5) cannot exist because the left-hand side can vanish only for a finite set of (normalized) numerical vectors $\bar{Z}_0$. Thus our construction must fail in this case.

7 Regularization via perturbation

Contrary to algebra-based approaches (including Buchberger's method [1]), homotopy methods are able to reach, at least, individual points on solution manifolds as has recently been established by

Zulehner ([4]) under rather weak assumptions. On the other hand, homotopy methods are likely to be inefficient when the zeros of the initial problem - which is then continuously deformed into the given problem (1.1) - are far away from the zeros of (1.1). Therefore, the following combination of our approach and the homotopy approach has been considered:

We may use our proposed elimination-eigenvalue method as the tool to compute the zeros of an initial problem which is quite close to (1.1) but sufficiently non-degenerate (no solution manifolds, no zeros at infinity). Then a homotopy code takes over and finds the exact location of the zeros of the original problem, or of some zeros (as many as its multiplicity) in the case of connected solution manifolds. (Normally a homotopy has to start at an initial problem with "known" solutions which may force one to start far away.) Note that for most regular zeros one homotopy step, i.e. a few Newton steps for (1.1), will suffice if we start from a nearby perturbed problem. Only in the vicinity of "difficult" zeros, several homotopy steps and a higher computational effort should be necessary. Also one may choose to process only those zeros of the initial problem further which lie in some particular region of the $\mathbb{R}^n$.

Some first positive experiences in this approach have been gained in collaboration with W. Zulehner whose help has been highly appreciated. The problem to be analyzed further (analytically and experimentally) is the selection of a perturbation of (1.1) which is optimal in covering the degeneracies in the original problem without moving the problem too far away in terms of its zeros. Also the behavior of Newton-like methods in the vicinity of zeros with a singular Jacobian (see e.g. [7]) will have to be taken into account.

Finally, let us remark that even in the regular case it will generally be more efficient to use one Newton step as an "iterative improvement" of the computed zeros than to resort to special algorithmic provisions in the elimination and eigenproblem phases of the algorithm specified at the end of section 5. If necessary this Newton step may even be executed with result verification (cf. e.g. [8]) to obtain a guaranteed inclusion of the zero. Thus the above approach for degenerate problems is quite natural.

8 Conclusion

We have developed a direct algorithm for the numerical computation of all zeros of a multivariate polynomial system (1.1), at least for the case without degeneracies. The algorithm uses standard Gauss elimination for a (relatively) large, sparse matrix and eigenvector computation for a matrix of dimension m, the number of different zeros. In the regular case, the algorithm works well; our numerical experiences will be reported in the forthcoming paper [5].

For the treatment of degenerate cases - which are discovered in the elimination phase - we have indicated several options: So far, perturbation of (1.1) into a regular problem and subsequent use of a homotopy code has been used successfully. In particular, it is the only approach which will also yield zeros on solution manifolds. Other tools are transformation of variables, including homogenization, and the construction of a reduced basis representation for PPs. More research will be necessary to use these techniques in a well-controlled manner. Ideas from [2] may prove useful in the basis reduction.

Certainly, our algorithm is not intended for the computation of some particular zero of a polynomial system, with sufficient information about its location. Rather, our algorithm will provide the information about the location of the complete solution set which may then be used to compute particular zeros more accurately. For this purpose, the approach of this paper which strongly utilizes the algebraic structure of (1.1) appears suitable and competitive.

Finally, it should be emphasized that, whenever we have spoken of computation and numerical determination, we have always meant common floating-point arithmetic. Therefore, there will be numerical errors in both phases of the algorithm, and these effects will depend on the condition of the problem. For a discussion of this aspect, see [5].

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