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A NOTE ON LYAPUNOV TRANSFORMATION AND EXPONENTIAL DECAY IN LINEAR ODE SYSTEMS*

W. AUZINGER† and A. EDER

Institut für Angewandte und Numerische Mathematik,
Technische Universität Wien, Wiedner Hauptstraße 8–10/1152,
A-1040 Wien, Austria

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In this paper we consider a class of matrices A where all eigenvalues have negative real parts and are of a common magnitude $O(1)$. Concerning the behavior of e^{tA} we provide a necessary and sufficient condition, via Lyapunov transformation, for an estimate of the form $\|e^{tA}\|_2 \le \mathcal{K}_0 \exp(-t/\mathcal{K}_1)$ to be valid uniformly for $t > 0$ with moderate-sized constants $\mathcal{K}_0$ and $\mathcal{K}_1$. All relevant relations are quantitatively specified.

1. Introduction and Background

In this paper we give a quantitative criterion for exponential decay in initial value problems for linear ODE systems

$$\begin{aligned} y'(t) &= Ay(t)\,, \qquad t \ge 0\,, \\ y(0) &= y_0 \end{aligned} \tag{1}$$

for a certain class of matrices $A \in \mathbb{C}^{n\times n}$ to be specified below.

The motivation for our considerations is twofold and deserves some introductory remarks. First of all, concerning the historical context, let us recall the well-known Kreiss Matrix Theorem (KMT) on linear L_2-stability (cf. Ref. 7), which provides necessary and sufficient conditions for the uniform boundedness of matrix powers A^ν, i.e. for the existence of a constant[a] $\mathcal{C}$ such that $\|A^\nu\|_2 \le \mathcal{C}$ uniformly for $\nu \to \infty$. For a fixed $n \times n$ matrix A such a condition for power-boundedness can be formulated in an elementary way as an eigenstructure criterion. The essential point behind the KMT is that it refers to *families* $\mathbf{A}$ of $n \times n$ matrices A, which makes it a strong tool in the stability theory for ODE and PDE discretizations. In

*Dedicated to the memory of Richard Weiss.
†E-mail: w.auzinger@tuwien.ac.at
[a]In this introductory section, the notation $\mathcal{C}$ is to be understood in a generic sense.

this general context, the straightforward eigenstructure criterion is of course too weak. In the formulation of the KMT, other criteria of a more quantitative nature appear, like the resolvent condition, $\|(zI-A)^{-1}\|_2 \le \mathcal{C}/(|z|-1)$ for all $z \in \mathbb{C}$ with $|z|>1$. Another of these equivalent conditions characterizes stability by means of a linear transformation H, with $\|H\|_2\|H^{-1}\|_2 \le \mathcal{C}$, such that $\|A^\nu\|_H \le 1$ in the corresponding "elliptic" norm $\|\cdot\|_H$ induced by H (see Refs. 9 and 10).

The essential point is that the relevant conditions have to be satisfied uniformly[b] for all $A \in \mathbf{A}$. In the original formulation of the KMT, the relations involved were not explicitly quantified. Within the past years, the theorem and conditions involved have further been sharpened, quantified and generalized by a number of authors (cf. e.g. Refs. 3, 8–10 and many others).

The problem of characterizing the behavior of *matrix exponentials* e^{tA} uniformly w.r.t. $t>0$ has been investigated in the same spirit; in particular, in Refs. 3 and 8 criteria are formulated (based on appropriate resolvent conditions) for $\|e^{tA}\|_2 \le \mathcal{C}$ (or, more generally, $\|e^{tA}\|_2 \le \mathcal{C}e^{kt}$) to be valid. In this context, the formulation of another equivalent criterion involving the concept of an elliptic norm (as mentioned above) would involve an appropriate *Lyapunov transformation*. However, a look at the relevant literature reveals that this approach seems not to have been taken so far.

The purpose of this paper is to present a criterion of this type, based on Lyapunov transformation, for a particular class of matrices A, with eigenvalues of a common magnitude $O(1)$ having negative real parts. At first sight, this is a rather trivial situation; however, in the case that A is not normal,[c] the exponential decay (generally) only occurs after an initial phase where these solutions may (even strongly) *grow* in $\|\cdot\|_2$. Such a behavior can be characterized by an estimate of the form

$$\|e^{tA}\|_2 \le \mathcal{K}_0 \exp(-t/\mathcal{K}_1) \quad \text{for arbitrary} \quad t \ge 0 \tag{2}$$

with certain constants $\mathcal{K}_0$ and $\mathcal{K}_1$.

Our essential result directly relates these constants to the condition number of a certain *Lyapunov transformation* V "normalizing" A, (cf. (4) below): We shall show that $\mathcal{K}_0$ and $\mathcal{K}_1$ can be estimated in terms of the norms of V and its inverse, *and vice versa* (with an additional dependence on the size of the eigenvalues of A and on the dimension n). The essential point is that all relations between the constants involved will be *quantitatively specified*.

The first part in our proof (estimation of $\|e^{tA}\|_2$ using bounds for V and its inverse; see Sec. 2) is elementary and is provided only for the sake of self-containedness. The second part, however, where estimates for $\|V\|_2$ and $\|V^{-1}\|_2$

[b]On the other hand, an assertion of a nature like the KMT may also be useful for the case of a fixed matrix A, in particular if the relevant constants $\mathcal{C}$ are quantitatively specified.
[c]Cf. also Refs. 1 and 2 for important remarks concerning the effect of non-normality, in particular for stiff problems.

are derived proceeding from (2), requires careful estimates based on Cauchy integrals (see Sec. 3). The latter gives the interesting information — a condition (on V) which is not only sufficient but also necessary (for (2) to hold) is a natural one.

Secondly, this quantitative criterion for exponential decay serves as an essential motivation in our numerical convergence theory for a general class of nonlinear stiff initial value problems $y' = f(y)$. As argued in Ref. 2, non-normality of the Jacobian f_y causes significant technical difficulties in the stability and convergence theory for discretizations like e.g. implicit Runge–Kutta schemes. One way around this difficulty is to base the analysis on a semi-global linearization concept which relates points in the phase space with corresponding points on an invariant manifold containing smooth solutions. In this context, the use of an appropriate (local) Lyapunov transformation (on the fast, stiff time scale) plays an essential role. A more detailed description of this concept is out of the scope of this paper; we refer to Refs. 2 and 4.

In the following, we restrict our considerations to matrices A satisfying the following assumption concerning the spectrum. We shall refer to this class of matrices simply as "*class* (3)":

Assumption 1.1. The eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ of A are contained in the half disc

$$\tilde{\sigma}(A) := \{z \in \mathbb{C} : \operatorname{Re} z \le -1, |1+z| \le \mathcal{K}\} \tag{3}$$

with radius $\mathcal{K}$ and location left to -1; cf. Fig. 1.

Notation 1.1. Throughout $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{C}^n$ and $\|\cdot\|_2$ the induced (Euclidean) vector (resp. matrix) norm. A^* denotes the Hermitian adjoint of $A \in \mathbb{C}^{n\times n}$.

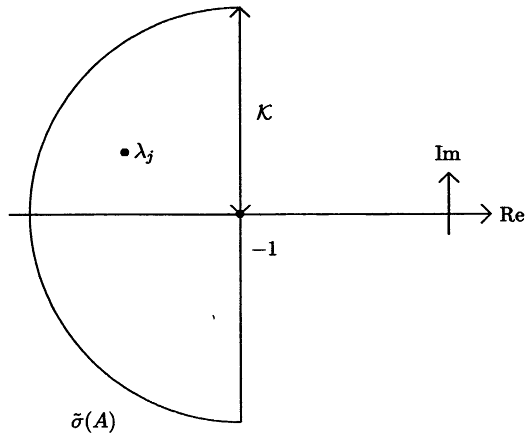

Fig. 1. Location of the spectrum of A.

2. The Lyapunov Equation. Estimation of $\|e^{tA}\|_2$ via a Related Elliptic Norm

Consider the Lyapunov equation

$$A^*V + VA = -I\,. \tag{4}$$

Since we have assumed that all eigenvalues of A have negative real parts, there exists a unique positive definite solution $V \in \mathbb{C}^{n\times n}$ of (4) (cf. Ref. 5). This matrix induces a so-called elliptic norm

$$\|y\|_V := \|V^{1/2}y\|_2 = \langle Vy, y\rangle^{1/2}\,. \tag{5}$$

Now we provide a simple estimate of the form (2) (valid for arbitrary matrices A for which the spectrum is contained in the left half-plane); the constants $\mathcal{K}_0$ and $\mathcal{K}_1$ in (2) will be bounded in terms of the matrix V defined by (4).

To this end we insert an arbitrary solution $y(t)$ of (1) and differentiate the outcome $\|y(t)\|_V^2$:

$$\begin{aligned}\frac{d}{dt}\langle Vy(t), y(t)\rangle &= \langle (A^*V + VA)y(t), y(t)\rangle \\ &= -\|y(t)\|_2^2 \le -\|V\|_2^{-1}\langle Vy(t), y(t)\rangle\,.\end{aligned} \tag{6}$$

Here we have used the Lyapunov equation (4) and $\langle Vy, y\rangle \le \|V\|_2\|y\|_2^2$. The inequality (6) implies

$$\|y(t)\|_V \le \exp\Big(-\frac{t}{2\|V\|_2}\Big)\|y(0)\|_V\,; \tag{7}$$

furthermore,

$$\begin{aligned}\|y(t)\|_2 &\le \|V^{1/2}\|_2\|V^{-1/2}\|_2 \exp\Big(-\frac{t}{2\|V\|_2}\Big)\|y(0)\|_2 \\ &= (\|V\|_2\|V^{-1}\|_2)^{1/2}\exp\Big(-\frac{t}{2\|V\|_2}\Big)\|y(0)\|_2\,.\end{aligned} \tag{8}$$

We obtain

Proposition 2.1. *For all $t \ge 0$, $\|e^{tA}\|_2$ can be bounded in terms of the Lyapunov transform V (defined by (4)) in the following way:*

$$\|e^{tA}\|_2 \le (\|V\|_2\|V^{-1}\|_2)^{1/2}\exp\Big(-\frac{t}{2\|V\|_2}\Big)\,.$$

I.e. the estimate (2) holds with $\mathcal{K}_0 = (\|V\|_2\|V^{-1}\|_2)^{1/2}$ and $\mathcal{K}_1 = 2\|V\|_2$.

3. Estimation of the Lyapunov Transform in Terms of a Bound for $\|e^{tA}\|_2$

In this section we reverse the argument from above: We assume that for A from class (3) an estimate (2) is valid, and derive a norm bound for V (defined by (4)) and its inverse. These bounds will be explicitly given in terms of $\mathcal{K}_0$, $\mathcal{K}_1$ from (2), $\mathcal{K}$ from assumption (3) and the matrix dimension n.

To this end we use the fact that V from (4) can be written as an integral (cf. Ref. 5):

$$V = \int_0^\infty e^{tA^*}e^{tA}dt\,. \tag{9}$$

We have assumed that (2) holds. Together with (9) and observing $\|e^{tA^*}\|_2 = \|e^{tA}\|_2$ we are led to

$$\|V\|_2 \le \int_0^\infty \mathcal{K}_0^2 \exp\Big(-\frac{2}{\mathcal{K}_1}t\Big)dt = \frac{\mathcal{K}_0^2\mathcal{K}_1}{2}\,. \tag{10}$$

Furthermore, multiplication of (4) with V^{-1} from left and right yields

$$V^{-1}A^* + AV^{-1} = -V^{-2}\,, \tag{11}$$

giving

$$\|V^{-1}\|_2^2 = \|V^{-2}\|_2 \le \|V^{-1}\|_2\|A^*\|_2 + \|A\|_2\|V^{-1}\|_2 = 2\|A\|_2\|V^{-1}\|_2\,; \tag{12}$$

hence

$$\|V^{-1}\|_2 \le 2\|A\|_2\,. \tag{13}$$

To draw the desired conclusion – namely to bound the norm of V and V^{-1} (in terms of $\mathcal{K}$, $\mathcal{K}_0$ and $\mathcal{K}_1$) – we need a *bound for $\|A\|_2$ valid under the given assumption* (2). This is the nontrivial step in our proof, and the rest of this paper is devoted to it. For this purpose we write A as a Cauchy integral

$$A = \frac{1}{2\pi i}\oint_\Gamma z(z-A)^{-1}dz\,, \tag{14}$$

where the positively oriented path $\Gamma \subset \mathbb{C}$ encloses all eigenvalues λ_j, $j = 1, \ldots, n$ of A, which are contained in the half disc $\tilde{\sigma}(A)$ (cf. Figs. 1 and 2).

We choose a special path Γ of length $|\Gamma| = O(1)$ on which an estimate of the norm of the resolvent $R(z) := (z-A)^{-1}$ will turn out to be $O(1)$ and $|z| = O(1)$ for $z \in \Gamma$. Quantification and combination of these estimates will yield the desired bound for $\|A\|_2$. The idea to construct this path is to partition Γ into a path Γ_0 parallel to the imaginary axis and two paths Γ_1, Γ_2 symmetric w.r.t. the real axis (cf. Fig. 2), in a way such that the resolvent on Γ_1, Γ_2 can be bounded with the help of a bound for the resolvent on Γ_0. First we derive a resolvent bound for the appropriately chosen Γ_0. To this end we use the Laplace transform formula

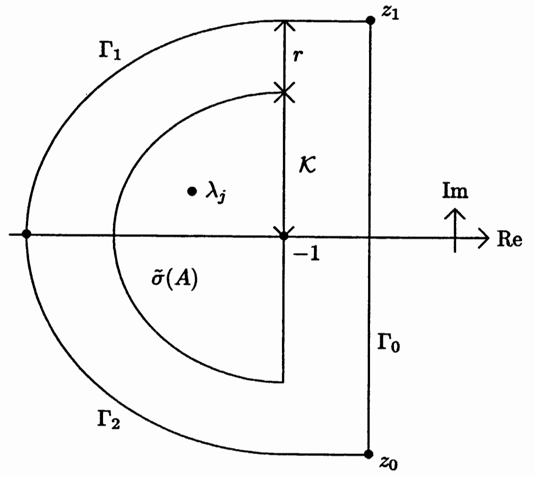

Fig. 2. A special path $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2$ for the Cauchy integral (14).

$(z-A)^{-1} = \int_0^\infty e^{-tz}e^{tA}dt$ for $\mathrm{Re}(\lambda_j - z) < 0$, $j = 1, \ldots, n$. Under the given assumption (2) this yields

$$\|(z-A)^{-1}\|_2 = \left\| \int_0^\infty e^{-tz}e^{tA}dt \right\|_2 \le \int_0^\infty e^{-(\mathrm{Re}\, z)t}\|e^{tA}\|_2 dt$$

$$\le \mathcal{K}_0 \int_0^\infty e^{-(\mathrm{Re}\, z + \frac{1}{\mathcal{K}_1})t} dt = \frac{\mathcal{K}_0}{\mathrm{Re}\, z + \frac{1}{\mathcal{K}_1}} \tag{15}$$

for $\frac{1}{\mathcal{K}_1} + \mathrm{Re}\, z > 0$. Now we define Γ_0 as

$$\Gamma_0 := \{z_0 + (z_1 - z_0)s : s \in [0,1]\}\,, \tag{16}$$

where $z_{0,1} := -\frac{1}{2\mathcal{K}_1} \mp (\mathcal{K}+r)i$, with

$$r := 1 - \frac{1}{2\mathcal{K}_1} \ge \frac{1}{2}\,. \tag{17}$$

We have

$$|z| \le \sqrt{(2\mathcal{K}_1)^{-2} + (\mathcal{K}+r)^2} =: \mathcal{B}_0 \quad \text{for } z \in \Gamma_0\,, \tag{18}$$

$$|\Gamma_0| = 2(\mathcal{K}+r) =: \mathcal{G}_0\,. \tag{19}$$

Since $\frac{1}{\mathcal{K}_1} + \mathrm{Re}\, z > 0$ for $z \in \Gamma_0$ we can apply (15) to obtain

$$\|R(z)\|_2 = \|(z-A)^{-1}\|_2 \le 2\mathcal{K}_0\mathcal{K}_1 \quad \text{for } z \in \Gamma_0\,. \tag{20}$$

The bounds (18)–(20) imply an estimate for the norm of (14) restricted to Γ_0:

$$\left\| \frac{1}{2\pi i}\int_{\Gamma_0} zR(z)dz \right\|_2 \le \frac{1}{\pi}\mathcal{G}_0\mathcal{B}_0\mathcal{K}_0\mathcal{K}_1 =: \mathcal{C}_0\,. \tag{21}$$

Now we define the path

$$\Gamma_1 := \{z_1 - sr : s \in [0,1]\} \cup \left\{ -1 + (\mathcal{K}+r)e^{i\varphi} : \varphi \in \left[\frac{\pi}{2}, \pi\right] \right\} \tag{22}$$

and the path Γ_2 symmetric to Γ_1 w.r.t. the real axis. For Γ_1 and Γ_2 we have

$$|z| \le 1 + \mathcal{K} + r =: \mathcal{B}_{1,2}\,, \quad z \in \Gamma_{1,2}\,, \tag{23}$$

$$|\Gamma_{1,2}| = (\mathcal{K}+r)\frac{\pi}{2} + r =: \mathcal{G}_{1,2}\,. \tag{24}$$

In order to bound the resolvent on Γ_1 we consider the resolvent equation

$$R(z) = (z-A)^{-1} = \Big(\underbrace{(z_1 - A)}_{=R(z_1)^{-1}} - (z_1 - z) \Big)^{-1}$$

$$= \Big(R(z_1)^{-1} \underbrace{(I - (z_1 - z)R(z_1))}_{\text{regular matrix!}} \Big)^{-1}$$

$$= (I - (z_1 - z)R(z_1))^{-1}R(z_1) \quad \text{for } z \in \Gamma_1\,. \tag{25}$$

The norm of the matrix $(I - (z_1 - z)R(z_1))^{-1}$ can be bounded by (cf. Ref. 6)

$$\|(I - (z_1 - z)R(z_1))^{-1}\|_2 \le \frac{\|I - (z_1 - z)R(z_1)\|_2^{n-1}}{|\det(I - (z_1 - z)R(z_1))|}\,. \tag{26}$$

We note that

$$|z_1 - z| \le \sqrt{(\mathcal{K}+2r)^2 + (\mathcal{K}+r)^2} =: \mathcal{R} \quad \text{for } z \in \Gamma_1 \tag{27}$$

and estimate with the help of (20) (note that z_1 is the point where Γ_0 and Γ_1 intersect):

$$\|I - (z_1 - z)R(z_1)\|_2 \le 1 + |z_1 - z|\|R(z_1)\|_2 \le 1 + 2\mathcal{R}\mathcal{K}_0\mathcal{K}_1 =: \mathcal{N} \tag{28}$$

for $z \in \Gamma_1$. To derive a bound for

$$|\det(I - (z_1 - z)R(z_1))|^{-1} = \prod_{j=1}^{n} \left| 1 - \frac{z_1 - z}{z_1 + c_j} \right|^{-1} = \prod_{j=1}^{n} \frac{|z_1 + c_j|}{|z + c_j|} \tag{29}$$

for $z \in \Gamma_1$, we use

$$|z + c_j| \ge r \ge \frac{1}{2} \quad \text{for } z \in \Gamma_1\,, \tag{30}$$

$$|z_1 + c_j| \le \sqrt{(\mathcal{K}+r)^2 + (2\mathcal{K}+r)^2} =: \mathcal{Z} \tag{31}$$

for $j = 1, \ldots, n$. This implies

$$\frac{|z_1 + c_j|}{|z + c_j|} \le 2\mathcal{Z} =: \mathcal{D} \qquad \text{for } z \in \Gamma_1, \qquad j = 1, \ldots, n\,. \tag{32}$$

The resolvent equation (25) together with the estimates (20), (26), (28) and (32) result in the desired resolvent bound

$$\|R(z)\|_2 \leq \|(I-(z_1-z)R(z_1))^{-1}\|_2 \|R(z_1)\|_2 \leq 2\mathcal{D}^n \mathcal{N}^{n-1} \mathcal{K}_0 \mathcal{K}_1 \tag{33}$$

for $z \in \Gamma_1$. By symmetry, the same bound is valid on Γ_2. As a consequence of (23), (24) and (33) we obtain

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_{1,2}} zR(z)dz \right\|_2 \leq \frac{1}{\pi} \mathcal{G}_{1,2} \mathcal{B}_{1,2} \mathcal{D}^n \mathcal{N}^{n-1} \mathcal{K}_0 \mathcal{K}_1 =: \mathcal{C}_{1,2}\,. \tag{34}$$

Combination of (21) and (34) now yields the desired estimate

$$\|A\|_2 \leq \sum_{j=0}^{2} \left\| \frac{1}{2\pi i} \int_{\Gamma_j} z(z-A)^{-1} dz \right\|_2 \leq \mathcal{C}_0 + 2\mathcal{C}_{1,2}\,. \tag{35}$$

Finally, the estimates (10) and (13) together with (35) give us the desired result.

Proposition 3.1. *If the matrix A from class* (3) *satisfies* (2), *then*

$$\|A\|_2 \leq 2(\mathcal{C}_0 + 2\mathcal{C}_{1,2})\,;$$

the Lyapunov transform V (defined by (4)) *and its inverse can be estimated by*

$$\|V\|_2 \leq \frac{\mathcal{K}_0^2 \mathcal{K}_1}{2}\,, \qquad \|V^{-1}\|_2 \leq 2(\mathcal{C}_0 + 2\mathcal{C}_{1,2})\,.$$

The constants $\mathcal{C}_i$ in these bounds can be explicitly expressed in terms of $\mathcal{K}$ and $\mathcal{K}_0$, $\mathcal{K}_1$ as in the above proof. In addition, $\mathcal{C}_{1,2}$ depends exponentially on the matrix dimension n (cf. (34)).

Remark 3.1. The sharpness of our results (in particular concerning the exponential dependence on n in Proposition 3.1) may be worth investigating.

For the case of a general matrix A (without an *a priori* assumption concerning the distribution of eigenvalues), the question on whether and in what way the behavior of e^{tA} can be characterized in the same spirit is also of interest.

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