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A Note on Convergence Concepts for Stiff Problems

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Dedicated to Professor Hans J. Stetter on the occasion of his 60th birthday.

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Abstract — Zusammenfassung

A Note on Convergence Concepts for Stiff Problems. Most convergence concepts for discretizations of nonlinear stiff initial value problems are based on one-sided Lipschitz continuity. Therefore only those stiff problems that admit moderately sized one-sided Lipschitz constants are covered in a satisfactory way by the respective theory. In the present note we show that the assumption of moderately sized one-sided Lipschitz constants is violated for many stiff problems. We recall some convergence results that are not based on one-sided Lipschitz constants; the concept of singular perturbations is one of the key issues. Numerical experience with stiff problems that are not covered by available convergence results is reported.

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Über Konvergenzkonzepte für steife Probleme. Die meisten Konvergenzkonzepte für Diskretisierungen nichtlinearer steifer Anfangswertprobleme basieren auf dem Begriff der einseitigen Lipschitz-Stetigkeit. Folglich sind durch diese theoretischen Konzepte nur steife Probleme mit moderater einseitiger Lipschitzkonstante abgedeckt. In der vorliegenden Arbeit zeigen wir, daß die Annahme moderater einseitiger Lipschitzkonstanten für viele steife Probleme verletzt ist. Wir weisen auf einige Konvergenzresultate hin, die nicht auf einseitigen Lipschitzkonstanten basieren; die Konzepte der singulären Störungstheorie sind hier von wesentlicher Relevanz. Wir berichten über einige numerische Erfahrungen mit steifen Problemen, die durch keine existierende Konvergenztheorie abgedeckt sind.

1. Convergence Based on One-Sided Lipschitz Continuity: A Critical Discussion

During the past fifteen years a considerable progress has taken place in the convergence theory for discretizations of nonlinear stiff initial value problems. The developments were initiated by G. Dahlquist in his lecture at the 1975 Dundee Conference [6], where he introduced the concept of one-sided Lipschitz continuity into the analysis of numerical methods for stiff systems and where he defined the concept of G-stability for multistep methods. Also in 1975, J. C. Butcher developed the concept of B-stability for Runge-Kutta methods [5], which is based on one-sided Lipschitz continuity, too. As an important consequence, it has been possible to derive more realistic global error bounds for nonlinear stiff equations which remain unaffected by conventional Lipschitz constants—which are inevitably large in the stiff case. (Cf. for instance the B-convergence papers [9, 10, 11].)

If, for instance, the given stiff initial value problem

$$\begin{aligned} &y' = f(t,y), \qquad t \in [0,T], \qquad f\colon \mathbb{G} \to \mathbb{R}^n, \qquad \mathbb{G} \subset \{[0,T] \times \mathbb{R}^n\} \\ &y(0) = y_0 \end{aligned} \tag{1.1}$$

is solved by the implicit Euler scheme,

$$\begin{aligned} &\frac{\eta_\nu - \eta_{\nu-1}}{h} = f(t_\nu, \eta_\nu), \qquad t_\nu = \nu h, \qquad \nu = 1, 2, \ldots, T/h, \\ &\qquad \eta_0 = y_0 + s_0, \end{aligned} \tag{1.2}$$

then the well-known B-convergence estimate for the global discretization error reads

$$\|\eta_\nu - y(t_\nu)\|_2 \le \begin{cases} \|s_0\|_2 + \frac{h}{2} M_2 \cdot t_\nu, & m = 0, \\ \|s_0\|_2 \cdot \left(\frac{1}{1-hm}\right)^\nu + \frac{h}{2} M_2 \cdot \left(\frac{\left(\frac{1}{1-hm}\right)^\nu - 1}{m}\right), & m \ne 0; \end{cases} \tag{1.3}$$

here M_2 denotes a bound for $\|y''(t)\|_2$ on $[0,T]$ (where $y(t)$ is the exact solution of (1.1)), and m denotes a one-sided Lipschitz constant for f, i.e. it is assumed that f satisfies a one-sided Lipschitz condition

$$\langle f(t,y_1) - f(t,y_2), y_1 - y_2 \rangle \le m \|y_1 - y_2\|_2^2, \qquad (t,y_1),(t,y_2) \in \mathbb{G}, \tag{1.4}$$

where $\langle \cdot,\cdot \rangle$ denotes the scalar product corresponding to the Euclidean norm $\|\cdot\|_2$. The starting perturbation s_0 in (1.2) has been introduced in order to cover the effects of nonequidistant grids.[1]

Due to (1.3) the order of B-convergence of the implicit Euler scheme is $p = 1$. Within the last years many similar results have been derived for higher order schemes; note, however, that the order of B-convergence is often below the conventional order, a phenomenon called *order reduction.*

The assumption that the stiff problems under consideration admit one-sided Lipschitz constants m which are not strongly positive is indispensable within the B-theory. Any B-convergence bound blows up for $m \to +\infty$. This would be quite natural if the optimal (i.e. the smallest possible) one-sided Lipschitz constant would characterize the error sensitivity of the given problem: If, for any stiff problem with $m \gg 0$, dramatic effects w.r.t. starting perturbations or perturbations of the right hand side would indeed occur, this would mean that B-convergence bounds blowing

[1] The interval $[0,T]$ in (1.1) can be interpreted as a subinterval with constant stepsize h of the whole integration interval, and s_0 can be identified with the global discretization error at $t = 0$—accumulated from the preceding subintervals with constant stepsizes. The inductive application of (1.3) eventually leads to the desired error bound for nonequidistant grids. Thus, the B-convergence estimate (1.3) shows that the implicit Euler scheme performs satisfactorily for those stiff problems that admit not strongly positive one-sided Lipschitz constants m, as long as grids are used which are *locally adjusted to the smoothness of the true solution* $y(t)$.

up for $m \to +\infty$ would be natural, because moderately sized global errors cannot be expected for any discretization method applied to such a problem. So—attempting to appraise whether B-convergence bounds are optimal and natural—the crucial question is whether m indeed *characterizes* the error sensitivity of a given initial value problem. For a discussion of this point, let us have a short look at available condition estimates. In [14], for instance, the well-known estimate

$$\|\tilde{y}(t) - y(t)\|_2 \leq \|\tilde{y}_0 - y_0\|_2 e^{Lt} + \delta \frac{e^{Lt} - 1}{L} \tag{1.5}$$

can be found, where $\tilde{y}(t)$ denotes the solution of an initial value problem with a perturbed starting value $\tilde{y}_0$ and with a perturbed right hand side $\tilde{f}(t, y)$ satisfying

$$\|\tilde{f}(t, y) - f(t, y)\|_2 \leq \delta. \tag{1.6}$$

L is a (conventional) Lipschitz constant for the right hand side $f(t, y)$ of (1.1). For stiff problems, where always $L \gg 0$, (1.5) is of course highly unrealistic; but there exists a similar but sharper condition estimate (cf. e.g. [8]) based on the one-sided Lipschitz constant m of $f(t, y)$:

$$\|\tilde{y}(t) - y(t)\|_2 \leq \|\tilde{y}_0 - y_0\|_2 e^{mt} + \delta \frac{e^{mt} - 1}{m} \tag{1.7}$$

with δ from (1.6).

Note that the B-convergence bound (1.3) for the implicit Euler scheme is an immediate discrete analogon to (1.7) (the same is true for any B-convergence bound).

Moreover, it can be shown that (1.7) is sharp in the following sense (cf. [7], [8]): There always exists a perturbed starting value $\tilde{y}_0$ such that in a right neighborhood of $t = 0$ the solution $\tilde{y}(t)$ of the perturbed problem $y' = f(t, y)$, $y(0) = \tilde{y}_0$ satisfies, in first approximation,

$$\|\tilde{y}(t) - y(t)\|_2 \simeq \|\tilde{y}_0 - y_0\|_2 e^{mt}, \tag{1.8}$$

m denoting the locally optimal[2] one-sided Lipschitz constant of f w.r.t. spectral norm $\|\cdot\|_2$. Thus, at least in a local sense the one-sided Lipschitz constant w.r.t. the L_2-norm *characterizes* the error sensitivity of an initial value problem (1.1), and from this point of view B-convergence estimates seem to be natural and almost ideal.

On the other hand, there are many stiff real-life problems with $m \gg 0$ which are well-conditioned and show an uncritical error sensitivity. At first sight this seems to contradict (1.8); but the point is that for many stiff problems there is a strong discrepancy between the 'local' and the 'global' condition in the following sense: An extremely rapid growth of perturbation effects occurs only on very short intervals (Fig. 1). A very simple example illustrating such a situation can be found in [8] and is discussed more extensively in [7], Example 1.4.5.

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[2] 'Locally optimal' means: the smallest possible one-sided Lipschitz constant in an infinitesimal right neighborhood of $(0, y_0)$.

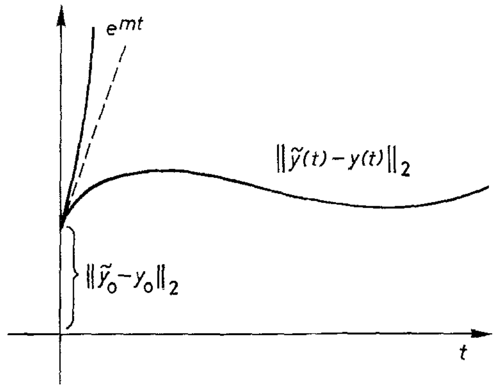

Figure 1. Local vs. global condition

Now the question arises whether such a situation—a globally well-conditioned stiff problem with $m \gg 0$—must be expected to occur frequently in practice or whether this is rather the exception, most stiff problems showing moderate one-sided Lipschitz constants (covered by the B-theory). The main purpose of the present note is to provide some illustration to this question. Let us first consider the very simple linear constant coefficient case $f(t, y) = Ay$. Here the optimal one-sided Lipschitz constant can be expressed by the *logarithmic norm* of A (cf. for instance [7]), defined by

$$\mu(A) := \lim_{\tau \to 0} \frac{\|I + \tau A\| - 1}{\tau}; \tag{1.9}$$

for the spectral norm $\|\cdot\|_2$ this is given by

$$\mu(A) \equiv \mu_2(A) = \lambda_{\max}\left[\frac{A + A^T}{2}\right]. \tag{1.10}$$

Assume $A = A(\varepsilon)$ is a stiff 2×2-matrix of the form

$$A = S\Lambda S^{-1}, \qquad \Lambda = \begin{pmatrix} c_1 & 0 \\ 0 & -\dfrac{c_2}{\varepsilon} \end{pmatrix} \tag{1.11}$$

with moderate, generally ε-dependent real data $S = S(\varepsilon)$,[3] $S^{-1} = S^{-1}(\varepsilon)$, $c_1 = c_1(\varepsilon)$, and $c_2 = c_2(\varepsilon) \geq \kappa > 0$. ε is a small positive parameter characterizing the order of magnitude of the stiff eigenvalue $-c_2/\varepsilon$. Analyzing this case we find the following, rather surprising result (cf. section 2, Lemma 2.1): *The logarithmic norm $\mu_2(A)$ is of moderate size, i.e., not affected by a factor $+1/\varepsilon$, if and only if A is 'ε-symmetric', that means, symmetric at the $O(\varepsilon^{-1})$-level*[4] (see section 2, Definition 2.1 for the precise formulation).

[3] Let S be scaled such that the L_2-norm of its columns (= eigenvectors of A) is 1 and the angle σ between its columns satisfies $\sin\sigma > 0$ ($\sigma \in (0, \pi)$).

[4] Note that, in general, all entries a_{ij} in A are of magnitude $O(\varepsilon^{-1})$.

For constant coefficient problems $y' = Ay = S\Lambda S^{-1}y$ for which S and S^{-1} are moderately sized it is very simple to circumvent this problem and to establish satisfactory B-convergence bounds by changing the norm: Instead of using the spectral norm, as we did in (1.4), we may work with the 'elliptic' vector norm

$$\|y\|_s := \langle S^{-1}y, S^{-1}y \rangle^{1/2} \tag{1.12}$$

such that the corresponding logarithmic norm $\mu_s(A)$ specializes to the spectral abscissa of A (cf. for instance Theorem 1.4.7 of [7]). In other words: We simply transform the problem to diagonal form $\bar{y}' = \Lambda\bar{y}$ ($\bar{y} = S^{-1}y$).

If, however, we have in mind the more general case

$$\begin{aligned} y' &= A(t)y, \qquad t \in [0, T] \\ y(0) &= y_0, \end{aligned} \tag{1.13}$$

$A(t) = S(t)\Lambda(t)S^{-1}(t)$, the above procedure does not help: If one defines an elliptic vector norm in $\mathbb{R}^2$ such that, analogously as above for constant A, the corresponding logarithmic norm of $A(t_0)$ at some point $t_0 \in [0, T]$ reduces to the spectral abscissa of $A(t_0)$, then this choice is not appropriate for other points $t \neq t_0$: It is easily seen that the corresponding logarithmic norm of $A(t)$ usually blows up to the $O(+1/\varepsilon)$-level for points $t \neq t_0$. (Cf. section 2 for a discussion of this point.)

One might think that, to circumvent this restriction, it is sufficient to transform (1.13) according to $\bar{y}(t) := S^{-1}(t)y(t)$, such that (1.13) takes the form

$$\bar{y}' = (\Lambda(t) - S^{-1}(t)S'(t))\bar{y}, \tag{1.14}$$

where $\Lambda(t) - S^{-1}(t)S'(t)$ is ε-symmetric under appropriate smoothness assumptions w.r.t. $S(t)$. The point is, however, that the discretization errors of a given numerical scheme applied to the original problem (1.13) are *not* directly related to the discretization errors of the same scheme applied to the transformed problem (1.14).

Thus, problems of the type (1.13) with a time-dependent, not ε-symmetric stiff matrix cannot be satisfactorily covered by the B-theory.[5] The same limitations hold, of course, for nonlinear stiff problems where the Jacobian is not ε-symmetric.

2. A Property of the Logarithmic Matrix Norm

In the following Lemma we state a simple property of logarithmic matrix norms already mentioned in section 1. Although very elementary, this property is of considerable interest for the theoretical analysis of stiff equations: It shows that all convergence concepts based on one-sided Lipschitz constants are of a restrictive applicability (cf. the discussion of section 1). Surprisingly enough, this simple property seems not to have been mentioned so far in the literature on stiff problems.

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[5] Numerical experience indicates that the same limitations as in the two-dimensional case hold for higher-dimensional stiff systems. We do, however, restrict our analysis to the 2×2-case (cf. section 2); the generalization to higher dimensions would be highly nontrivial.

For a symmetric matrix A of the form (1.11), it is trivial that $\mu_2(A)$ is of moderate size independent of the stiffness (i.e., uniformly for $\varepsilon \to 0$) since in this case, $\mu_2(A) =$ *spectral abscissa of* A. Lemma 2.1 below shows that symmetry in a slightly weaker sense (ε-symmetry) is even necessary. We define

Definition 2.1. A 2×2-matrix A of the form (1.11) is called ε-symmetric if it satisfies

$$A = \varepsilon^{-1} A_0 + O(\varepsilon^{-1/2}) \tag{2.1}$$

with a symmetric and moderately sized matrix A_0 and with a moderate O-constant.

Definition 2.2. A 2×2-matrix S is called ε-orthogonal if

$$S^T S = \begin{pmatrix} 1 & O(\varepsilon^{1/2}) \\ O(\varepsilon^{1/2}) & 1 \end{pmatrix} \tag{2.2}$$

with a moderate O-constant.

For an ε-orthogonal matrix the angle $\sigma \in (0, \pi)$ between its columns satisfies $\sigma = \pi/2 + O(\varepsilon^{1/2})$ and $\sin \sigma = 1 - O(\varepsilon)$.

Lemma 2.1. *The logarithmic norm* $\mu_2(A)$ *of a* 2×2*-matrix* A *of the form* (1.11) *is of moderate size iff* A *is* ε*-symmetric.*

Proof. The logarithmic norm $\mu_2(A)$ is given by (1.10). After appropriate scaling of S, the matrix $W := S^T S$ has diagonal elements equal 1 and non-diagonal elements $\cos \sigma$ where $\sigma \in (0, \pi)$ is the angle between the columns of S. The characteristic equation for the eigenvalues λ of $(A + A^T)/2$ is equivalent to

$$\det\left[\frac{1}{2}(W\Lambda + \Lambda W) - \lambda W\right] = 0 \tag{2.3}$$

or

$$\lambda^2 - \lambda\left(c_1 - \frac{c_2}{\varepsilon}\right) + \frac{1}{4}\left[\left(c_1 - \frac{c_2}{\varepsilon}\right)^2 - \left(c_1 + \frac{c_2}{\varepsilon}\right)^2 \frac{1}{\sin^2 \sigma}\right] = 0 \tag{2.4}$$

where c_1 and $-c_2/\varepsilon$ are the eigenvalues of A (cf. (1.11)). Solving this equation we obtain for the maximal eigenvalue of $(A + A^T)/2$:

$$\mu_2(A) = \lambda_{\max}\left[\frac{A + A^T}{2}\right] = \frac{1}{2}\left(c_1 - \frac{c_2}{\varepsilon}\right) + \frac{1}{2}\left(c_1 + \frac{c_2}{\varepsilon}\right)\frac{1}{\sin \sigma}, \tag{2.5}$$

where $\sin \sigma \in (0, 1]$. Thus, $\mu_2(A)$ is of moderate size iff S is ε-orthogonal in the sense of Definition 2.2. It remains to be shown that this is equivalent to A being ε-symmetric in the sense of Definition 2.1. To this end we express the s_{ij} as functions of the angle $\sigma \in (0, \pi)$ between the columns of S, using the relation

$$s_{11} s_{12} + s_{21} s_{22} = \cos \sigma; \tag{2.6}$$

without loss of generality we may write $s_{12} = \cos \alpha$, $s_{22} = \sin \alpha$ and $s_{11} = \cos(\alpha - \sigma)$, $s_{21} = \sin(\alpha - \sigma)$, $\alpha \in [0, \pi]$ arbitrary. Now a short calculation shows that the off-diagonal elements of a matrix $A = (a_{ij}) = S \Lambda S^{-1}$ of the form (1.11) satisfy

$$a_{21} = a_{12} + \left(c_1 + \frac{c_2}{\varepsilon}\right)\cot\sigma \tag{2.7}$$

(independent of α). This proves Lemma 2.1 since S is ε-orthogonal iff $\cot\sigma = O(\varepsilon^{1/2})$. □

As already mentioned in section 1, the above restrictive property is of no real relevance in the case of a constant matrix $A = A\Lambda S^{-1}$, since the underlying problem can be transformed to diagonal form $\bar{y}' = \Lambda\bar{y}$ via $\bar{y} = S^{-1}y$ or, in other words, the choice of the elliptic vector norm (1.12) leads to a moderately sized logarithmic norm $\mu_s(A)$.

If, however, the matrix $A(t) = S(t)\Lambda(t)S^{-1}(t)$ is time dependent for $t \in [0, T]$, one may think of using an appropriately transformed scalar product

$$\langle y, y\rangle_x = \langle X^{-1}y, X^{-1}y\rangle^{1/2}, \tag{2.8}$$

and it is tempting to choose $X := S(0)$. The corresponding logarithmic norm $\mu_x(A(t))$ can be expressed as

$$\mu_x(A(t)) = \mu_2(X^{-1}A(t)X) = \mu_2(X^{-1}S(t)\Lambda(t)(X^{-1}S(t))^{-1}). \tag{2.9}$$

Thus, according to the proof of Lemma 2.1, $\max_{t\in[0,T]}\mu_x(A(t))$ is of moderate size if $X^{-1}S(t)$ is ε-orthogonal for all $t \in [0, T]$. Such a relation cannot, of course, be expected in general (unless $S(t)$ itself is ε-orthogonal for all $t \in [0, T]$).

3. Further Remarks and Examples

The discussion of sections 1 and 2 shows that all convergence concepts for discretizations of stiff problems that are based on one-sided Lipschitz continuity are often useless because many stiff problems only admit strongly positive one-sided Lipschitz constants *despite their (globally) well-conditioned behavior.*

In recent years, however, some convergence results have been derived that do not suffer from this limitation (cf. for instance [1], [2], [3], [12], [13], [15], [16]). The concept of *singular perturbations* is one of the key issues here. Some of these results apply to the singularly perturbed model equation[6]

$$\begin{aligned} y_1' &= \varphi_1(t, y_1, y_2), \\ y_2' &= \frac{1}{\varepsilon}\varphi_2(t, y_1, y_2), \quad 0 < \varepsilon \ll 1, \qquad \frac{\partial}{\partial y_2}\varphi_2(t, y_1, y_2) \le -\kappa < 0; \end{aligned} \tag{3.1}$$

further results refer to the case where the underlying stiff problem takes the form (3.1) after an appropriate smooth transformation.

Whereas more general in the sense that large one-sided Lipschitz constants are admitted, all these results are, on the other hand, more restrictive in another sense:

[6] Note that, in general, a problem of the form (3.1) is indeed not covered by the B-theory since its Jacobian is not ε-symmetric (cf. Lemma 2.1).

It is assumed that the stiffness is characterized by only one large parameter $1/\varepsilon$; furthermore, $1/\varepsilon$ does only appear in a linear way, i.e., as factor on the right hand side.

As examples for more general situations, where $1/\varepsilon$ appears in nonlinear way, let us now consider the following stiff test problems:

Example 1:

$$y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = f(t, y) = \begin{pmatrix} \varphi_1(t, y_1, y_2) \\ \varphi_2(t, y_1, y_2) - \left(\frac{1}{\varepsilon}\right)^t y_2 \end{pmatrix} \tag{3.2}$$

Example 2:

$$y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = f(t, y) = \begin{pmatrix} \varphi_1(t, y_1, y_2) \\ \varphi_2(t, y_1, y_2) - \left(\frac{1}{\varepsilon}\right)^{y_1^2} y_2 \end{pmatrix} \tag{3.3}$$

with smooth functions φ_1 and φ_2. Although not covered by the singular perturbation theory, equations (3.2) and (3.3) have moderate one-sided Lipschitz constants (for Example 1 this can be easily seen by a look at its Jacobian; concerning Example 2 cf. Table 3.1 below). Thus, Examples 1 and 2 are covered by the B-theory.

These problems can be made 'more difficult' by applying a time-dependent transformation $S(t)$. In particular, we consider

Example 3:

$$y' = S(t) f(t, S^{-1}(t) y), \qquad f \text{ from Example 2}, \tag{3.4}$$

such that the resulting Jacobian $S(t) f_y(t, S^{-1}(t) y(t)) S^{-1}(t)$ is strongly unsymmetric and, consequently, the one-sided Lipschitz constant is positive and very large (cf. section 2 and Table 3.1 below). Example 3 is neither covered by the singular perturbation theory nor by the B-convergence theory.

We will now report some numerical experience with Examples 1–3. For these experiments we have chosen (multistep) BDF methods (which are frequently used in practice) and h^2-extrapolation based on the (one-step) implicit trapezoidal rule without smoothing (which is hardly used in applications but is also very efficient in many cases). Note that the convergence theory for extrapolation presented in [4] does not apply here because it strongly relies on singular perturbation arguments. For the higher order BDF schemes, a satisfactory nonlinear convergence theory does not exist either. The main purpose of our experiments was to gain some insight to the convergence properties of these methods in such non-standard situations, and to check what level of accuracy is achieved depending on the stepsizes used. (It was not our intention to compare production codes.)

The following experiments were performed in IEEE double precision arithmetic. The integration interval is $[1, 2]$, the initial values $y(1)$ were chosen to define smooth solutions of the resp. equations. The (multistep) BDF schemes were started using exact solution values. The functions φ_1 and φ_2 were chosen as

$$\varphi_1(t, y_1, y_2) := e^{-t}\sin(y_1 + y_2), \quad \varphi_2(t, y_1, y_2) := e^{t}\cos(y_1 - y_2) \quad \text{(Examples 1–3)}. \tag{3.5}$$

The other problem data were specified as follows:

Example 1:

$$\varepsilon = e^{-10} \approx 4.5 \cdot 10^{-5}, \qquad y(1) = \begin{pmatrix} 1.60472828203259\ldots \\ -0.00000418689967\ldots \end{pmatrix} \tag{3.6}$$

Example 2.

$$\varepsilon = e^{-3} \approx 5 \cdot 10^{-2}, \qquad y(1) = \begin{pmatrix} 1.59996626858355\ldots \\ -0.00003651196912\ldots \end{pmatrix} \tag{3.7}$$

Example 3:

$$\varepsilon = e^{-3} \approx 5 \cdot 10^{-2}, \qquad y(1) = \begin{pmatrix} 2.41941110741754\ldots \\ -1.84212599689155\ldots \end{pmatrix}, \qquad S(t) = \begin{pmatrix} e^{0.1t} & -\sin t \\ -\sin t & e^{0.1t} \end{pmatrix} \tag{3.8}$$

Table 3.1 illustrates the degree of stiffness of these examples, expressed by the eigenvalues λ_1, λ_2 of the Jacobian $J(t, y(t))$ at the solution point $(2, y(2))$, and displays the logarithmic norm $\mu_2(J)$ at this point.

Table 3.1	Example 1	Example 2	Example 3
λ_1	−3.52E−02	−3.46E−02	−6.06E−02
λ_2	−4.85E+08	−2.30E+04	−2.50E+05
$\mu_2(J)$	−3.52E−02	−3.30E−02	+3.10E+05

Tables 3.2 and 3.3 display the results for Example 1.

Table 3.2: Example 1, BDF (order p): L2-norm of global error at t = 2

h	p = 1	2	3	4	5	6
1/8	1.417E−02	9.793E−04	6.753E−05	1.049E−06	1.371E−06	4.992E−07
1/16	7.166E−03	2.739E−04	1.004E−05	1.109E−08	8.146E−08	1.701E−08
1/32	3.604E−03	7.203E−05	1.341E−06	3.263E−09	3.437E−09	3.632E−10
1/64	1.807E−03	1.845E−05	1.726E−07	3.714E−10	1.238E−10	6.491E−12
1/128	9.047E−04	4.666E−06	2.187E−08	2.927E−11	4.201E−12	8.593E−14
observed order						
	0.98	1.84	2.75	6.56	4.07	4.88
	0.99	1.93	2.90	1.76	4.57	5.55
	1.00	1.97	2.96	3.14	4.79	5.81
	1.00	1.98	2.98	3.67	4.88	6.24

Table 3.3: Example 1, extrapolated ITR: L2-norm of global error at t = 2

h	ITR	1st EX	2nd EX	3rd EX	4th EX
1/8	3.021E−04				
1/16	7.552E−05	4.746E−09			
1/32	1.888E−05	3.002E−10	3.740E−12		
1/64	4.720E−06	1.881E−11	5.840E−14	2.743E−17	
1/128	1.180E−06	1.178E−12	2.221E−15	1.332E−15	1.332E−15
observed order					
	2.00				
	2.00	3.98			
	2.00	4.00	6.00		
	2.00	4.00	4.72	−5.60	

Tables 3.4 and 3.5 display the results for Example 2.

Table 3.4: Example 2, BDF (order p): L2-norm of global error at t = 2

h	p = 1	2	3	4	5	6
1/8	1.417E−02	9.771E−04	6.698E−05	9.499E−07	1.381E−06	4.963E−07
1/16	7.167E−03	2.733E−04	9.950E−06	3.122E−09	8.166E−08	1.680E−08
1/32	3.604E−03	7.186E−05	1.330E−06	3.789E−09	3.432E−09	3.573E−10
1/64	1.807E−03	1.840E−05	1.711E−07	4.045E−10	1.235E−10	6.366E−12
1/128	9.048E−04	4.655E−06	2.168E−08	3.133E−11	4.190E−12	1.057E−13
observed order						
	0.98	1.84	2.75	8.25	4.08	4.88
	0.99	1.93	2.90	−0.28	4.57	5.56
	1.00	1.97	2.96	3.23	4.80	5.81
	1.00	1.98	2.98	3.69	4.88	5.91

Table 3.5: Example 2, extrapolated ITR: L2-norm of global error at t = 2

h	ITR	1st EX	2nd EX	3rd EX	4th EX
1/8	3.014E−04				
1/16	7.535E−05	4.696E−09			
1/32	1.884E−05	2.977E−10	2.270E−11		
1/64	4.709E−06	2.028E−11	7.219E−12	6.978E−12	
1/128	1.177E−06	1.221E−12	1.398E−13	2.566E−13	2.849E−13
observed order					
	2.00				
	2.00	3.98			
	2.00	3.88	1.65		
	2.00	4.05	5.69	4.77	

Tables 3.6 and 3.7 display the results for Example 3.

Table 3.6: Example 3, BDF (order p): L2-norm of global error at t = 2

h	p = 1	2	3	4	5	6
1/8	7.060E−02	1.072E−02	4.521E−03	2.521E−03	4.938E−03	2.848E−03
1/16	3.531E−02	2.390E−03	6.392E−06	4.903E−05	4.376E−05	9.905E−06
1/32	1.760E−02	5.388E−04	3.006E−05	5.274E−07	2.670E−06	2.354E−07
1/64	8.783E−03	1.258E−04	5.280E−06	1.152E−07	8.998E−08	3.155E−09
1/128	4.386E−03	3.024E−05	7.470E−07	9.395E−09	2.831E−09	4.297E−11
observed order						
	1.00	2.17	9.47	5.68	6.82	8.17
	1.00	2.15	−2.23	6.54	4.03	5.39
	1.00	2.10	2.51	2.19	4.89	6.22
	1.00	2.06	2.82	3.62	4.99	6.20

Table 3.7: Example 3, extrapolated ITR: L2-norm of global error at t = 2

h	ITR	1st EX	2nd EX	3rd EX	4th EX
1/8	1.956E−03				
1/16	4.702E−04	2.519E−05			
1/32	1.163E−04	1.615E−06	4.293E−08		
1/64	2.901E−05	9.906E−08	1.972E−09	2.684E−09	
1/128	7.248E−06	6.527E−09	3.576E−10	3.946E−10	4.067E−10
observed order					
	2.06				
	2.01	3.96			
	2.00	4.03	4.44		
	2.00	3.92	2.46	2.77	

As can be seen from the above tables, both methods considered perform satisfactorily for these examples. In the case of the extrapolated ITR, the order observed at the higher extrapolation steps is below the 'conventional' order, a phenomenon well-known from [4]. However, the extrapolated ITR yields a similar accuracy as the BDF schemes.

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