B-Convergence

The theory of B-convergence predicts the convergence properties of discretization methods, *implicit Runge-Kutta* (IRK) methods in particular, applied to initial value problems (IVPs) for a nonlinear ODE systems $y'(t) = f(t, y(t))$.

Problem class and historical background. Concerning the class of ODEs considered, the underlying assumption is that f satisfies a *one-sided Lipschitz condition*

$$\langle y - z, f(t,y) - f(t,z) \rangle \le m \, \|y - z\|^2.$$

If m (the so-called one-sided Lipschitz constant of f) is of moderate size, the IVP is easily proved to be well-conditioned throughout. Of particular interest are *stiff* problems. Stiffness means that that the underlying ODE admits smooth solutions, with moderate derivatives, together with nonsmooth ('transient') solutions rapidly converging towards smooth ones. In the stiff case, the conventional Lipschitz constant L of f inevitably becomes large; therefore the classical error bounds for discretization methods - which depend on L - are of no use for an appropriate characterization and analysis of methods which are able to efficiently integrate a stiff problem. Thus there is a need for a special convergence theory applicable in the presence of stiffness.

The idea to use m (instead of L) as the problem-characterizing parameter goes back to [7] where it was used in the analysis of multistep methods. The point is that stiffness is often compatible with moderate values of m, while $L \gg 0$. In the same spirit, the concept of B-stability was introduced in [4], [5], [6] in the context of IRK methods, and an algebraic criterion on the IRK coefficients entailing B-stability was derived ('algebraic stability'). The notion of *B*-stability enables realistic estimates of the propagation of inevitable perturbations like local discretization errors.

The concept of B-convergence. In the convergence theory of IRK methods applied to stiff problems, stability is essential but also the analysis of local errors is nontrivial: straightforward estimates are affected by $L \gg 0$ and do not reflect reality. For a simple scalar model class, the local error of IRK schemes was studied in [13]. It turned out that the order observed for practically relevant stepsizes is usually reduced compared to smooth, non-stiff situations.

In [9], [10], [11], the convergence properties of IRK schemes are studied and the notion of B-convergence is introduced. A B-convergence result is nothing but a realistic global error estimate based on the parameter m but unaffected by L. Besides relying on B-stability, the essential point are sharp local error estimates which require a special internal stability property called BS-stability. The latter can be concluded from a certain algebraic condition on the IRK coefficients ('diagonal stability'). Explicit error bounds are derived for Gauss, Radau IA and Radau IIA schemes; the corresponding 'B-convergence order' is in accordance with the observations from [13]. B-convergence results for Lobatto IIIC schemes are given in [14].

An overview on the 'B-theory' of IRK methods is presented in [8]. Another relevant text is [12].

Further developments. Concerning the relevance of the B-theory for stiff problems, there remains a gap. The point is that for most stiff problems there is a strong discrepancy between the local and the global condition: Neighboring solutions may locally strongly diverge, such that the problem is locally ill-conditioned. This is a transient effect, leaving the good global condition unaffected. But it inevitably implies that the one-sided Lipschitz constant m is strongly positive (like L). For details cf. [1], where it is shown that m remains moderate only for a restricted class of stiff problems, namely with Jacobians f_y which are 'almost normal'. In general, however, m is large and positive, and the B-convergence bounds based on m become unrealistically large.

As a consequence, not even linear stiff problems $y' = A(t)y$ are satisfactorily covered. In [2] the B-theory is extended to semilinear stiff problems of the form $y' = A(t)y + \varphi(t, y)$ where $A(t)$ has a smoothly varying eigensystem and $\varphi(t, y)$ is smooth. However, this does not cover sufficiently large class of nonlinear problems. Current work concentrates on a more natural, geometric characterization of stiffness; cf. e.g. [3].

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AMS 1980 Subject Classification: 65L05