D E F E C T C O R R E C T I O N S

A N D M U L T I G R I D I T E R A T I O N S

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from: "Multigrid Methods", Proceedings Köln-Porz, Nov.1981
Lecture Notes in Mathematics, Springer, Berlin 1982

DEFECT CORRECTIONS AND MULTIGRID ITERATIONS

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Introduction

In several of his publications since 1978, Brandt describes a modification of the general multigrid algorithm which be calls "truncation extrapolation" or "τ-extrapolation" (see, e.g., [1]). In their analysis of this idea, the authors of this paper reinterpreted Brandt's τ-extrapolation as a defect correction step ([2]). Independently, W. Hackbusch had suggested a combination of the multigrid and defect correction approaches in [3], see also [4].

In the following, we will at first demonstrate how any defect correction iteration may be combined with an iterative procedure for its effective solution operator and derive a bound on the convergence factor for the combined iterative process. We will then consider the situation when the iterative solution procedure consists of multigrid cycles. Various implementation versions of a defect correction multigrid cycle will be discussed, particularly for the case where defect correction is to achieve a higher order approximation to the differential equation. We will also regard some algorithmic and quantitative details in the context of model problems. Finally we will present some numerical experiences.

Our approach permits the use of standard multigrid software in situations more general than those for which the software was designed. This will extend the applicability of existing and coming multigrid packages and thus reduce programming costs for many applications.

1. Defect Correction with an Iterative Solution Procedure

The principle of defect correction may be summarized thus (cf., e.g., [5]): We wish to solve the problem

$$F\,u = c \tag{1.1}$$

but we only have an efficient solution procedure for the related problem

$$\tilde{F}\, v = \tilde{c}\ ; \tag{1.2}$$

however, we are able to evaluate the direct mapping F . Then the following two iterative processes are natural (see [5]):

$$\begin{aligned} &\text{(A)} \quad u^{(i+1)} := u^{(i)} - [\tilde{F}^{-1} F\, u^{(i)} - \tilde{F}^{-1} c] \\ &\text{or} \\ &\text{(B)} \quad \tilde{F}\, u^{(i+1)} := \tilde{F}\, u^{(i)} - [F\, u^{(i)} - c]\ . \end{aligned} \tag{1.3}$$

If $\tilde{F}$ is a linear mapping (F may still be nonlinear), then (A) and (B) collapse into the familiar

$$u^{(i+1)} := u^{(i)} - \tilde{F}^{-1}\, [F\, u^{(i)} - c]\ . \tag{1.4}$$

Convergence of these processes depends on the contractivity of the mappings

$$\text{(A)} \quad I - \tilde{F}^{-1} F \quad \text{or} \quad \text{(B)} \quad I - F\, \tilde{F}^{-1}\ ;$$

in the case of linear F these two mappings are related by the similarity transformation

$$I - \tilde{F}^{-1} F = \tilde{F}^{-1} (I - F\, \tilde{F}^{-1})\, \tilde{F}\ .$$

(Here and in the following, we always assume that solutions of equations exist and are unique at least locally so that the notion of an inverse mapping makes sense.)

We will now assume that the efficient solution procedure for problems (1.2) is itself an iterative process and of the defect correction type; let us denote it by

$$v_{j+1} := v_j - K\, (\tilde{F} v_j - \tilde{c})\,. \tag{1.5}$$

In conjunction with the iterative process (1.4) or (1.3), one will not want to use more than one or a few iterations (1.5) - with an appropriate initial value v_o - in place of $\tilde{F}^{-1}$, even if (1.5) does not converge very rapidly towards the solution $\tilde{F}^{-1}\tilde{c}$ of (1.2).

In the case of (1.4), i.e. when $\tilde{F}$ is linear, we have to compute the correction $\Delta u^{(i)}$ for $u^{(i)}$ from

$$\tilde{F}\ \Delta u^{(i)} = F\ u^{(i)} - c =: d(u^{(i)})\ ; \tag{1.6}$$

therefore our iterative solution procedure (1.5) will take the form

$$\begin{aligned} v_0^{(i)} &:= 0\ , \\ v_{j+1}^{(i)} &:= v_j^{(i)} - K\ (\tilde{F}\ v_j^{(i)} - d(u^{(i)}))\ , \quad j = 0(1)r-1\ , \end{aligned} \tag{1.7}$$

and the next iterate in (1.4) will be

$$u^{(i+1)} := u^{(i)} - v_r^{(i)}\ . \tag{1.8}$$

For r = 1, (1.7) and (1.8) collapse into

$$u^{(i+1)} := u^{(i)} - K\ d(u^{(i)})\ ; \tag{1.9}$$

this is (1.4), with $\tilde{F}^{-1}$ replaced by the operation K.

In the case of (1.3B), where we have to solve

$$\tilde{F}\ u^{(i+1)} = \tilde{F}\ u^{(i)} - d(u^{(i)})\ ,$$

the corresponding iterative solution procedure is

$$\begin{aligned} u_0^{(i+1)} &:= u^{(i)}\ , \\ u_{j+1}^{(i+1)} &:= u_j^{(i+1)} - K\ [\tilde{F}\ u_j^{(i+1)} - \tilde{F}\ u^{(i)} + d(u^{(i)})]\ , \quad j = 0(1)r-1, \end{aligned} \tag{1.10}$$

and $u^{(i+1)} := u_r^{(i+1)}$. For r = 1, this becomes (1.9) again.

The case of (1.3A) is slightly different through the occurence of $F^{-1}c =: \bar{u}$ which is normally used as $u^{(0)}$ in version A of the nonlinear defect correction procedure, see [5]. If we apply one iteration of (1.5), with starting value $\bar{u}$, in the computation of $\tilde{F}^{-1}F\ u^{(i)}$ in (1.3A), we arrive once more at (1.9). (If we start with $u^{(i)}$, which is more natural, we must replace both occurences of $\tilde{F}^{-1}$ in (1.3A) by one iteration step (1.5) in order to obtain (1.9).)

In any case, (1.9) appears as the natural combination of iterative

defect correction and an iterative solution procedure for $\tilde{F}$ even if F is nonlinear. Because of its simple structure, the iterative procedure (1.9) admits also other natural interpretations; but in our context it seems most intuitive to regard it as a <u>composite iterative procedure</u> with the constituents

defect correction for (1.1) and

iterative solution for (1.2).

It is clear that (1.9) converges to the same limit as (1.4) resp. (1.3) if it converges. If we know the rates of convergence of (1.4) (or (1.3)) and (1.5), what can we tell about the rate of convergence of (1.9)?

<u>Proposition</u>: Assume that q and $\tilde{q}$ are the (local) convergence rates of (1.4) and (1.5) resp. Then the convergence rate $\bar{q}$ of (1.9) satisfies

$$\bar{q} \leq q + \tilde{q} + q\tilde{q}. \tag{1.11}$$

<u>Proof</u>: If all operations are linear we simply observe that

$$I - KF = (I - K\tilde{F}) + (I - \tilde{F}^{-1}F) - (I - K\tilde{F})(I - \tilde{F}^{-1}F) \tag{1.12}$$

which implies

$$\|I - KF\| \leq \|I - K\tilde{F}\| + \|I - \tilde{F}^{-1}F\| + \|I - K\tilde{F}\|\,\|I - \tilde{F}^{-1}F\| \ .$$

If one or both of the operations are nonlinear, the convergence rates are local Lipschitz constants rather than norms. We have

$$\|u^{(i+1)} - u^*\| \leq q\,\|u^{(i)} - u^*\| \ ,$$
$$\|\tilde{u}^{(i+1)} - u^{(i+1)}\| \leq \tilde{q}\,\|u^{(i)} - u^{(i+1)}\| \ ,$$

where u^* is the limit of (1.3), $u^{(i+1)}$ and $\tilde{u}^{(i+1)}$ are the results of (1.3) and (1.9) resp. ; the second inequality holds because the iterative processes (1.7) or (1.10) have $u^{(i+1)}$ as a limit and $u^{(i)} - u^{(i+1)}$ as initial error. Hence

$$\begin{aligned}\|\tilde{u}^{(i+1)} - u^*\| &\le \|\tilde{u}^{(i+1)} - u^{(i+1)}\| + \|u^{(i+1)} - u^*\| \\ &\le \tilde{q}\,\|u^{(i)} - u^{(i+1)}\| + q\,\|u^{(i)} - u^*\| \\ &\le \tilde{q}(1+q)\,\|u^{(i)} - u^*\| + q\,\|u^{(i)} - u^*\| . \qquad \square\end{aligned}$$

Obviously, $q < 1$ and $\tilde{q} < 1$ are not sufficient for $\bar{q} < 1$. On the other hand, $q \ll 1$ and $\tilde{q} \ll 1$ imply $\bar{q} \approx q + \tilde{q} < 1$. If more than one iteration of (1.5) is employed $\tilde{q}$ is replaced by $\tilde{q}^r$; for $\tilde{q} < 1$, $\bar{q} \to q$ as $r \to \infty$.

2. Application to Multigrid Algorithms

We will now apply the composite iterative defect correction procedure introduced in section 1 to the following situation:

(1.2): a standard discretization of a standard differential problem on a given (fine) grid;

(1.1): a more sophisticated discretization of the same problem on the same grid, or

a discretization on the same grid of a weakly nonlinear version of the same problem, or

another related discrete problem more complicated than (1.2);

(1.5): one cycle of a multigrid algorithm for (1.2).

We assume that an efficient implementation of a multigrid algorithm for the standard problem (1.2) is available; thus our approach extends the use of such software to the wider range of problems (1.1). This extension of standard algorithms to wider classes of problems by means of defect correction may become an important tool in the design of easy-to-use multigrid software packages.

In accordance with the above interpretation, we will from now on assume that $\tilde{F}$ is a linear operation, i.e. we will consider (1.4) and (1.6) - (1.9). Note that the linearity of F has not been assumed in these equations.

In the more detailed "model problem analysis" of sections 4 and 5, (1.2) will be the standard five-point $O(h^2)$-discretization of a Helm-

holtz equation on a rectangle. (1.1) will either be the nine-point $O(h^4)$-discretization of the same differential equation problem or an $O(h^2)$-discretization of a nonlinear Helmholtz equation. It may be useful to have these interpretations in mind as a reference in this and the next section.

At first, we have to verify that a multigrid cycle for a linear problem (1.2) has indeed the structure (1.5). This is not immediately obvious from the common notation*) for an 1-level multigrid cycle for $L_1 v = c_1$ which is

$$v^{j+1} := M_1 \, v^j + N_1 \, c_1 \tag{2.1}$$

where the 1-grid iteration operator M_1 is recursively defined by

$$\begin{aligned} M_1 &:= [I_1 - I_0^1 \, L_0^{-1} \, I_1^0 \, L_1] \, S_1^\nu \, , \\ M_k &:= [I_k - I_{k-1}^k \, (I_{k-1} - M_{k-1}^\gamma) \, L_{k-1}^{-1} \, I_k^{k-1} \, L_k] \, S_k^\nu \, , \qquad k = 2(1)1. \end{aligned} \tag{2.2}$$

However, (2.2) does not display the fact that the smoothing operations S_k are relaxations w.r.t. the corrector equations $L_k v_k = c_k$ on the levels k, i.e.

$$\begin{aligned} v_k \xrightarrow{S_k} \bar{v}_k &:= v_k - R_k \, (L_k v_k - c_k) \, , \\ v_k \xrightarrow{S_k^\nu} \bar{v}_k &:= v_k - R_k^\nu \, (L_k v_k - c_k) \, , \end{aligned} \tag{2.3}$$

where R_k^ν, $\nu \geq 2$, is recursively defined by $R_k^\nu := R_k^{\nu-1} + R_k - R_k L R_k^{\nu-1}$. With (2.3) and the abbreviation $(L_1^{l-1})^{-1} := I_{l-1}^l \, L_{l-1}^{-1} \, I_l^{l-1}$, the two-grid algorithm for $L_1 v = c_1$ takes the form

$$\begin{aligned} v_1^{j+1} &:= \bar{v}_1^j - (L_1^{l-1})^{-1} \, (L_1 \bar{v}_1^j - c_1) \\ &= v_1^j - R_1^\nu \, (L_1 v_1^j - c_1) - (L_1^{l-1})^{-1} \, (L_1 (v_1^j - R_1^\nu (L_1 v_1^j - c_1)) - c_1) \\ &= v_1^j - [R_1^\nu + (L_1^{l-1})^{-1} - (L_1^{l-1})^{-1} \, L_1 \, R_1^\nu] \, (L_1 v_1^j - c_1) \, , \end{aligned} \tag{2.4}$$

which has the defect correction structure of (1.5).

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*) Cf. the introductory paper by U. Trottenberg in this volume. For simplicity, smoothing after correction is not considered here.

We now replace the operation L_{l-1}^{-1} in $(L_l^{l-1})^{-1}$ by another two-grid step to obtain a three-grid algorithm ($\gamma = 1$): The correction $e_{l-1} = L_{l-1}^{-1} d_{l-1}$ is approximatively computed by

$$e_{l-1}^{o} = 0 \xrightarrow{S_{l-1}^{\nu}} \bar{e}_{l-1} = e_{l-1}^{o} - R_{l-1}^{\nu} (L_{l-1} e_{l-1}^{o} - d_{l-1})$$

$$= R_{l-1}^{\nu} d_{l-1},$$

$$e_{l-1} = \bar{e}_{l-1} - (L_{l-1}^{l-2})^{-1} (L_{l-1} \bar{e}_{l-1} - d_{l-1})$$

$$= [R_{l-1}^{\nu} + (L_{l-1}^{l-2})^{-1} - (L_{l-1}^{l-2})^{-1} L_{l-1} R_{l-1}^{\nu}] d_{l-1} .$$

Hence, in (2.4), $(L_{l-1}^{l})^{-1}$ is simply replaced by

$$I_{l-1}^{l} [R_{l-1}^{\nu} + (L_{l-1}^{l-2})^{-1} - (L_{l-1}^{l-2})^{-1} L_{l-1} R_{l-1}^{\nu}] I_l^{l-1}$$

which leaves the structure of (2.4) unaltered.

It is now clear that this argument can be continued recursively and that $\gamma = 2$ (W-cycle) and smoothing after correction lead to the same structure (1.5) for (2.1). (The use of a notation displaying this structure - like the one employed in (2.3) and (2.4) - may also be helpful in the analysis of other aspects of multigrid algorithms.)

In the context described at the beginning of this section, one step of the composite iterative defect correction procedure (1.9)

$$u^{(i+1)} := u^{(i)} + K\, d(u^{(i)})$$

consists the following operations:

Computation of defect $d(u^{(i)}) := Fu^{(i)} - c$
on the given (fine) grid

Multigrid cycle for $\tilde{F}v = -d(u^{(i)})$:

- Smoothing of the initial correction $v_o^{(i)} = 0$ by relaxation w.r.t. $\tilde{F}v = -d(u^{(i)})$
- Transfer of the defect $F\bar{v}_o^{(i)} + d(u^{(i)})$ of the smoothed correction $\bar{v}_o^{(i)}$ to coarser grid
- . . .
- Generation of approximate solution $\tilde{v}^{(i+1)} = -K\, d(u^{(i)})$ of $\tilde{F}v = -d(u^{(i)})$ on given (fine) grid

Computation of $u^{(i+1)} := u^{(i)} - \tilde{v}^{(i+1)}$

The appearance of this defect correction multigrid cycle ("DCMG-cycle") becomes more familiar when we set $u := u^{(i)} - v$ and use the linearity of $\tilde{F}$ to obtain the right hand version of

one multigrid cycle for

$$\tilde{F}v = -d(u^{(i)}) \qquad\qquad \tilde{F}u = \tilde{F}u^{(i)} - d(u^{(i)}) \qquad (2.5)$$

starting with

$$v_o^{(i)} := 0 \qquad\qquad u_o^{(i)} := u^{(i)}$$

yields

$$\tilde{v}^{(i+1)} \qquad\qquad \tilde{u}^{(i+1)}$$

which is used to form

$$u^{(i+1)} := u^{(i)} - \tilde{v}^{(i+1)} \qquad\qquad u^{(i+1)} := \tilde{u}^{(i+1)} \ .$$

This right hand version also corresponds to the iterative procedure (1.10) which must be used for nonlinear $\tilde{F}$. (2.5) may also be written in the forms

$$\begin{aligned} \tilde{F}u &= \tilde{c} - [(Fu^{(i)} - c) - (\tilde{F}u^{(i)} - \tilde{c})] \\ &= \tilde{c} - [d(u^{(i)}) - \tilde{d}(u^{(i)})] \\ &= c + \Delta Fu^{(i)}, \qquad \text{with} \quad \Delta F := \tilde{F} - F \ , \end{aligned} \qquad (2.6)$$

which display various aspects of our defect correction approach.

In this version, the DCMG-cycle is distinct from the original multigrid cycle for $\tilde{F}u = \tilde{c}$ only by the replacement of $\tilde{c}$ by $\tilde{F}u^{(i)} - d(u^{(i)})$. The evaluation of $d(u^{(i)})$ and the replacement of $\tilde{c}$ can be implemented into a given multigrid program without much effort; in particular, none of the data structures and of the grid transfer operations (including the definition of $\tilde{F}$ on the coarser grids) are affected!

This DCMG-cycle may also be used like the original multigrid cycle: It may be repeated or it may be recursively embedded into a Full Multigrid Algorithm (FMG). No further changes in the respective standard algorithms are required.

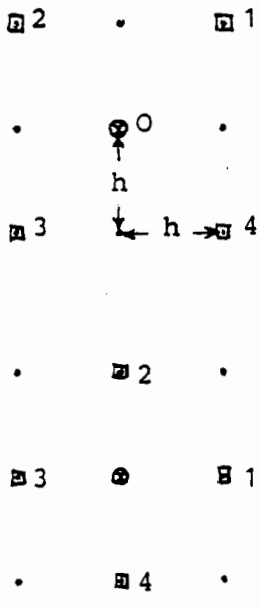
In a FMG-algorithm sufficient care has to be exerted in the interpolation for an initial approximation on a next finer grid. Several authors have suggested that a higher order interpolation should be employed which makes use either of the difference equation or of the differential equation. In our context, the difference or differential equa-

tion connected with Fu = c must be used. Particularly in the case of nonlinear F, the following use of the differential equation leads to a simple procedure:

Assume that we wish to approximate the solution of

$$-\Delta u(x,y) = g(x,y,u(x,y)); \qquad (2.7)$$

then the following two interpolation formulas with the substitution (2.7) permit an easy and accurate transfer of an approximate solution from a 2h-grid to an h-grid:



$$u_o = \frac{1}{4}\sum_{i=1}^{4} u_i - \frac{h^2}{8}\sum_{i=1}^{4} (\Delta u)_i + O(h^4)$$

with $(\Delta u)_i := -g(x_i,y_i,u_i)$

$$u_o = \frac{1}{4}\sum_{i=1}^{4} u_i - \frac{h^2}{16}\sum_{i=1}^{4} (\Delta u)_i + O(h^4)$$

So far we have taken the convergence of our DGMG-procedure for granted. It is clear that the contractivity of the multigrid cycle proper will remain unchanged, i.e. $\tilde{q}$ of (1.11) will be the convergence rate of the multigrid cycle. However, the convergence rate q of the defect correction component of the composite process has to be determined and checked:

With $\Delta\tilde{F} := \tilde{F} - F$, the iteration operator of iterative defect correction is (cf. section 1)

$$I - \tilde{F}^{-1} F = \tilde{F}^{-1} \Delta F \ .$$

In many typical applications (see the beginning of this section), $\tilde{F}^{-1}\Delta Fu$ will be small for smooth gridfunctions u but $\|F^{-1}\Delta F\|$ will not be small. E.g., if F and $\tilde{F}$ are $O(h^4)$ and $O(h^2)$-discretizations of the same smooth,

2nd order differential equation problem, $\Delta F\, u = O(h^2)$ for smooth u because it is essentially the principal term of the local discretization error of $\tilde{F}$. But $\|\Delta F\| = O(h^{-2})$ and it seems at first that there is no uniform (in h) bound for $\|\tilde{F}^{-1}\Delta F\|$ at all.

Fortunately, a closer analysis reveals that - in this situation - ΔF is h^{-2} × (4th order difference operator) so that the application of the "summation operator" $\tilde{F}^{-1}$ compensates the h^{-2}; thus there exists a constant q independent of h such that $\|\tilde{F}^{-1}\Delta F\, u\| \leq q\|u\|$ or $\|\tilde{F}^{-1}\Delta F\| \leq q$. Yet q may not be sufficiently small in view of (1.11).

In these and similar cases the convergence rate q of the defect correction part of our procedure may be improved by an a-priori reduction of the non-smooth components in the current approximation $u^{(i)}$, i.e. by a smoothing of $u^{(i)}$ before the formation of $d(u^{(i)})$. In the next section we will consider various implementations of such an "a-priori smoothing" and their effect on the performance of a DCMG-algorithm.

3. Defect correction and smoothing

Through the considerations of section 1, we had arrived at the following form of a DCMG-cycle (see also (2.5)):

Implementation (i)

Form $d(u^{(i)}) := F\, u^{(i)} - c$

Perform one multigrid cycle for $\tilde{F}\, u = \tilde{F}\, u^{(i)} - d(u^{(i)})$ starting with $u^{(i)}$

This procedure which takes $u^{(i)}$ into $u^{(i+1)}$ has the structure (1.9); hence it is clear that its fixed point is the desired solution u* of $F\, u = c$ and that - if it contracts - it contracts towards u*.

An a-priori smoothing*) takes $u^{(i)}$ into $\bar{u}^{(i)}$, then we proceed as in implementation (i), with $u^{(i)}$ replaced by $\bar{u}^{(i)}$. Since our goal is an approximation of u*, it is natural to use relaxation w.r.t. (1.1) for smoothing. Hence, with our notation (2.3),

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*) This term is to indicate that the smoothing occurs outside the multigrid cycle proper and in addition to the smoothing inside the multigrid cycle.

$$\bar{u}^{(i)} := u^{(i)} - R\,(F\,u^{(i)} - c)\,, \tag{3.1}$$

where R is now the relaxation operator of the a-priori smoothing w.r.t. (1.1).

Implementation (ii)

Smooth $u^{(i)}$ by relaxation w.r.t. $F\,u = c$
to obtain $\bar{u}^{(i)}$

Form $d(\bar{u}^{(i)}) := F\,\bar{u}^{(i)} - c$

Perform one multigrid cycle for $\tilde{F}\,u = \tilde{F}\,\bar{u}^{(i)} - d(\bar{u}^{(i)})$
starting with $\bar{u}^{(i)}$

We will now consider the properties of this DCMG-cycle: Substitution of (3.1) into (1.9) yields

$$\begin{aligned} u^{(i+1)} &:= \bar{u}^{(i)} - K\,(F\,\bar{u}^{(i)} - c) \\ &= u^{(i)} - R\,(F\,u^{(i)} - c) - K\,(F(u^{(i)} - R\,(Fu^{(i)} - c)) - c)\,. \end{aligned} \tag{3.2}$$

If F is linear, (3.2) becomes

$$u^{(i+1)} := u^{(i)} - [R + K - K\,F\,R]\,(F\,u^{(i)} - c) \tag{3.3}$$

$$=: u^{(i)} - \bar{K}\,(F\,u^{(i)} - c) \tag{3.4}$$

which is again of type (1.9). The iteration operator is now

$$I - [R + K - K\,F\,R]F = (I - K\,F)\,(I - RF)\,;$$

but if our smoothing has been successful we must have

$$\|I - \bar{K}\,F\| \ll \|I - K\,F\| \cdot \|I - R\,F\| \approx \|I - K\,F\| \tag{3.5}$$

because $I - R\,F$ reduces the unsmooth components and $I - K\,F$ the smooth ones. It is clear from (3.3) that the fixed point u^* has remained unaltered.

Even when F is nonlinear, the invariance of u^* follows immediately from (3.2) and $F\,u^* - c = 0$ because K and R are linear and map the origin into itself. The Frechet derivative of the right hand side of (3.2) at $u^{(i)}$ becomes $(I - K\,F'(u^{(i)}))(I - R\,F'(u^{(i)}))$ so that we may expect a similar effect as in (3.5) for the contraction rate of (3.2) which is

now defined by (local) Lipschitz bounds which may be estimated by norms of Frechet derivatives.

Thus implementation (ii) is perfect except for the fact that it uses a smoother which is not the one contained in the program for a multigrid cycle for (1.2). Although the implementation of (3.1) should require little effort beyond the defect computation which we need anyway, it seems worthwile to consider how we would get away with a smoother for (1.2) as an a-priori smoother, i.e. with the standard smoother of our multigrid software:

Implementation (iii)

Smooth $u^{(i)}$ by relaxation w.r.t. $\tilde{F}\, u = \tilde{c}$
to obtain $\bar{u}^{(i)}$

Form $d(\bar{u}^{(i)}) := F\, \bar{u}^{(i)} - c$

Perform one multigrid cycle for $\tilde{F}\, u = \tilde{F}\, \bar{u}^{(i)} - d(\bar{u}^{(i)})$
starting with $\bar{u}^{(i)}$

Intuitively, the exchange of the a-priori smoother should not strongly affect the performance of the DCMG-cycle: The smooth components which would suffer from a contraction towards the "wrong" solution $\tilde{u}^*$ of $\tilde{F}\, u = \tilde{c}$ are hardly affected by a typical relaxation smoother and the unsmooth components need only be reduced. It is true that u^* will no longer be a fixed point of the DCMG-cycle (iii) because it is not reproduced by the a-priori smoother; but this will not hurt if the actual fixed point $\bar{u}^*$ differs from u^* by no more than the magnitude of the discretization error of (1.1).

With a relaxation smoother w.r.t. (1.2)

$$\bar{u}^{(i)} := u^{(i)} - \tilde{R}\, (\tilde{F}\, u^{(i)} - \tilde{c}) \tag{3.6}$$

we obtain for version (iii) of the DCMG-cycle in analogy to (3.2)

$$u^{(i+1)} := u^{(i)} - \tilde{R}\, (\tilde{F}\, u^{(i)} - \tilde{c}) - K\, (F(u^{(i)} - \tilde{R}(\tilde{F} u^{(i)} - \tilde{c})) - c) \tag{3.7}$$

Assuming F to be linear, we may subtract the identity

$$u^* = u^* - \tilde{R}\, (\tilde{F}\, u^* - \tilde{c}) - K\, (F(u^* - \tilde{R}(\tilde{F}\, u^* - \tilde{c})) - c)$$
$$+ (I - K\, F)\, \tilde{R}\, (\tilde{F}\, u^* - \tilde{c})$$

from the fixed point equation for $\bar{u}^*$ (cf. (3.7)) to obtain, with $\bar{e} := \bar{u}^* - u^*$,

$$\bar{e} = (I - K\,F)(I - \tilde{R}\,\tilde{F})\,\bar{e} - (I - KF)\,\tilde{R}\,(\tilde{F}u^* - \tilde{c}) \ .$$

If our DCMG-cycle is contractive, $\tilde{M} := (I - KF)(I - \tilde{R}\,\tilde{F})$ has a norm smaller than 1 and

$$\bar{e} = -(I - \tilde{M})^{-1}\,(I - K\,F)\,\tilde{R}\,(\tilde{F}\,u^* - \tilde{c}) \ . \qquad (3.8)$$

(3.8) yields the desired characterization of the "fixed point shift" $\bar{e}$: E.g., if (1.1) and (1.2) are $O(h^4)$ and $O(h^2)$ discretizations of the same elliptic equation, we have

$$\tilde{F}\,u^* - \tilde{c} = O(h^2) \quad \text{(it is approx. the discr. error of (1.2))}$$
$$\|\tilde{R}\| = O(h^2) \quad \text{for standard relaxation smoothers}$$

while the first two operators are $O(1)$. Thus $\bar{e} = O(h^4)$ which is the order of the discretization error of (1.1). Thus the solution u^* of (1.1) may be approximated by our DCMG-cycle (iii) to within the magnitude of its discretization error. Actually, $\bar{e}$ will be a small multiple of h^4 because $I - K\,F$ reduces smooth components considerably.

For nonlinear F, a more complicated analysis leads to a result analogous to (3.8).

In its version (iii), the DCMG-cycle begins with a smoothing and a defect computation step which are immediately followed - within the multigrid cycle proper - by another smoothing and defect computation step, all of them on the finest grid level; both smoothers refer to (1.2). Thus it is natural to identify the defect correction part of the DCMG-cycle (iii) with the beginning of the multigrid cycle proper:

Smooth $u^{(i)}$ w.r.t. $\tilde{F}\,u = \tilde{c} \to \bar{u}^{(i)}$
Form $d(\bar{u}^{(i)}) := F\,\bar{u}^{(i)} - c$

Smooth $\bar{u}^{(i)}$ w.r.t. $\tilde{F}u = \tilde{F}\bar{u}^{(i)} - d(\bar{u}^{(i)}) \to \bar{\bar{u}}^{(i)}$	Smooth $u^{(i)}$ w.r.t. $\tilde{F}u = \tilde{c} \to \bar{u}^{(i)}$
Form $d_1 := \tilde{F}\bar{\bar{u}}^{(i)} - \tilde{F}u^{(i)} + d(\bar{u}^{(i)})$	Form $d_1 := F\bar{u}^{(i)} - c$
. . .	. . .
(remainder of MG-cycle)	

Implementation (iv)

Perform one multigrid cycle for $\tilde{F}u = \tilde{c}$
starting with $u^{(i)}$

where the defect d_l on the finest grid level is
computed w.r.t. $F\,u = c$

Obviously, this version requires the least change in a standard program for the multigrid cycle; only the defect computation on the finest grid has to be changed. This version has been suggested by Hackbusch ([3]); it is also essentially identical to Brandt and Dinar's τ-extrapolation approach.

From the point of view of our analysis, version (iv) is the least transparent: The two components of our composite iteration procedure have now been merged. If we continue to identify the initial smoothing and the computation of $d(\bar{u}^{(i)})$ as the defect correction phase then we no longer have a full l-level multigrid cycle but only an (l-1)-level one.

Actually, using the equivalence considered in connection with (2.5), we may relate implementations (iii) and (iv) in the following way (cf. implementation (iii)):

(iii)	(iv)
Smooth $u^{(i)}$ by relaxation w.r.t. $\tilde{F}u = \tilde{c}$ to obtain $\bar{u}^{(i)}$	
Form $d_l(u^{(i)}) := F_l\,\bar{u}^{(i)} - c$	
Perform one l-grid cycle for $\tilde{F}_l\, v = -d_l(\bar{u}^{(i)})$ starting with $v_l = 0$	Perform one (l-1)-grid cycle for $\tilde{F}_{l-1}\, v = -I_l^{l-1}\, d_l(\bar{u}^{(i)})$ starting with $v_{l-1} = 0$
Form $u^{(i+1)} := u^{(i)} - \tilde{v}_l^{(i+1)}$	Form $u^{(i+1)} := u^{(i)} - I_{l-1}^{l}\, \tilde{v}_{l-1}^{(i+1)}$

Thus version (iv) should behave rather like version (iii) and its fixed point should satisfy (3.8) with a slightly different multigrid operator $\tilde{M}$. Hackbusch ([3]) has shown that - under suitable assumptions - his defect correction process and hence our version (iv) of the DCMG-iteration converges to a limit $\bar{u}^*$ which satisfies $\bar{u}^* - u^* = O(h^4)$ in the case which we have considered for version (iii).

Remark*):

Actually, the authors feel that version (iv) should not be regarded as a member of the DCMG-family but as a first step towards a more general multigrid algorithm concept which is just emerging in the work of Brandt (see his contributions in these Proceedings):

In this concept, the role of smoothing (of "relaxation") is seen like that of other stabilizing devices in classical discretizations of p.d.e.: They are necessary to control high frequency effects, and it is accepted that they may change the mathematical model slightly (like artificial viscosity). This means that - generally and throughout a multigrid cycle - the smoother need not be restricted by the request that it must have the solution of the corrector equation at the respective grid level as a fixed point.

In our context, this means that F takes the role of $\tilde{F}$ (in the form pertaining to the momentary grid level) throughout the multigrid cycle except in the smoothing steps. This implies that the direct solution on the coarsest grid level is also done with the respective version of F instead of $\tilde{F}$. This seems acceptable for two reasons:

a) At that stage, the size of the system is sufficiently small so that the more complicated F-system is cheaply solvable as well.

b) At the coarsest grid level, the stepsize parameter h is sufficiently large so that the F-system is stable.

Although the convergence analyses for multigrid algorithms (e.g. in Hackbusch's presentation in these Proceedings) are immediately extendable to such algorithms, it seem desirable to tailor their analysis more immediately to the new concept. In particular, the fixed point of such an MG-cycle has to be suitably characterized. Some progress in this direction will be reported by the authors in a separate paper.

4. Model problem analysis

We will now amplify some of the preceding discussion with a simple model problem

$$\begin{aligned} -\Delta u(x,y) + u(x,y) &= 1 \quad \text{on} \quad G = [0,1] \times [0,1] \ , \\ u(x,y) &= 0 \quad \text{on} \quad \partial G \ . \end{aligned} \tag{4.1}$$

*) This remark was not part of the presentation at the conference but arose as an afterthought on various remarks made by Achi B. during the conference.

We consider uniform grids $\mathbb{E}_h$, with $h = \frac{1}{n}$, $n = 2^{-m}$, $m \in \mathbb{N}$. The grid subscripts μ,ν vary over $0(1)n$ in $\mathbb{E}_h$ and over $1(1)n-1$ on the interior of $\mathbb{E}_h$.

At first we consider the computation of an $O(h^4)$-approximation to (4.1) by means of an $O(h^2)$ multigrid algorithm for (4.1). Thus (cf. sections 2 and 3), for $\mu,\nu = 1(1)n-1$,

$$(\tilde{F}u)_{\mu\nu} := \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} u_{\mu\nu} + \begin{bmatrix} & 1 & \end{bmatrix} u_{\mu\nu} ,$$

$$(Fu)_{\mu\nu} := \frac{1}{6h^2}\begin{bmatrix} -1 & -4 & -1 \\ -4 & 20 & -4 \\ -1 & -4 & -1 \end{bmatrix} u_{\mu\nu} + \frac{1}{12}\begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix} u_{\mu\nu} ,$$

$$(\Delta Fu)_{\mu\nu} := \underbrace{\frac{1}{6h^2}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} u_{\mu\nu}}_{\approx h^4 (u_{xxyy})_{\mu\nu} \text{ for smooth } u} + \frac{1}{12}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} u_{\mu\nu} .$$

Because of their simple Fourier representations we consider damped Jacobi smoothers only ($\omega < 1$):

$$\tilde{S}_\omega : u \to u - \frac{\omega}{4} h^2 (\tilde{F} u - \tilde{c}) ,$$

$$S_\omega : u \to u - \frac{\omega}{10/3} h^2 (F u - c) ,$$

$$(I - \tilde{R}_\omega \tilde{F}) u_{\mu\nu} = \frac{1}{4}\left(\begin{bmatrix} & \omega & \\ \omega & 4(1-\omega) & \omega \\ & \omega & \end{bmatrix} - h^2 \omega \begin{bmatrix} & 1 & \end{bmatrix}\right) u_{\mu\nu} ,$$

$$(I - R_\omega F) u_{\mu\nu} = \frac{1}{20}\left(\begin{bmatrix} \omega & 4\omega & \omega \\ 4\omega & 20(1-\omega) & 4\omega \\ \omega & 4\omega & \omega \end{bmatrix} - \frac{h^2}{2} \omega \begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix}\right) u_{\mu\nu} .$$

With respect to the well-known orthogonal basis

$$(s_{kl})_{\mu\nu} := \sin k \frac{\mu}{n} \pi \cdot \sin l \frac{\nu}{n} \pi , \qquad k,l = 1(1)n-1 , \quad (4.2)$$

for the gridfunctions which vanish on ∂G, the representations of the above operations become, with

$$c_k := \cos \frac{k}{n} \pi , \qquad k = 1(1)n-1 , \tag{4.3}$$

$$\tilde{F}\ s_{kl} = \frac{2}{h^2} \left[(1-c_k) + (1-c_l) + \frac{h^2}{2} \right] s_{kl}$$

$$F\ s_{kl} = \frac{2}{h^2} \left[(1-c_k) + (1-c_l) - \frac{1}{3}(1-c_k)(1-c_l) + \frac{h^2}{6} \left(2 + \frac{c_k+c_l}{2}\right) \right] s_{kl}$$

$$\Delta F\ s_{kl} = \frac{2}{3h^2} \left[(1-c_k)(1-c_l) + \frac{h^2}{4} (2-c_k-c_l) \right] s_{kl}$$

$$\tilde{F}^{-1}\ \Delta F\ s_{kl} = \frac{1}{3} \frac{(1-c_k)(1-c_l) + h^2(2-c_k-c_l)/4}{(1-c_k) + (1-c_l) + h^2/2}\ s_{kl} =: \rho_{kl}\ s_{kl}$$

$$(I - \tilde{R}_\omega\ \tilde{F})\ s_{kl} = \left[(1-\omega) + \frac{1}{2}\ \omega(c_k+c_l) - \frac{h^2}{4}\ \omega \right] s_{kl}$$

$$(I - R_\omega\ F)\ s_{kl} = \left[(1-\omega) + \frac{2}{5}\ \omega(c_k+c_l) + \frac{\omega}{5}\ c_k c_l - \frac{h^2}{10}\ \omega \left(2 + \frac{c_k+c_l}{2}\right) \right] s_{kl}$$

It is easily checked that the individual contraction factors ρ_{kl} of the defect correction operator $\tilde{F}^{-1}\Delta F$ increase with k and l as we have assumed at the end of section 2. Although

$$\rho_{1,1} = \frac{h^2}{12} \frac{\pi^2(\pi^2+1)}{\pi^2 + 1/2} + O(h^4) \doteq h^2 ,$$

the convergence rate $q^{(i)}$ of the unsmoothed defect correction in version (i) of the DCMG-cycle is only

$$q^{(i)} = \| \tilde{F}^{-1}\Delta F \|_2 = \max_{k,l} \rho_{kl} = \rho_{n-1,n-1}$$

$$= \frac{1}{3} \left(1 - \frac{2\pi^2-1}{8}\ h^2 + O(h^4)\right) \approx \frac{1}{3} .$$

These numbers confirm the strong contraction of the defect correction for smooth components but also the poor overall contraction. Although $q = \frac{1}{3}$ would not destroy the convergence of the composite cycle for any reasonable multigrid algorithm for (4.1) it would still lead to an intolerably poor convergence rate, cf. (1.11).

It is true that - for FMG-algorithms - the poor convergence may not materialize because the starting approximations $u^{(o)}$ on a new finer grid level are formed by interpolation and may not contain a good deal of high frequency error components with a suitable interpolation procedure. But this smoothing effect of the interpolation may depend crucially on details of the algorithm and the problem.

We will now consider the effect of a-priori smoothing on the con-

traction rate of a DCMG-cycle. According to (3.3) we have the iteration operator $(I - KF)(I - RF)$ for version (ii) where $I - KF$ is the iteration operator for version (i). If we decompose $I - KF$ as in (1.12) we find

$$(I - KF)(I - RF) = (I - K\tilde{F})(I - RF) + (I - \tilde{F}^{-1}F)(I - RF) - (I - K\tilde{F})(I - \tilde{F}^{-1}F)(I - RF) .$$

Obviously, we must have

$$\|(I - \tilde{F}^{-1}F)(I - RF)\| << \|I - \tilde{F}^{-1}F\|$$

in order to achieve (3.5) because the contraction rate $\tilde{q} = \|I - K\tilde{F}\|$ of the multigrid cycle for (1.2) may be assumed to be small in any case and $\|I - RF\| \approx 1$.

From the above Fourier representations of $I - \tilde{F}^{-1}F = \tilde{F}^{-1}\Delta F$ and $I - R_\omega F$ we obtain

$$(I - \tilde{F}^{-1}F)(I - R_\omega F)\, s_{kl} = \frac{1}{3}\,\frac{(1-c_k)(1-c_l) + O(h^2)}{(1-c_k)+(1-c_l)+O(h^2)}\,\left[(1-\omega) + \frac{2}{5}\,\omega(c_k+c_l) + \frac{\omega}{5}\,c_k c_l + O(h^2)\right] s_{kl}$$

For $\omega = \frac{5}{8}$, where $[..] = O(h^2)$ for $c_k = c_l = -1$, we obtain

$$(I - \tilde{F}^{-1}F)(I - R_{\frac{5}{8}}F)\, s_{kl} = \left\{\frac{(1-c_k)(1-c_l)(3+2(c_k+c_l)+c_k c_l)}{24\,(2-c_k-c_l)} + O(h^2)\right\} s_{kl}$$

and the principal term of $\{...\}$ remains below 0.065 for $(c_k,c_l) \in [-1,+1]^2$. Thus we have been able to reduce q in (1.11) approximately by a factor 5.

It is to be expected that other standard smoothers achieve comparable reductions in the effective contraction rate q of the defect correction component in the DCMG-cycle. For version (iii), we have to replace $I - R_\omega F$ by $I - \tilde{R}_\omega \tilde{F}$ and Jacobi relaxation with $\omega = \frac{1}{2}$ leads to

$$\|(I - \tilde{F}^{-1}F)(I - \tilde{R}_{\frac{1}{2}}\tilde{F})\|_2 = \frac{1}{12} + O(h^2).$$

Thus, from the contraction point of view, version (iii) with a suitable smoother should be equally effective.

Since version (iv) has been established as a simplified variant of version (iii) it should behave similarly. Here, the contraction rate $\tilde{q}$ of the multigrid component may be slightly smaller due to the truncated

multigrid cycle.

5. Weakly nonlinear problems

To exhibit the possibilities of a DCMG algorithm for nonlinear problems, we consider a boundary value problem for the semilinear Helmholtz equation

$$- \Delta u(x,y) + p(x,y,u(x,y))\, u(x,y) = g(x,y) \ , \tag{5.1}$$

with $p(x,y,u) \geq 0$, and its standard $O(h^2)$-discretization. As indicated at the beginning of section 2, (1.1) will now signify this nonlinear discretization (= system of algebraic equations) while (1.2) will denote the analogous discretization of a linear equation

$$- \Delta u(x,y) + \tilde{p}(x,y)\, u(x,y) = g(x,y) \ , \tag{5.2}$$

where $\tilde{p}(x,y) \geq 0$ is a suitable approximation of $p(x,y,u(x,y))$ for the anticipated values of u. If the variation of p and/or u is small, $\tilde{p}$ may be constant (see below).

Our defect correction iteration operator $\tilde{F}^{-1}\Delta F$ is now nonlinear, with

$$(\Delta F u)_{\mu\nu} = [\tilde{p}(x_\mu,y_\nu) - p(x_\mu,y_\nu,u_{\mu\nu})]\, u_{\mu\nu} \ . \tag{5.3}$$

A Lipschitz bound q for $\tilde{F}^{-1}\Delta F$ is furnished by a bound on the Frechet derivative of $\tilde{F}^{-1}\Delta F$

$$q = \sup_u \| \tilde{F}^{-1} (\Delta F)'(u) \| \leq \| \tilde{F}^{-1} \| \cdot \sup_u \| (\Delta F)'(u) \|$$

where the sup is taken over a suitable convex domain of grid functions which contains the iterates and their limit. For (5.3),

$$[(\Delta F)'(u)v]_{\mu\nu} = [\tilde{p}(x_\mu,y_\nu) - \frac{\partial}{\partial u} p(x_\mu,y_\nu,u_{\mu\nu})u_{\mu\nu} - p(x_\mu,y_\nu,u_{\mu\nu})]\, v_{\mu\nu} \ ; \tag{5.4}$$

$\| \tilde{F}^{-1} \|$ for (5.2) can be bounded by the well-known bound $(2\pi^2)^{-1} \approx 0.05$ for $\tilde{p} = 0$.

Due to this relatively low value for $\| \tilde{F}^{-1} \|$, the choice of $\tilde{p}(x,y)$ is not too crucial: If we can keep the Lipschitz bound for ΔF below 2, we have $q \approx 0.1$. Naturally, some a-priori estimate on the solution u

will normally be needed for the selection of an appropriate $\tilde{p}$.

Since now ΔF is not a difference quotient there is no particular need for a-priori smoothing. Thus implementation (i) of our DCMG-cycle should be satisfactory. Due to the simple structure of ΔF (cf. (5.3)), it is advantageous to form the right hand side of the adjusted problem $\tilde{F}u = \tilde{F}u^{(i)} - d(u^{(i)})$ as $c + \Delta F\, u^{(i)}$ so that no explicit defect correction occurs:

Implementation (i)'

Form $\Delta F\, u^{(i)}$

Perform one multigrid cycle for $\tilde{F}u = c + \Delta F\, u^{(i)}$ starting with $u^{(i)}$.

Thus we have returned to our original approach in its most elementary form: Take the standard defect correction pattern (cf. (1.3B))

Form $\Delta F\, u^{(i)}$

Solve $\tilde{F}u = c + \Delta F\, u^{(i)}$ for $u^{(i+1)}$

and replace the direct solution by one multigrid cycle. Note that $\tilde{F}\, u = c + \Delta F\, u^{(i)}$ is normally not a local linearization of $Fu = c$ in the usual sense of the word, i.e. a Frechet derivative of the original problem; this would be more complicated in the present situation, cf. (5.4).

The following concrete example will show some details of the approach more clearly: Consider the boundary value problem

$$\begin{aligned} -\Delta u(x,y) + e^{u(x,y)} u(x,y) &= 1 \quad \text{on} \quad G = [0,1]^2 , \\ u(x,y) &= 0 \quad \text{on} \quad \partial G . \end{aligned} \tag{5.5}$$

It is easily seen that $u \geq 0$ and $\Delta u \leq 0$ in G, hence $0 \leq e^u u < 1$ or $0 \leq u < 0.57$ must hold for the true solution of (5.5). Therefore

$$\tilde{p}(x,y) \equiv 1$$

is a reasonable approximation for $p(x,y,u) = e^u$. For it,

$$(\Delta F\, u)_{\mu\nu} = (1 - e^{u_{\mu\nu}})\, u_{\mu\nu}$$

and a Lipschitz bound for ΔF in a neighborhood of the true solution is

(see (5.4))

$$\max_{u\in[0,0.6]} |1 - e^u\, u - e^u| < 2 \,. \qquad (5.6)$$

This leads to a bound 0.1 for the contraction rate q of the defect correction as explained above.

But from (5.6) we also realize that a constant value $\tilde{p}$ at the upper end of the range of e^u would have been a better choice: For $\tilde{p} = 1.8$, we obtain a Lipschitz bound near 1 and $q \approx 0.05$. Note that the linear operator with $\tilde{p} = 1$ could have been constructed as $F'(u_o)u$ with $u_o \equiv 0$ while the linear operator with $\tilde{p} = 1.8$, although it may formally be backinterpreted as $F'(u_o)u$ with $u_o \equiv 0.314...$, has been obtained by a completely different reasoning.

All that remains to be done is to take a multigrid program which is suitable for $-\Delta u + 1.8\, u = 1$ and to insert a few lines of code which update the right hand side of the difference equation to

$$1 + (1.8 - e^{u_{\mu\nu}^{(i)}})\, u_{\mu\nu}^{(i)}$$

at the beginning of each multigrid cycle where $u^{(i)}$ is the approximation available at this time on the current finest grid. This modified program will compute an approximate solution for (5.5), with hardly any extra effort beyond that for a comparable linear problem.

It is obvious that the present approach may be combined with that of the previous sections to obtain a DCMG-cycle which computes an $O(h^4)$-approximation for (5.5), again with not much additional effort.

6. Numerical results

In a first preliminary set of test runs, we used the code MGOOD2 from the multigrid package MGOO of the GMD-IMA ([6]). It solves the standard $O(h^2)$-discretization of a Dirichlet problem for a Helmholtz equation on a rectangle in a fixed MG or an FMG mode, using checkered relaxation for smoothing. We used the V-cycle mode throughout. A subroutine for computing defects w.r.t. the $O(h^4)$-discretization of section 4 was added as well as various a priori smoothing procedures.

The following test problem was used:

$$-\Delta u(x,y) + (1+x^2+y^2)\, u(x,y) = g(x,y) \quad \text{on} \quad G = [0,1]^2$$
$$u(x,y) = 0 \quad \text{on} \quad \partial G \ ; \tag{6.1}$$

g was chosen such that (6.1) yielded the true solution

$$U(x,y) = \sin \pi x \sin \pi y + 0.2 \sin 5\pi x \sin 5\pi y. \tag{6.2}$$

To exhibit the efficiency of the various DCMG implementations of section 3 (see Table 1), we employ the accuracy of the exact $O(h^2)$-solution u_h^* has a reference, i.e. we define its maximal truncation error $\|u_h^* - U_h\|_\infty$ as 1. This accuracy level is reached by the FMG algorithm implemented in MGOOD2 with one MG-cycle per grid level; hence we count computing times in multiples of the time needed by this code to compute $\approx u_h^*$. Both references are used separately for finest grids with $h = \frac{1}{16}$ and $h = \frac{1}{32}$. All accuracies refer to the distance from the true solution (6.2); thus, in terms of the reference system for $h = \frac{1}{16}$, u_h^* for $h = \frac{1}{32}$ would have an accuracy of $\approx .25$ and need an effort ≈ 4.

In Table 1, implementation (iia) employs a-priori smoothing w.r.t. the $O(h^4)$-discretization by means of one $\frac{5}{8}$ - damped Jacobi relaxation (cf. section 4), implementation (iib) uses one "4 color relaxation". In implementation (iii), one $\frac{1}{2}$ - damped Jacobi relaxation w.r.t. the $O(h^2)$-discretization is used. Implementation (iv) employs only the original smoothing of the MG cycle, cf. section 3.

The following facts are obvious from table 1:

- All versions lead to an $O(h^4)$ approximation of U, but only (i) and (ii) produce the solution determined by the defect operation.
- While the fixed point shift is harmless for version (iii), it is quite significant for version (iv) (which had been suggested by Hackbusch and Brandt/Dinar).
- The improvement in convergence with a-priori smoothing is significant With a suitable smoother (e.g. (iib)), one V-cycle per grid level in an FMG-algorithm suffices for a close approximation of the $O(h^4)$-solution determined by the defect operation. Without a-priori smoothing, two V-cycles per grid level are necessary, which is more expensive.
- Defect correction achieves more accuracy than further grid refinement at less cost. Note that - at $h = \frac{1}{32}$ - the 50% time increase (over the original MGOOD2) of implementation (iib) buys a reduction of the max

error of the computed approximation of (6.2) by a factor of 150-200, from a poor $.5 \times 10^{-2}$ to a decent $.3 \times 10^{-4}$!

We also determined contraction rates by iterating the MG cycles on a fixed grid ($h = \frac{1}{32}$). Table 2 contains the reduction of the max error within the 5th cycle. The numbers have to be interpreted cautiously because in all versions the rates were much lower in the first cycles, probably due to the smooth error of our initial approximation $u_h \equiv 0$.

Finally, we tested the approach of section 5 on problem (5.5), again on the basis of the code MGOOD2. No a-priori smoothing was used (cf. section 5). Both $\tilde{p} = 1$ and $\tilde{p} = 1.8$ were tried.

The early contraction rate was $\approx .1$ for $p = 1$ and $\approx .05$ for $p = 1.8$, which confirms our considerations. In the FMG mode, with one V-cycle per grid level, however, defect correction with $\tilde{p} = 1$ produced a better approximation to the exact $O(h^2)$-solution than with $\tilde{p} = 1.8$.

In any case, the truncation error level was reached with one V-cycle per grid level. Although we have run no test, we believe that the time needed by our DCMG-version of a linear MG code remains below that of a nonlinear FAS-code for the same job.

Conclusions

We have considered some ways how defect correction may be combined with multigrid algorithms. We have indicated how effective algorithms may be designed and what is necessary to obtain the full power of the approach. Such algorithms need only minor modifications of standard multigrid software for linear problems.

Since several such modifications may be standarized they could be built into future multigrid software as modes or options. This would greatly increase the range and flexibility of such software at the cost of little extra code.

DCMG - Implementation

		(i)		(iia)		(iib)		(iii)		(iv)	
	NCYCL	Accuracy	Effort	Acc.	Eff.	Acc.	Eff.	Acc.	Eff.	Acc.	Eff.
$h = \frac{1}{16}$	1	.3459	1.35	.1001	1.53	.0308	1.49	.1218	1.45	.3518	1.35
	2	.0268	1.79	.0233	2.07	.0245	2.08	.0393	1.89	.3720	1.78
T.E. = $.1980_{10}{}^{-1}$	3	.0248	2.14	.0243	2.63	.0244	2.62	.0376	2.47	.3825	2.14
$h = \frac{1}{32}$	1	.0980	1.33	.0078	1.52	.0050	1.52	.0097	1.47	.2664	1.42
	2	.0160	1.71	.0066	2.10	.0067	2.03	.0077	2.00	.1021	1.91
T.E. = $.4797_{10}{}^{-2}$	3	.0070	2.10	.0067	2.78	.0067	2.66	.0076	2.43	.0984	2.35

Table 1

MGOOD2	(i)	(iia)	(iib)	(iii)	(iv)
.108	.299	.073	.106	.300	.068

Table 2: Contraction rates

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