Modified Defect Correction Algorithms for ODEs.
Part I: General Theory

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Abstract

The well-known method of Iterated Defect Correction (IDeC) is based on the following idea: Compute a simple, basic approximation and form its defect w.r.t. the given ODE via a piecewise interpolant. This defect is used to define an auxiliary, neighboring problem whose exact solution is known. Solving the neighboring problem with the basic discretization scheme yields a global error estimate. This can be used to construct an improved approximation, and the procedure can be iterated. The fixed point of such an iterative process corresponds to a certain collocating solution.

We present a variety of modifications to this algorithm. Some of these have been proposed only recently, and together they form a family of iterative techniques, each with its particular advantages. These modifications are based on techniques like defect quadrature (IQDeC), defect interpolation (IPDeC), and combinations thereof. We investigate the convergence on (locally) equidistant and nonequidistant grids and show how superconvergent approximations can be obtained. Numerical examples illustrate our considerations.

The application to stiff initial value problems will be discussed in Part II of this paper.

1 Introduction

Since its introduction in the 1970's, cf. e.g. [7], [12], [13], the idea of iterated defect correction (IDeC) has been successfully applied to various classes of differential equations.

The purpose of this work is to describe some useful, recently developed modifications of the IDeC procedure, and to present them in a unified framework. In particular, their performance when applied to stiff problems will be discussed. The material is subdivided into two papers. In present Part I we describe the algorithmic details of the major variants of the procedures and and present a convergence theory valid for smooth problems, with a moderately Lipschitz continuous right hand side. Convergence proofs for two main versions are given in Appendix A. Here we will to some extent refer to previous work, cf. [4], [11], but we try keep the presentation self-contained. Numerical experiments illustrate our results.

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In principle, the algorithms developed here can be applied to linear or nonlinear ODE systems with initial or boundary conditions. In this paper, we consider the case of initial value problems (IVPs) in view of Part II [3], which will be devoted to stiff IVPs.

2 The method of Iterated Defect Correction (IDeC)

2.1 Review of basic ideas and definitions

Consider a system of ordinary differential equations (ODE) with given initial condition,

$$y'(t) = f(t, y(t)), \tag{2.1a}$$
$$y(t_0) = y_0, \tag{2.1b}$$

and exact solution $y^* : [t_0, t_{end}] \to \mathbb{R}^n$. Assume that a first approximation (grid function) $\{\eta_i^{[0]}\}$ has been computed on a given grid $\{t_i,\ i = 0, 1, \dots\}$ using a basic discretization scheme. We concentrate on a particular case, namely that the $\eta_i^{[0]}$ are computed by the backward Euler scheme,

$$\frac{\eta_i^{[0]} - \eta_{i-1}^{[0]}}{t_i - t_{i-1}} = f(t_i, \eta_i^{[0]}), \tag{2.2}$$

for $i = 1, 2, \dots$. This choice is not really essential; for other simple one-step schemes like the forward Euler scheme or the trapezoidal rule, for instance, the procedure can be realized in an analogous way. For the general framework see [12].

Now we assume that the integration interval $[t_0, t_{end}]$ is divided into subintervals $\mathbf{I}_j = [\mathbf{t}_{j-1}, \mathbf{t}_j],\ j = 1 \dots N$, each of them containing $m+1$ grid points (including the end points $\mathbf{t}_{j-1}, \mathbf{t}_j$). Let

$$\mathbf{h}_j := \mathbf{t}_j - \mathbf{t}_{j-1}, \qquad \text{and} \qquad \mathbf{h} := \max_j \mathbf{h}_j. \tag{2.3}$$

Let us briefly motivate and describe the classical IDeC procedure. The $\eta_i^{[0]}$ are interpolated by a continuous piecewise polynomial function,

$$p^{[0]}(t) := p_j^{[0]}(t), \qquad \text{for} \quad t \in \mathbf{I}_j, \tag{2.4}$$

where the $p_j^{[0]}$ are polynomials of degree $\le m$. The resulting *defect*,

$$d^{[0]}(t) := \frac{d}{dt} p^{[0]}(t) - f(t, p^{[0]}(t)), \tag{2.5}$$

which may expected to be small, is used to define a so-called *neighboring problem*,

$$y'(t) = f(t, y(t)) + d^{[0]}(t), \tag{2.6a}$$
$$y(t_0) = y_0, \tag{2.6b}$$

with exact solution $p^{[0]}(t)$, by construction. Solving (2.6) by means of the basic scheme (backward Euler) yields approximations $\pi_i^{[0]} \approx p^{[0]}(t_i)$ and a global error estimate

$$\text{error estimate} = \pi_i^{[0]} - p^{[0]}(t_i) \ \approx\ \eta_i^{[0]} - y^*(t_i) = \text{error}, \tag{2.7}$$

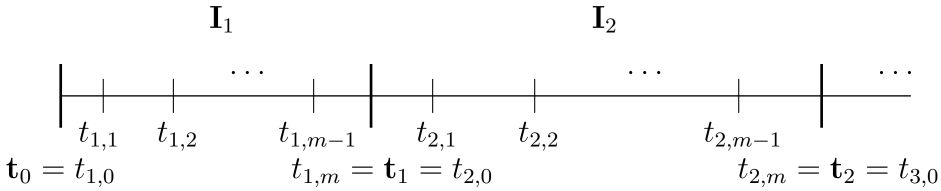

Figure 1: Grid and interpolation intervals.

which is used to define an improved approximation $\eta_i^{[1]} := \eta_i^{[0]} - (\pi_i^{[0]} - p^{[0]}(t_i))$ for the $y^*(t_i)$. This procedure can be continued inductively in a straightforward manner. Its general form is

$$\begin{aligned} \eta_i^{[\nu+1]} &:= \eta_i^{[0]} - (\pi_i^{[\nu]} - p^{[\nu]}(t_i)) \\ &= \eta_i^{[0]} - (\pi_i^{[\nu]} - \eta_i^{[\nu]}), \qquad \text{for} \quad \nu = 0, 1, \ldots . \end{aligned} \tag{2.8}$$

The $\pi_i^{[\nu]}$ are the approximate solutions (obtained by backward Euler) of the neighboring problems

$$y'(t) = f(t, y(t)) + d^{[\nu]}(t), \tag{2.9a}$$

$$y(t_0) = y_0, \tag{2.9b}$$

with the defect

$$d^{[\nu]}(t) := \frac{d}{dt} p^{[\nu]}(t) - f(t, p^{[\nu]}(t)), \tag{2.10}$$

where $p^{[\nu]}(t)$ interpolates the $\eta_i^{[\nu]}$.

In the sequel, we assume that the relative position of the interior points t_i is the same in each interval $\mathbf{I}_j$. It is characterized by $m+1$ parameters c_ℓ, $\ell = 0 \ldots m$, with $0 = c_0 < c_1 < \cdots < c_{m-1} < c_m = 1$. Accordingly, we use a notation for the grid points t_i involving a double index,

$$t_{j,\ell} := \mathbf{t}_{j-1} + c_\ell \, \mathbf{h}_j, \quad j = 1, 2, \ldots, \quad \ell = 0 \ldots m, \tag{2.11}$$

including the redundant but convenient notation $t_{j,m} \equiv \mathbf{t}_m \equiv t_{j+1,0}$, cf. Figure 1. For the 'inner stepsizes' (on the t-grid) we write

$$h_{j,\ell} := t_{j,\ell} - t_{j,\ell-1} = (c_\ell - c_{\ell-1}) \mathbf{h}_j. \tag{2.12}$$

Unless otherwise stated, the index range is always $j = 1 \ldots N$, $\ell = 1 \ldots m$. In this sense, it is clear what $\mathbf{h} \to 0$ means and how an asymptotic relation $\mathcal{O}(\mathbf{h}^{\cdots})$ is to be interpreted (cf. e.g. (2.17) below).

With this denotation, the backward Euler equations defining the $\pi_i^{[\nu]} \equiv \pi_{j,\ell}^{[\nu]}$ read

$$\frac{\pi_{j,\ell}^{[\nu]} - \pi_{j,\ell-1}^{[\nu]}}{h_{j,\ell}} = f(t_{j,\ell}, \pi_{j,\ell}^{[\nu]}) + d^{[\nu]}(t_{j,\ell}), \qquad \text{for} \quad \nu = 0, 1, \ldots, \tag{2.13}$$

with the defect values

$$d^{[\nu]}(t_{j,\ell}) = \frac{d}{dt} p^{[\nu]}(t_{j,\ell}) - f(t_{j,\ell}, p^{[\nu]}(t_{j,\ell})). \tag{2.14}$$

The resulting approximations are

$$\eta_{j,\ell}^{[\nu+1]} := \eta_{j,\ell}^{[0]} - (\pi_{j,\ell}^{[\nu]} - \eta_{j,\ell}^{[\nu]}), \qquad \text{for} \quad \nu = 0, 1, \ldots, \tag{2.15}$$

cf. (2.8).

Remark 1 The IDeC iteration may be organized in a slightly different way based on local instead of global error estimates ('version B' in the terminology of Stetter [12]). Moreover, the process may be restarted at the begin of each interval I_j by setting $\eta_{j,0} := \eta^{[K]}_{j-1,m}$, assuming that K IDeC steps have been performed in the interval I_{j-1}. We do not consider these modifications here, because they are usually less effective for stiff problems (cf. e.g. [2]).

We will illustrate our results using the (nonstiff) nonlinear ODE system

$$\begin{aligned} y_1' &= -y_2 + y_1(1-y_1^2-y_2^2), \\ y_2' &= y_1 + 3y_2(1-y_1^2-y_2^2), \end{aligned} \tag{2.16}$$

as a test example, with initial condition $y(0) = (1,0)^T$ and exact solution $y^*(t) = (\cos t, \sin t)^T$. All numerical experiments were performed in MATLAB 6 using IEEE double precision arithmetic and Newton iteration with stringent tolerances.

2.2 Convergence behavior on equidistant and nonequidistant grids

It is well known that, under appropriate smoothness assumptions, the IDeC scheme is convergent for $\mathbf{h} \to 0$. This means that if, for instance, an Euler scheme is used as the basic discretization on a *piecewise equidistant* grid (equidistant in each $\mathbf{I}_j$), and if m is the degree of interpolation, then

$$\| \eta^{[\nu]}_{j,\ell} - y^*(t_{j,\ell}) \| = \mathcal{O}(\mathbf{h}^{\nu+1}), \qquad \text{for} \quad \nu = 0 \ldots m-1, \tag{2.17}$$

and $\mathcal{O}(\mathbf{h}^m)$ is the maximal achievable order in general. The original proof of (2.17), cf. e.g. [7], is based on an asymptotic expansion for the global error of the basic scheme. Results for other basic schemes, including faster convergence rates depending on the basic order, have also been obtained.

Remark 2 Whenever we use the symbol $\mathcal{O}(\mathbf{h}^{\cdots})$, this means the 'classical order' of a discretization method, i.e. the asymptotic order for $\mathbf{h} \to 0$ in the conventional sense. In particular, the error constant involved is allowed to depend on the right hand side f, in particular on a Lipschitz constant for f, and on certain higher derivatives of f and y^* which are assumed to be sufficiently smooth. The convergence results presented in this paper (Propositions 1 and 2) are to be understood in this sense. In the same spirit, we use the denotation $\mathcal{C}$ for generic constants.

For stiff equations this classical order is usually not relevant, because order reductions occur. Therefore the theoretical results presented in this paper have no immediate consequence for stiff problems, which will be investigated in Part II [3].

Concerning convergence results like (2.17) it is important to note that, for the classical IDeC method, the assumption of piecewise equidistant grids is essential, i.e., the stepsize for the basic discretization (Euler) is required to be constant in each of the subintervals $\mathbf{I}_j$. This fact is remarkable in the sense that the heuristic idea underlying the IDeC procedure is independent of the particular grid used. However, the above result is sharp, as a simple experiment carried out on a nonequidistant grid shows.

Example 1 We consider the test problem (2.16) and divide the interval $[t_0, t_{end}] = [0,3]$ into $n = 3/\mathbf{h}$ subintervals $\mathbf{I}_j$ of equal length $\mathbf{h}_j \equiv \mathbf{h}$. We choose $m = 4$, and in each $\mathbf{I}_j$ we apply the backward Euler scheme (BEUL) with *nonequidistant* interior grid points chosen 'at random': $t_{j,\ell} = \mathbf{t}_{j-1} + c_\ell \mathbf{h}$ with $c_1 = 0.0185,\ c_2 = 0.4565,\ c_3 = 0.7721,\ c_4 = 1.0$.

Table 1 shows the resulting global errors and the observed orders for the IDeC iterates at $T = 3.0$. Furthermore, the column labelled COLL shows the results for the collocation solution of degree 4 (piecewise polynomial function satisfying the given ODE at the grid points $t_{j,\ell}$) which is the fixed point of the IDeC iteration, cf. Section 3.

We observe that the first IDeC step results in a slight increase of accuracy, but further iterations stall, and no increase of order compared to the basic order 1 is observed.

n	**h**	BEUL	IDeC 1	IDeC 2	IDeC 3	IDeC 4	COLL
15	0.200	1.19E-02	1.26E-03	4.20E-03	3.31E-03	3.71E-04	1.07E-06
30	0.100	6.07E-03	3.42E-04	1.36E-03	1.55E-03	2.04E-04	6.68E-08
60	0.050	3.06E-03	1.03E-04	4.81E-04	7.64E-04	1.04E-04	4.17E-09
120	0.025	1.54E-03	3.66E-05	1.94E-04	3.83E-04	5.88E-05	2.61E-10
15	0.200						
		0.98	1.88	1.63	1.10	0.87	4.00
30	0.100						
		0.99	1.73	1.50	1.02	0.96	4.00
60	0.050						
		0.99	1.49	1.31	1.00	0.83	4.00
120	0.025						

Table 1: IDeC results for Example 1 ('random' grid)

In the following sections we will consider certain modifications of the IDeC procedure, overcoming the restriction of piecewise equidistant grids. Here, the structure (2.15) of the iterative scheme remains the same as for the conventional IDeC algorithm, but the defect terms $d^{[\nu]}(t_{j,\ell})$ from (2.13) will be replaced by alternative variants. In all these cases, we will use the notation $\eta_{j,\ell}^{[\nu]}$, $\pi_{j,\ell}^{[\nu]}$ and $p^{[\nu]}(t)$ introduced above, but we will indicate the modified defect versions using different accents, cf. (3.9), (4.1), (5.2), (5.6).

3 IDeC with defect quadrature (IQDeC)

3.1 IDeC and collocation

The IDeC scheme introduced in Section 2 is an iterative nonlinear correction scheme based on residuals (= defects). Iterative schemes of this type are frequently used in the design of numerical algorithms. A characteristic property of the IDeC method is that, due to the interpolation $p^{[\nu]}(t_{j,\ell}) = \eta_{j,\ell}^{[\nu]}$, the given ODE (2.1) is directly involved in the definition of the defect. The resulting neighboring problems (2.9) are perturbed versions of (2.1).

This way of using the defect is related to the idea of *collocation*: The fixed point of an IDeC iteration (on a given, fixed grid) is characterized by the property that the defect vanishes at certain grid points. Let $p^{[C]}(t)$ denote the continuous collocating solution defined by

$$p^{[C]}(t) := p_j^{[C]}(t), \qquad \text{for} \quad t \in \mathbf{I}_j, \tag{3.1}$$

where the $p_j^{[C]}(t)$ are polynomials of degree $\leq m$ satisfying the collocation relations

$$\frac{d}{dt}\, p_j^{[C]}(t_{j,\ell}) = f(t_{j,\ell}, p_j^{[C]}(t_{j,\ell})), \quad j = 1 \ldots N, \quad \ell = 1 \ldots m. \tag{3.2}$$

Collocation solutions feature the uniform classical order $\mathcal{O}(\mathbf{h}^m)$, cf. e.g. [8]. Relation (3.2) means that the defect of $p^{[C]}(t)$ vanishes at all these grid points, and thus an IDeC step starting from the grid function $\{p^{[C]}(t_{j,\ell})\}$ maps it onto itself (cf. (2.8), (2.13)). Thus, IDeC may simply be considered as an iterative collocation solver.

Except for special choices of the collocation nodes, the order of the global error of such a collocation is $\mathcal{O}(\mathbf{h}^m)$, and this order is also achieved by the IDeC method on the piecewise equidistant grid after $m-1$ iterations (cf. (2.17)). However, as Example 1 shows, this is *not* the case for general, nonequidistant grids.

3.2 IDeC, Runge-Kutta, and local defect quadrature

It is well known that collocation methods form a subclass of implicit Runge-Kutta methods (cf. e.g. [9]). Here, we consider the collocation method from (3.2) and identify the collocation solution $p^{[C]}(t)$ with a mesh function $\{\mathbf{p}_j \equiv p^{[C]}(\mathbf{t}_j)\}$ on the mesh $\{\mathbf{t}_j\}$, while the intermediate values $p_{j,\ell} = p^{[C]}(t_{j,\ell})$ play the role of the intermediate unknowns in the corresponding m-stage Runge-Kutta system

$$p_{j,\ell} \;=\; \mathbf{p}_{j-1} + \mathbf{h}_j \sum_{\mu=1}^{m} a_{\ell,\mu}\, f(t_{j,\mu}, p_{j,\mu}), \quad \ell = 1 \ldots m, \tag{3.3a}$$

$$\mathbf{p}_j \;:=\; \mathbf{p}_{j-1} + \mathbf{h}_j \sum_{\ell=1}^{m} b_\ell\, f(t_{j,\ell}, p_{j,\ell}), \tag{3.3b}$$

for $j = 1, 2, \ldots$. As usual, we characterize such a scheme by its so-called Butcher array,

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \;=\; \begin{array}{c|ccc} c_1 & a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & & \vdots \\ c_m & a_{m,1} & \cdots & a_{m,m} \\ \hline & b_1 & \cdots & b_m \end{array} \tag{3.4}$$

Here we have $c_m = 1$, $b_\ell \equiv a_{m\ell}$, $\mathbf{p}_j = p_{j,m}$ because the endpoint $\mathbf{t}_j = t_{j,m}$ is assumed to be a collocation node (cf. (3.2)). In the context of collocation, the $a_{\ell,\mu}$ have a straightforward interpretation as interpolatory quadrature coefficients (cf. e.g. [9]).

As already discussed in [12], IDeC can in a more abstract setting be interpreted as an iterative method to solve a high order scheme $F(p) = 0$ (in our case: collocation) by means of a simpler scheme $\tilde{F}(\eta) = 0$, (in our case: backward Euler). We now adopt this more abstract point of view, and identify $F(\mathbf{p})$ not with the original collocation formulation (3.2) as before, but with its Runge-Kutta equivalent (3.3) and check the outcome. Still we have to be careful: The ansatz for the iteration makes sense only if $\tilde{F}$ is a reasonable approximation for F at each grid point. Therefore we rewrite the Runge-Kutta equations (3.3) by means of a linear combination of the stages in the following way:

$$\frac{p_{j,\ell} - p_{j,\ell-1}}{h_{j,\ell}} = \sum_{\mu=1}^{m} \alpha_{\ell,\mu}\, f(t_{j,\mu}, p_{j,\mu}), \quad \ell = 1 \ldots m, \tag{3.5}$$

and $\mathbf{p}_j := p_{j,m}$. Here the coefficients $\alpha_{\ell,\mu}$ are given by

$$\begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,m} \\ \alpha_{2,1} & \cdots & \alpha_{2,m} \\ \vdots & & \vdots \\ \alpha_{m,1} & \cdots & \alpha_{m,m} \end{bmatrix} = \begin{bmatrix} \frac{a_{1,1}}{c_1} & \cdots & \frac{a_{1,m}}{c_1} \\ \frac{a_{2,1}-a_{1,1}}{c_2-c_1} & \cdots & \frac{a_{2,m}-a_{1,m}}{c_2-c_1} \\ \vdots & & \vdots \\ \frac{a_{m,1}-a_{m-1,1}}{c_m-c_{m-1}} & \cdots & \frac{a_{m,m}-a_{m-1,m}}{c_m-c_{m-1}} \end{bmatrix} \tag{3.6}$$

Comparing (3.5) with the backward Euler scheme on the grid $\{t_{j,\ell}\}$,

$$\frac{\eta_{j,\ell}-\eta_{j,\ell-1}}{h_{j,\ell}} = f(t_{j,\ell},\eta_{j,\ell}), \tag{3.7}$$

we see that the left hand sides, i.e. the discretizations of the first derivative, coincide. The schemes differ only by their weighting of f-values on the right hand side (trivial weighting for backward Euler). Due to the nature of the $a_{\ell,\mu}$, the weights $\alpha_{\ell,\mu}$ are easy to characterize:

Lemma 1 *The $\alpha_{\ell,\mu}$ from* (3.6) *are the unique coefficients for which*

$$\sum_{\mu=1}^{m} a_{\ell,\mu}\, q(c_\mu) = \frac{1}{c_\ell - c_{\ell-1}} \int_{c_{\ell-1}}^{c_\ell} q(s)\, ds, \quad \ell = 1 \ldots m, \tag{3.8}$$

holds for arbitrary polynomials $q(s)$ of degree $\leq m-1$.

The equations (3.5) are locally integrated (weighted) versions of the collocation equations (3.2), where on the right hand side the set of quadrature rules given by the $\alpha_{\ell,\mu}$ has been used. With collocation rewritten in this way, we now consider the following modification of the IDeC procedure based on the backward Euler scheme:

Definition 1 (IDeC with defect quadrature, IQDeC) Consider the IDeC method described in Section 2 and modify it as follows: Replace the (pointwise) defect values $d^{[\nu]}(t_{j,\ell})$ (cf. (2.14)) by the residuals of the $p_{j,\ell}^{[\nu]} := p^{[\nu]}(t_{j,\ell}) \equiv \eta_{j,\ell}^{[\nu]}$ with respect to the modified Runge-Kutta equations (3.5),

$$\begin{aligned} \bar{d}_{j,\ell}^{[\nu]} &:= \frac{p_{j,\ell}^{[\nu]} - p_{j,\ell-1}^{[\nu]}}{h_{j,\ell}} - \sum_{\mu=1}^{m} \alpha_{\ell,\mu}\, f(t_{j,\mu}, p_{j,\mu}^{[\nu]}) \\ &= \frac{\eta_{j,\ell}^{[\nu]} - \eta_{j,\ell-1}^{[\nu]}}{h_{j,\ell}} - \sum_{\mu=1}^{m} \alpha_{\ell,\mu}\, f(t_{j,\mu}, \eta_{j,\mu}^{[\nu]}), \quad \ell = 1 \ldots m. \end{aligned} \tag{3.9}$$

We call such a method IQDeC: IDeC with defect quadrature.

All other algorithmic details remain unchanged. In particular, the $\eta_{j,\ell}^{[\nu]}$ are computed as before (cf. (2.15)), with neighboring solutions $\pi_{j,\ell}^{[\nu]}$ defined by

$$\frac{\pi_{j,\ell}^{[\nu]} - \pi_{j,\ell-1}^{[\nu]}}{h_{j,\ell}} = f(t_{j,\ell}, \pi_{j,\ell}^{[\nu]}) + \bar{d}_{j,\ell}^{[\nu]}. \tag{3.10}$$

This is the analog of (2.13), with $\bar{d}_{j,\ell}^{[\nu]}$ replacing the pointwise defect $d^{[\nu]}(t_{j,\ell})$.

The terminology 'defect quadrature' is motivated by the fact that, due to Lemma 1, the $\bar{d}_{j,\ell}^{[\nu]}$ can also be written as the weighted sums of the conventional defect values $d^{[\nu]}(t_{j,\ell})$, see (2.14):

$$\bar{d}_{j,\ell}^{[\nu]} = \sum_{\mu=1}^{m} \alpha_{\ell,\mu} \frac{d}{dt} p^{[\nu]}(t_{j,\mu}) - \sum_{\mu=1}^{m} \alpha_{\ell,\mu} f(t_{j,\mu}, p_{j,\mu}^{[\nu]}) = \sum_{\mu=1}^{m} \alpha_{\ell,\mu} d^{[\nu]}(t_{j,\ell}), \quad \ell = 1 \dots m. \tag{3.11}$$

This may also be interpreted as a kind of defect preconditioning.

For practical computation of the $\bar{d}_{j,\ell}^{[\nu]}$, (3.9) is preferable. (For any given distribution of the $t_{j,\ell}$, the coefficients $\alpha_{\ell,\mu}$ are easy to compute.)

Example 2 We repeat the experiment from Example 1, on the same nonequidistant grid, with IDeC replaced by the modified version IQDeC.

Table 2 shows the resulting global errors and the observed orders for the IQDeC iterates at $T = 3.0$. Obviously, the order sequence is $1, 2, 3, 4$. After 3 iteration steps, the order $m = 4$ of the collocation scheme (fixed point) has been reached. The fourth IQDeC iteration step yields no further increase in order but takes the iterates very close to the fixed point. This behavior is completely analogous to that of the conventional IDeC iteration on a piecewise equidistant grid.

n	**h**	BEUL	IQDeC 1	IQDeC 2	IQDeC 3	IQDeC 4	COLL
15	0.200	1.19E-02	2.44E-03	7.31E-05	7.98E-06	1.10E-06	1.07E-06
30	0.100	6.07E-03	5.99E-04	8.10E-06	4.94E-07	6.66E-08	6.68E-08
60	0.050	3.06E-03	1.48E-04	9.65E-07	3.07E-08	4.15E-09	4.17E-09
120	0.025	1.54E-03	3.69E-05	1.18E-07	1.91E-09	2.60E-10	2.61E-10
15	0.200						
		0.98	2.03	3.17	4.01	4.05	4.00
30	0.100						
		0.99	2.01	3.07	4.01	4.01	4.00
60	0.050						
		0.99	2.01	3.03	4.01	4.00	4.00
120	0.025						

Table 2: IQDeC results for Example 2 ('random' grid)

3.3 IQDeC convergence

The robust convergence behavior of the IQDeC iteration observed above generally holds for smooth problems, as the following result shows. It describes the asymptotic rate of convergence towards the fixed point $p^{[C]}$ for arbitrary grids.

Proposition 1 (IQDeC fixed point convergence) *Let $\eta_{j,\ell}^{[\nu]}$ denote the iterates of the* IQDeC *iteration according to Definition 1, based on the backward Euler scheme. On arbitrary grids of the type* (2.11)*, the error of the $\eta_{j,\ell}^{[\nu]} = p^{[\nu]}(t_{j,\ell})$ w.r.t. the corresponding fixed point $p^{[C]}(t)$ (collocation at $t_{j,\ell}$, $\ell = 1 \dots m$) satisfies*

$$\| \eta_{j,\ell}^{[\nu]} - p_{j,\ell}^{[C]} \| = \mathcal{O}(\mathbf{h}^{\nu+1}), \qquad \text{for all} \quad \nu \geq 0. \tag{3.12}$$

Proof: See Appendix A. □

As an immediate consequence of Proposition 1, the IQDeC errors $\eta_{j,\ell}^{[\nu]} - y^*(t_{j,\ell})$ satisfy

$$\| \eta_{j,\ell}^{[\nu]} - y^*(t_{j,\ell}) \| = \mathcal{O}(\mathbf{h}^{\nu+1}), \quad \text{for} \quad \nu = 0 \ldots m-1, \tag{3.13}$$

on arbitrary grids. The maximal achievable order is $\mathcal{O}(\mathbf{h}^m)$, which cannot be improved by further iteration in general.

Furthermore, due to Proposition 1, the IQDeC iteration may be applied in a way producing superconvergent solutions. Consider collocation schemes of the type Radau IIa or Gauss, for instance. Then we have $\|p^{[C]}(\mathbf{t}_j) - y^*(\mathbf{t}_j)\| = \mathcal{O}(\mathbf{h}^p)$ with $p = 2m-1$ or $p = 2m$, respectively (cf. [8], [9]). If we apply the IQDeC method an a grid where in each interval $\mathbf{I}_j$, the $t_{j,\ell}$ are the corresponding collocation nodes, the superconvergence order is attained at the points $\mathbf{t}_j$ after an appropriate number of IQDeC steps.

Example 3 We repeat the IQDeC experiment from Example 2, but with $m = 3$ and with $t_{j,\ell}$ corresponding to 'Radau right' nodes in each interval $\mathbf{I}_j$, i.e., $c_1 = (4-\sqrt{6})/10$, $c_2 = (4+\sqrt{6})/10$, $c_3 = 1$.

Table 3 shows the resulting global errors and the observed orders at $T = 3.0$. The order 5 is obtained after 4 iteration steps. This corresponds to the superconvergence order of the Radau IIa scheme with polynomial degree $m = 3$ which defines the fixed point of this iteration.

n	$\mathbf{h}$	BEUL	IQDeC 1	IQDeC 2	IQDeC 3	IQDeC 4	RADAU
15	0.200	1.40E-02	2.80E-03	7.56E-05	1.36E-05	1.72E-07	1.22E-07
30	0.100	6.99E-03	6.87E-04	8.82E-06	8.53E-07	6.36E-09	3.86E-09
60	0.050	3.51E-03	1.70E-04	1.09E-06	5.33E-08	2.10E-10	1.21E-10
120	0.025	1.76E-03	4.24E-05	1.36E-07	3.33E-09	6.68E-12	3.78E-12
15	0.200						
30	0.100	1.00	2.03	3.10	4.00	4.76	4.99
60	0.050	1.00	2.01	3.02	4.00	4.92	5.00
120	0.025	1.00	2.01	3.00	4.00	4.97	5.00

Table 3: IQDeC results for Example 3 (grid based on Radau IIa(3) nodes)

4 IDeC with defect interpolation (IPDeC)

4.1 Motivation and definition

In Section 3, we have shown how the IDeC method can be modified to converge on nonequidistant grids. This modification increases the grid flexibility, and it enables the realization of superconvergent approximations.

From an efficiency viewpoint, however, in particular concerning the application to stiff problems, a basic scheme operating on a nonequidistant grid is rather undesirable. For stiff problems, the use of an implicit basic scheme becomes indispensable, and in a practical implementation it will be important to optimize the effort in the linear algebra involved in a Newton iteration for the corresponding implicit equations. In particular, a LU-decomposition for the basic scheme is usually frozen as far as possible. But this is only possible as long as the stepsize does not change.

Therefore the question is how to design a version of the IDeC procedure where the basic scheme works on a piecewise equidistant grid but which converges towards a collocation scheme with nonequidistant nodes. This sounds paradoxical but can be realized in the following way. We use two different grids: A grid $\{t_{j,\ell}\}$ which is piecewise equidistant on the intervals $\mathbf{I}_j$, and a nonequidistant grid $\{\tilde{t}_{j,\kappa},\ \kappa = 1\ldots m\}$ where, for each j, the $\tilde{t}_{j,\kappa}$ correspond to the desired collocation nodes. Now we use both sets of grid points to design another variant of the IDeC procedure.

Definition 2 (IDeC with defect interpolation, IPDeC) Consider the IDeC method described in Section 2 and modify it as follows: Replace the (pointwise) defect values $d^{[\nu]}(t_{j,\ell})$ (cf. (2.14)) by the values $\tilde{d}^{[\nu]}(t_{j,\ell})$, where $\tilde{d}^{[\nu]}(t)$ is a piecewise polynomial function of degree $\leq m-1$ which interpolates $d^{[\nu]}(t)$ at the $\tilde{t}_{j,\kappa}$, i.e.,

$$\tilde{d}^{[\nu]}(t_{j,\ell}) \quad \text{is defined via} \quad \tilde{d}^{[\nu]}(\tilde{t}_{j,\kappa}) = d^{[\nu]}(\tilde{t}_{j,\kappa}), \quad \kappa = 1\ldots m. \tag{4.1}$$

We call such a method IPDeC: IDeC with defect interpolation.

Here the point is that due to (4.1), we have shifted the fixed point of our iteration: Consider an IPDeC step starting from the collocating solution $\tilde{p}^{[C]}(t)$ w.r.t. the collocation nodes $\tilde{t}_{j,\kappa}$ (continuous, piecewise polynomial of degree $\leq m$). Then, by definition, its defect $\frac{d}{dt}\,\tilde{p}^{[C]}(t) - f(t,\tilde{p}^{[C]}(t))$ vanishes for all $t = \tilde{t}_{j,\kappa}$. Therefore the resulting interpolated defect is identically zero, and so the IPDeC step starting from the grid function $\{\tilde{p}^{[C]}(t_{j,\ell})\}$ will map it onto itself. This means that $\{\tilde{p}^{[C]}(t_{j,\ell})\}$ *is a fixed point of the* IPDeC *iteration.*

This procedure may be interpreted as an alternative way of defect preconditioning, i.e., the $\tilde{d}^{[\nu]}(t_{j,\ell})$ can be written in the form (3.11) but with other coefficients $\alpha_{\ell,\mu}$ defined by (4.1).

Next we study the convergence of IPDeC towards its fixed point.

4.2 IPDeC convergence

For the IPDeC iteration, the analog of Proposition 1 holds.

Proposition 2 (IPDeC fixed point convergence) *Let $\eta_{j,\ell}^{[\nu]}$ denote the iterates of the IPDeC iteration according to Definition 2, based on the backward Euler scheme on a piecewise equidistant grid $\{t_{j,\ell}\}$. Moreover, let the defect be interpolated at arbitrary nodes $\tilde{t}_{j,\kappa} \in \mathbf{I}_j$. Then the error of the $\eta_{j,\ell}^{[\nu]} = p^{[\nu]}(t_{j,\ell})$ w.r.t. the corresponding fixed point $\tilde{p}^{[C]}(t)$ (collocation at $\tilde{t}_{j,\kappa}$, $\kappa = 1\ldots m$) satisfies*

$$\|\, \eta_{j,\ell}^{[\nu]} - \tilde{p}_{j,\ell}^{[C]} \,\| = \mathcal{O}(\mathbf{h}^{\nu+1}), \qquad \text{for all} \quad \nu \geq 0. \tag{4.2}$$

Proof: See Appendix A. □

Concerning [super]convergence towards $y^*(t)$, the same conclusions apply as for the IQDeC case (see Subsection 3.3).

Example 4 We illustrate the behavior of the IPDeC iteration by means of test problem (2.16). We choose $m = 3$ and execute the backward Euler scheme on an equidistant mesh first. In each iteration step, the defect $d(t)$ is replaced by its interpolant $\tilde{d}(t)$ of degree ≤ 2, with interpolation at the Radau nodes $\tilde{t}_{j,\kappa}$ which have played the role of the $t_{j,\ell}$ in Example 3.

Table 4 shows the resulting convergence history, which is qualitatively the same as for the IQDeC iteration from Example 3: Again the superconvergence order 5 of the fixed point is reproduced after 4 iteration steps.

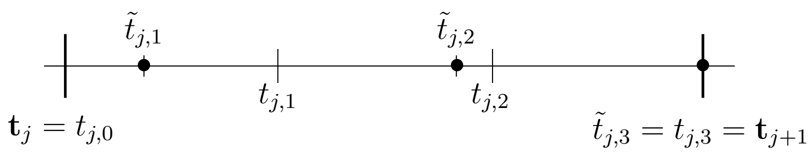

Figure 2: Equidistant and RadauIIa(3) nodes.

n	**h**	BEUL	IPDeC 1	IPDeC 2	IPDeC 3	IPDeC 4	RADAU
15	0.200	1.20E-02	9.13E-04	1.62E-04	1.50E-05	1.84E-06	1.22E-07
30	0.100	6.00E-03	2.47E-04	2.25E-05	1.14E-06	6.79E-08	3.86E-09
60	0.050	3.00E-03	6.41E-05	2.96E-06	7.82E-08	2.30E-09	1.21E-10
120	0.025	1.50E-03	1.63E-05	3.79E-07	5.10E-09	7.47E-11	3.78E-12
15	0.200						
30	0.100	1.00	1.89	2.85	3.71	4.76	5.26
60	0.050	1.00	1.95	2.93	3.87	4.88	5.22
120	0.025	1.00	1.97	2.96	3.94	4.94	5.12

Table 4: IPDeC results for Example 4 (equidistant combined with Radau IIa(3) grid)

Remark 3 Our main motivation for introducing IPDeC was its application to stiff problems. In contrast to equidistant collocation, Gauss or Radau schemes are superconvergent in the classical sense and have excellent stability properties also in the stiff case, and therefore the $\tilde{t}_{j,\kappa}$ will be chosen accordingly. Concerning the presence or failure of superconvergence in the stiff case, we refer the reader to Part II [3].

5 Further variants

We have not tried to describe our family of I*DeC methods in full generality. Rather, we conclude this presentation with a discussion of some further useful variants. These arise from an alternative choice of quadrature or interpolation nodes and of the underlying discretization scheme, and include a symmetric IPDeC method resulting in a faster convergence with an order sequence $2, 4, \ldots$. We show that this version is closely related to a defect correction method proposed and analyzed in [11]. In the final subsection we briefly remark on IQDeC as the basis for a-posteriori error estimation.

5.1 IQDeC with $m+1$ quadrature nodes

Let us take another look at the defect (3.9) in the definition of the IQDeC procedure. There, the $\alpha_{\ell,\mu}$ were chosen as the quadrature coefficients according to Lemma 1, based on the quadrature nodes $t_{j,\mu},\ \mu = 1 \ldots m$. But it is also perfectly reasonable to include the leftmost point $\mathbf{t}_{j-1} = t_{j,0}$ as a further quadrature node, and to use a set of coefficients $\{\hat{\alpha}_{\ell,\mu},\ \ell = 1 \ldots m,\ \mu = 0 \ldots m\}$ corresponding to m quadrature rules of higher precision. Here the $\hat{\alpha}_{\ell,\mu}$ are uniquely characterized by the property that

$$\sum_{\mu=0}^{m} \hat{\alpha}_{\ell,\mu}\, q(c_\mu) = \frac{1}{c_\ell - c_{\ell-1}} \int\limits_{c_{\ell-1}}^{c_\ell} q(s)\, ds, \quad \ell = 1 \ldots m, \tag{5.1}$$

holds for arbitrary polynomials $q(s)$ of degree $\leq m$.

This choice leads to a modified IQDeC algorithm, where the $\bar{d}_{j,\ell}^{[\nu]}$ from (3.9) are replaced by

$$\hat{d}_{j,\ell}^{[\nu]} := \frac{\eta_{j,\ell}^{[\nu]} - \eta_{j,\ell-1}^{[\nu]}}{h_{j,\ell}} - \sum_{\mu=0}^{m} \hat{\alpha}_{\ell,\mu}\, f(t_{j,\mu}, \eta_{j,\mu}^{[\nu]}), \quad \ell = 1 \ldots m. \tag{5.2}$$

It can be shown that this modified IQDeC iteration has essentially the same convergence properties as the original IQDeC version. The fixed point is a Runge-Kutta solution which can again be identified with a collocation polynomial, but not in a completely straightforward way (cf. [4]). Since the asymptotic accuracy of this fixed point is higher as for the original IQDeC version, the iterates $\eta_{j,\ell}^{[\nu]}$ of the modified IQDeC iteration can be shown to satisfy

$$\| \, \eta_{j,\ell}^{[\nu]} - y^*(t_{j,\ell}) \, \| = \mathcal{O}(\mathbf{h}^{\nu+1}), \quad \text{for} \quad \nu = 0 \ldots m, \tag{5.3}$$

which is to be compared with (3.13). Here, the best possible order of accuracy is $m+1 > m$, and it is achieved after m iteration steps. For further details concerning this modified IQDeC procedure we refer to [4], where also a proof of (5.3) can be found in the context of boundary value problems. This is related to the proof of Proposition 1 given in Appendix A.

Example 5 We repeat the simple experiment from Example 1, on the same nonequidistant grid, with the modified version of IQDeC including the left end points $\mathbf{t}_j = t_{j,0}$ of the I_j as quadrature nodes. This enables a direct comparison with Tables 1 and 2.

Table 5 displays the resulting global errors and the observed orders. Indeed, we observe order 5 after 4 iterations steps, which corresponds to the order of the Runge-Kutta scheme defining the fixed point of the iteration.

n	$\mathbf{h}$	BEUL	IQDeC 1	IQDeC 2	IQDeC 3	IQDeC 4
15	0.200	1.19E-02	1.92E-03	7.34E-05	6.13E-06	1.62E-07
30	0.100	6.04E-03	4.80E-04	8.58E-06	3.85E-07	4.68E-09
60	0.050	3.05E-03	1.20E-04	1.05E-06	2.41E-08	1.47E-10
120	0.025	1.53E-03	3.01E-05	1.30E-07	1.51E-09	4.71E-12

15	0.200					
30	0.100	0.98	2.00	3.10	3.99	5.12
60	0.050	0.99	2.00	3.03	4.00	4.99
120	0.025	0.99	2.00	3.01	4.00	4.97

Table 5: IQDeC results for Example 5 ('random' grid, defect according to (5.2))

5.2 Symmetric IPDeC

The IDeC method and its descendants can be adapted to work with different basic discretizations. So far we have only used an Euler scheme. We now consider the IPDeC method as introduced before but we use a basic method of order 2, namely the implicit trapezoidal rule (ITR) working on locally equidistant $t_{j,\ell}$. (The implicit midpoint rule would be another natural candidate.) Furthermore

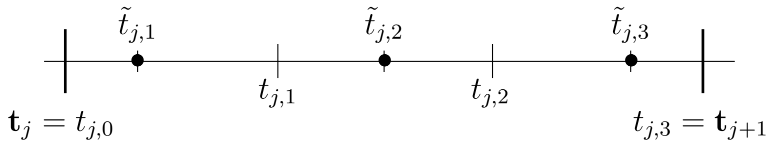

Figure 3: Equidistant and Gauss(3) nodes.

we choose a symmetric distribution for the quadrature nodes $\tilde{t}_{j,\kappa}$, i.e. $\tilde{t}_{j,\kappa} - \mathbf{t}_{j-1} = \mathbf{t}_j - \tilde{t}_{m+1-k}$ for $\kappa = 1 \ldots m$, and expect in this case an asymptotically faster convergence towards the fixed point.

The neighboring solutions $\pi^{[\nu]}_{j,\ell}$ are now computed from

$$\frac{\pi^{[\nu]}_{j,\ell} - \pi^{[\nu]}_{j,\ell-1}}{h_{j,\ell}} = \frac{1}{2}\big(f(t_{j,\ell}, \pi^{[\nu]}_{j,\ell-1}) + f(t_{j,\ell}, \pi^{[\nu]}_{j,\ell})\big) + \frac{1}{2}\big(\tilde{d}^{[\nu]}(t_{j,\ell-1}) + \tilde{d}^{[\nu]}(t_{j,\ell})\big), \tag{5.4}$$

with the interpolated defect $\tilde{d}^{[\nu]}(t)$ according to (4.1). The original fixed point argument from Subsection 4.1 remains valid, because the inhomogeneous term in (5.4) vanishes for $\tilde{d}^{[\nu]}(t) \equiv 0$. In particular, if the $\tilde{t}_{j,\kappa}$ are chosen as Gauss nodes (transformed zeros of the Legendre polynomials of degree m), then the fixed point of the IPDeC iteration corresponds to Gauss collocation with the optimal classical (super)convergence order $2m$.

Example 6 Table 6 shows the numerical results for the symmetric IPDeC iteration with $m = 3$ and Gaussian interpolation nodes (see Figure 3). The problem data are the same as before (see Example 1).

The optimal order $2m = 6$ is achieved after $m = 3$ iteration steps.

n	**h**	ITR	IPDeC 1	IPDeC 2	IPDeC 3	GAUSS
15	0.200	1.11E-03	1.29E-06	2.07E-08	1.75E-09	1.79E-09
30	0.100	2.78E-04	8.06E-08	3.26E-10	2.87E-11	2.88E-11
60	0.050	6.94E-05	5.04E-09	5.10E-12	4.53E-13	4.54E-13
120	0.025	1.74E-05	3.15E-10	7.99E-14	5.59E-15	6.57E-15
15	0.200					
30	0.100	2.00	4.00	5.99	5.93	5.96
60	0.050	2.00	4.00	6.00	5.99	5.99
120	0.025	2.00	4.00	6.00	6.34	6.11

Table 6: Symmetric IPDeC results for Example 6 (equidistant combined with Gauss(3) grid)

5.3 Combination of IQDeC and IPDeC. Schild's method

Consider the following combination of IQDeC and IPDeC: We proceed from the symmetric IPDeC algorithm based on the ITR as described in Subsection 5.2, but we modify (5.4) according to

$$\frac{\pi^{[\nu]}_{j,\ell} - \pi^{[\nu]}_{j,\ell-1}}{h_{j,\ell}} = \frac{1}{2}\big(f(t_{j,\ell-1}, \pi^{[\nu]}_{j,\ell-1}) + f(t_{j,\ell}, \pi^{[\nu]}_{j,\ell})\big) + \check{d}^{[\nu]}_{j,\ell}, \tag{5.5}$$

where the inhomogeneous term,

$$\breve{d}^{[\nu]}_{j,\ell} := \frac{1}{h_{j,\ell}} \int\limits_{t_{j,\ell-1}}^{t_{j,\ell}} \tilde{d}^{[\nu]}(t)\,dt = \sum_{\kappa=1}^{m} \breve{\alpha}_{\ell,\kappa}\, d^{[\nu]}(\tilde{t}_{j,\kappa}), \tag{5.6}$$

is the integral analog of the weighted defect $\frac{1}{2}(\tilde{d}^{[\nu]}(t_{j,\ell-1}) + \tilde{d}^{[\nu]}(t_{j,\ell}))$ from (5.4). In (5.6), $d^{[\nu]}(t)$ is the conventional (pointwise) defect, and the $\breve{\alpha}_{\ell,k}$ are quadrature coefficients defined analogously as in Lemma 1, but associated with the symmetrically placed quadrature nodes $\tilde{t}_{j,\kappa} = \mathbf{t}_{j-1} + \breve{c}_k \mathbf{h}_j,\ \kappa = 1 \ldots m$. The second identity in (5.6) follows from the fact that the interpolated defect (with interpolation nodes $\tilde{t}_{j,\kappa},\ \kappa = 1 \ldots m$) is a polynomial of degree $\leq m-1$ (cf. Definition 2) for which the quadrature rules based on the $\breve{\alpha}_{\ell,k}$ are exact. Again, as (5.6) shows, this way of defect evaluation can be interpreted as defect preconditioning, i.e., the $\breve{d}^{[\nu]}(t_{j,\ell})$ are weighted sums of values of the pointwise defect $d^{[\nu]}(t)$.

We have introduced this latter modification in order to relate our work to an earlier approach. Namely, for the case that the $\tilde{t}_{j,\kappa}$ are Gauss nodes, defect evaluation in the sense of (5.6) has been used in a defect correction algorithm proposed by K. H. Schild for the solution of non-stiff boundary value problems (see [11, (3.6)]). However, the motivation and some technical details of the procedure described in [11] differ from ours.

The convergence properties of our symmetric IPDeC algorithm based on (5.4) are very similar to that of Schild's method ([11]). We have repeated the experiment from Example 6 but using (5.5), (5.6) instead of (5.4). The results are nearly identical.

We do not present a convergence proof for our symmetric IPDeC algorithm in this paper. We simply note that the analysis given in [11] could easily be adapted to cover our version.

5.4 Remark on a-posteriori error estimation

The defect correction idea can also be applied for the construction of efficient a-posteriori error estimators for high order schemes, see [12]. However, if this is based on the classical, pointwise defect, this method fails even in the equidistant case if the global error to be estimated is not perfectly smooth.

In such cases, a modification based on a locally integrated defect in the spirit of IQDeC is significantly more robust, on equidistant as well as on nonequidistant grids. In [5] and [6], numerical results and a proof of the asymptotic correctness of such an estimator are presented in the context of collocation schemes for regular and singular boundary value problems.

A Appendix. Convergence proofs

In the proofs given below, a conventional stability argument for the backward Euler scheme, based on the smoothness and Lipschitz continuity of $f(t,y)$ and valid for sufficiently small $\mathbf{h}$, will be used several times. However, since this is a standard argument, we have refrained from explicitly writing down the corresponding inductive growth estimates.

Throughout, the range of indices is $j = 1 \ldots N$, $\ell = 1 \ldots m$, cf. Figure 1. Concerning the use of the $\mathcal{O}(\mathbf{h}^{\cdots})$ - symbol, cf. Remark 2 in Subsection 2.2.

Proof of Proposition 1. (IQDeC fixed point convergence.)

This is an extension of the proof of Theorem 2.1 in [4]. We use the compact denotation $\partial_h \eta_{j,\ell}$ for difference quotients of grid functions, i.e.,

$$\partial_h \eta_{j,\ell} := \frac{\eta_{j,\ell} - \eta_{j,\ell-1}}{h_{j,\ell}}. \tag{A.1}$$

The error of the $\eta_{j,\ell}^{[\nu]} \equiv p^{[\nu]}(t_{j,\ell})$ w.r.t. the $p_{j,\ell}^{[C]} := p^{[C]}(t_{j,\ell})$ (collocation solution = fixed point of the IQDeC iteration) is denoted by $\varepsilon_{j,\ell}^{[\nu]}$:

$$\varepsilon_{j,\ell}^{[\nu]} := \eta_{j,\ell}^{[\nu]} - p_{j,\ell}^{[C]} = p^{[\nu]}(t_{j,\ell}) - p^{[C]}(t_{j,\ell}). \tag{A.2}$$

We will prove Proposition 1 inductively by deriving the estimates

$$\| \varepsilon_{j,\ell}^{[\nu]} \| = \mathcal{O}(\mathbf{h}^{\nu+1}), \qquad \text{and} \qquad \| \partial_h \varepsilon_{j,\ell}^{[\nu]} \| = \mathcal{O}(\mathbf{h}^{\nu+1}), \tag{A.3}$$

for $\nu = 0, 1, \ldots$. First we introduce the auxiliary quantities

$$\xi_{j,\ell}^{[\nu]} := \pi_{j,\ell}^{[\nu]} - \eta_{j,\ell}^{[\nu]}. \tag{A.4}$$

We have $\varepsilon_{1,0}^{[\nu]} = \xi_{1,0}^{[\nu]} = 0$, and

$$\varepsilon_{j,\ell}^{[\nu]} = \varepsilon_{j,\ell}^{[0]} - \xi_{j,\ell}^{[\nu-1]}, \tag{A.5}$$

by definition of the iteration, cf. (2.15). Furthermore we use the notation

$$\phi^{[\nu]}(t) := f(t, p^{[\nu]}(t)) - f(t, p^{[C]}(t)), \qquad \text{for} \quad \nu = 0, 1, \ldots. \tag{A.6}$$

- *Let* $\nu = 0$. The $\varepsilon_{j,\ell}^{[0]}$ satisfy the difference equation (cf. Subsection 3.2)

$$\partial_h \varepsilon_{j,\ell}^{[0]} = f(t_{j,\ell}, \eta_{j,\ell}^{[0]}) - \sum_{\mu=1}^{m} \alpha_{\ell,\mu} f(t_{j,\mu}, p_{j,\mu}^{[C]}). \tag{A.7}$$

Using Taylor arguments and $\sum_{\mu=1}^{m} \alpha_{\ell,\mu} = 1$ for all ℓ (cf. Lemma 1), we write (A.7) in the form

$$\begin{aligned} \partial_h \varepsilon_{j,\ell}^{[0]} &= (f(t_{j,\ell}, \eta_{j,\ell}^{[0]}) - f(t_{j,\ell}, p_{j,\ell}^{[C]})) + (f(t_{j,\ell}, p_{j,\ell}^{[C]}) - \sum_{\mu=1}^{m} \alpha_{\ell,\mu} f(t_{j,\mu}, p_{j,\mu}^{[C]})) \\ &= \int_0^1 Df(t_{j,\ell}, p_{j,\ell}^{[C]} + \sigma \varepsilon_{j,\ell}^{[0]}) d\sigma \cdot \varepsilon_{j,\ell}^{[0]} + \mathcal{O}(\mathbf{h}). \end{aligned} \tag{A.8}$$

Now, by means of a conventional stability estimate, the bound

$$\| \varepsilon_{j,\ell}^{[0]} \| = \mathcal{O}(\mathbf{h}), \tag{A.9}$$

follows from (A.7) (classical convergence order 1 of the backward Euler scheme). Furthermore,

$$\| \partial_h \varepsilon_{j,\ell}^{[0]} \| = \mathcal{O}(\mathbf{h}), \tag{A.10}$$

follows immediately from (A.8). With (A.9) and (A.10) the proof of (A.3) for $\nu = 0$ is completed.

- *Induction step* $\nu-1 \to \nu$. We assume that (A.3) holds for $\nu-1 \geq 0$, i.e.,

$$\| \varepsilon_{j,\ell}^{[\nu-1]} \| = \mathcal{O}(\mathbf{h}^\nu), \tag{A.11a}$$

$$\| \partial_h \varepsilon_{j,\ell}^{[\nu-1]} \| = \mathcal{O}(\mathbf{h}^\nu). \tag{A.11b}$$

We shall also need an asymptotic bound for the defect $\bar{d}_{j,\ell}^{[\nu-1]}$ as defined in (3.9). This bound follows from (A.11a), (A.11b) by using the identity (cf. Subsection 3.2)

$$\partial_h p_{j,\ell}^{[C]} = \sum_{\mu=1}^{m} \alpha_{\ell,\mu} f(t_{j,\mu}, p_{j,\mu}^{[C]}), \tag{A.12}$$

and the Lipschitz continuity of f in the following way:

$$\begin{aligned}
\| \bar{d}_{j,\ell}^{[\nu-1]} \| &= \| \partial_h \eta_{j,\ell}^{[\nu-1]} - \sum_{\mu=1}^{m} \alpha_{\ell,\mu} \, f(t_{j,\mu}, \eta_{j,\mu}^{[\nu-1]}) \| \\
&\leq \| \partial_h \varepsilon_{j,\ell}^{[\nu-1]} \| + \| \sum_{\mu=1}^{m} \alpha_{\ell,\mu} \, f(t_{j,\mu}, \eta_{j,\mu}^{[\nu-1]}) - \partial_h p_{j,\ell}^{[C]} \| \\
&= \| \partial_h \varepsilon_{j,\ell}^{[\nu-1]} \| + \| \sum_{\mu=1}^{m} \alpha_{\ell,\mu} \, (f(t_{j,\mu}, \eta_{j,\mu}^{[\nu-1]}) - f(t_{j,\mu}, p_{j,\mu}^{[C]})) \| = \mathcal{O}(\mathbf{h}^\nu).
\end{aligned} \tag{A.13}$$

The $\xi_{j,\ell}^{[\nu-1]}$ (cf. (A.4)) satisfy a difference relation which follows from the equations (3.10) defining the $\pi_{j,\ell}^{[\nu-1]}$ together with (3.9):

$$\begin{aligned}
\partial_h \xi_{j,\ell}^{[\nu-1]} &= \partial_h \pi_{j,\ell}^{[\nu-1]} - \partial_h \eta_{j,\ell}^{[\nu-1]} \\
&= f(t_{j,\ell}, \pi_{j,\ell}^{[\nu-1]}) + \bar{d}_{j,\ell}^{[\nu-1]} - \partial_h \eta_{j,\ell}^{[\nu-1]} \\
&= f(t_{j,\ell}, \pi_{j,\ell}^{[\nu-1]}) - \sum_{\mu=1}^{m} \alpha_{\ell,\mu} f(t_{j,\mu}, \eta_{j,\mu}^{[\nu-1]}).
\end{aligned} \tag{A.14}$$

Now, from (A.5), (A.7) and (A.14) we obtain a difference equation for the $\varepsilon_{j,\ell}^{[\nu]}$ and rewrite it using the notation $\phi^{[\nu-1]}$ from (A.6):

$$\begin{aligned}
\partial_h \varepsilon_{j,\ell}^{[\nu]} &= \partial_h \varepsilon_{j,\ell}^{[0]} - \partial_h \xi_{j,\ell}^{[\nu-1]} \\
&= f(t_{j,\ell}, \eta_{j,\ell}^{[0]}) - \sum_{\mu=1}^{m} \alpha_{\ell,\mu} f(t_{j,\mu}, p_{j,\mu}^{[C]}) - f(t_{j,\ell}, \pi_{j,\ell}^{[\nu-1]}) + \sum_{\mu=1}^{m} \alpha_{\ell,\mu} f(t_{j,\mu}, \eta_{j,\mu}^{[\nu-1]}) \\
&= (f(t_{j,\ell}, \eta_{j,\ell}^{[0]}) - f(t_{j,\ell}, p_{j,\ell}^{[C]})) - (f(t_{j,\ell}, \pi_{j,\ell}^{[\nu-1]}) - f(t_{j,\ell}, \eta_{j,\ell}^{[\nu-1]})) - \\
&\quad - (\phi^{[\nu-1]}(t_{j,\ell}) - \sum_{\mu=1}^{m} \alpha_{\ell,\mu} \phi^{[\nu-1]}(t_{j,\mu})).
\end{aligned} \tag{A.15}$$

To estimate the right hand side of (A.15) we use Taylor arguments. First, we have

$$\| \phi^{[\nu-1]}(t_{j,\ell}) - \sum_{\mu=1}^{m} \alpha_{\ell,\mu} \phi^{[\nu-1]}(t_{j,\mu}) \| \leq \mathcal{C} \, \mathbf{h} \max_{t \in \mathbf{I}_j} \| \frac{d}{dt} \, \phi^{[\nu-1]}(t) \|, \tag{A.16}$$

for all ℓ, where

$$\| \frac{d}{dt}\,\phi^{[\nu-1]}(t)\,\| \;\le\; \|\,\frac{\partial}{\partial t}\,f(t,p^{[\nu-1]}(t)) - \frac{\partial}{\partial t}\,f(t,p^{[C]}(t))\,\| + \\ + \|\,\frac{\partial}{\partial y}\,f(t,p^{[\nu-1]}(t))\frac{d}{dt}\,p^{[\nu-1]}(t) - \frac{\partial}{\partial y}\,f(t,p^{[C]}(t))\frac{d}{dt}\,p^{[C]}(t)\,\| = \mathcal{O}(\mathbf{h}^{\nu}), \quad \text{(A.17)}$$

due to our inductive assumption (A.11a). Furthermore, to derive a bound for the difference of f-differences in (A.15) we write them in the form

$$f(t_{j,\ell},\eta_{j,\ell}^{[0]}) - f(t_{j,\ell},p_{j,\ell}^{[C]}) \;=\; \int_0^1 Df(t_{j,\ell},p_{j,\ell}^{[C]} + \sigma\varepsilon_{j,\ell}^{[0]})d\sigma\cdot\varepsilon_{j,\ell}^{[0]} =: J(t_{j,\ell})\,\varepsilon_{j,\ell}^{[0]}, \quad \text{(A.18a)}$$

$$f(t_{j,\ell},\pi_{j,\ell}^{[\nu-1]}) - f(t_{j,\ell},\eta_{j,\ell}^{[\nu-1]}) \;=\; \int_0^1 Df(t_{j,\ell},\eta_{j,\ell}^{[\nu-1]} + \sigma\xi_{j,\ell}^{[\nu-1]})d\sigma\cdot\xi_{j,\ell}^{[\nu-1]} =: \tilde{J}(t_{j,\ell})\,\xi_{j,\ell}^{[\nu-1]}, \quad \text{(A.18b)}$$

and use relation (A.5),

$$(f(t_{j,\ell},\eta_{j,\ell}^{[0]}) - f(t_{j,\ell},p_{j,\ell}^{[C]})) - (f(t_{j,\ell},\pi_{j,\ell}^{[\nu-1]}) - f(t_{j,\ell},\eta_{j,\ell}^{[\nu-1]})) \\ = \; J(t_{j,\ell})\,\varepsilon_{j,\ell}^{[\nu]} - (\tilde{J}(t_{j,\ell}) - J(t_{j,\ell}))\,\xi_{j,\ell}^{[\nu-1]}. \quad \text{(A.19)}$$

To estimate $\|\,\tilde{J}(t_{j,\ell}) - J(t_{j,\ell})\,\|$, we rewrite the difference of the integrands in (A.18) according to the identity

$$(\eta_{j,\ell}^{[\nu-1]} - p_{j,\ell}^{[C]}) + \sigma(\xi_{j,\ell}^{[\nu-1]} - \varepsilon_{j,\ell}^{[0]}) = (1-\sigma)\varepsilon_{j,\ell}^{[\nu-1]} + \sigma(\pi_{j,\ell}^{[\nu-1]} - \eta_{j,\ell}^{[0]}), \quad \text{(A.20)}$$

and observe that the Euler equations defining the $\eta_{j,\ell}^{[0]}$ and the $\pi_{j,\ell}^{[\nu-1]}$ (cf. (3.10)) together with (A.13) yield

$$\|\,\pi_{j,\ell}^{[\nu-1]} - \eta_{j,\ell}^{[0]}\,\| = \mathcal{O}(\mathbf{h}^{\nu}), \quad \text{(A.21)}$$

by a conventional stability argument for the backward Euler scheme. Now, since Df is Lipschitz continuous, (A.20) together with (A.3) and (A.21) implies

$$\|\,\tilde{J}(t_{j,\ell}) - J(t_{j,\ell})\,\| = \mathcal{O}(\mathbf{h}^{\nu}). \quad \text{(A.22)}$$

We also note that

$$\|\,\xi_{j,\ell}^{[\nu-1]}\,\| \le \|\,\pi_{j,\ell}^{[\nu-1]} - \eta_{j,\ell}^{[0]}\,\| + \|\,\eta_{j,\ell}^{[0]} - \eta_{j,\ell}^{[\nu-1]}\,\| = \mathcal{O}(\mathbf{h}), \quad \text{(A.23)}$$

and combining (A.15)–(A.23) we obtain

$$\partial_h\varepsilon_{j,\ell}^{[\nu]} = J(t_{j,\ell})\,\varepsilon_{j,\ell}^{[\nu]} - \underbrace{(\tilde{J}(t_{j,\ell}) - J(t_{j,\ell}))\xi_{j,\ell}^{[\nu-1]}}_{=\mathcal{O}(\mathbf{h}^{\nu+1})} + \mathcal{O}(\mathbf{h}^{\nu+1}). \quad \text{(A.24)}$$

We can interpret (A.24) as a linear difference equation of the backward Euler type, with inhomogeneous part $\mathcal{O}(\mathbf{h}^{\nu+1})$ and initial value $\varepsilon_{1,0}^{[\nu]} = 0$. (A.24) together with a stability argument for the backward Euler scheme yields (A.3).

This completes the proof of Proposition 1.

□

Proof of Proposition 2. (IPDeC fixed point convergence, linear analysis.)

We confine ourselves to the case of linear initial value problems

$$y'(t) = A(t)y(t) + g(t), \quad y(t_0) = y_0, \tag{A.25}$$

with sufficiently smooth data functions $A(t)$ and $g(t)$. (Using standard linearization techniques similarly as in the proof of Proposition 1 above, the results can be extended to problems with a smooth nonlinearity, but we do not work out the details in this case.)

We use the convention that, if a piecewise (polynomial) function $p(t)$ is mentioned, this refers to a function, defined via N pieces $p_1(t), \ldots, p_N(t)$, such that $p(t) = p_j(t),\ t \in \mathbf{I}_j = [\mathbf{t}_{j-1}, \mathbf{t}_j],\ j = 1 \ldots N$ (not necessarily continuous at the $\mathbf{t}_j$). The space of all such piecewise polynomial functions of degree $\leq k$ is denoted by $\mathcal{P}_{\mathbf{h}}^k$.

We will make use of the following auxiliary results.

(i) For $p \in \mathcal{P}_{\mathbf{h}}^m$, let $\partial_h p \in \mathcal{P}_{\mathbf{h}}^{m-1}$ be defined by

$$(\partial_h p)_j(t) := \frac{p_j(t) - p_j(t - h_j)}{h_j}. \tag{A.26}$$

Then there holds

$$\| p_j^{(k)}(t) \| \leq \mathcal{C} \max_{t \in \mathbf{I}_j} \| (\partial_h p)_j^{(k-1)}(t) \|, \quad k = 1 \ldots m, \tag{A.27}$$

with a constant $\mathcal{C}$ independent of p and $\mathbf{h}$. The estimate (A.27) easily follows from the representation of the derivative of a polynomial $q(t)$ (degree $\leq m$) by a linear combination of difference quotients,

$$q'(t) = \sum_{\ell=1}^{m} \beta_\ell(t)\, (\partial_h q)(t_\ell), \tag{A.28}$$

where the coefficients $\beta_\ell(t)$ can be expressed using Lagrange representation; cf. e.g. [4, (2.20)].

(ii) If $q \in \mathcal{P}_{\mathbf{h}}^{m-1}$ interpolates the values of a function $f(t)$ (with m times continuously differentiable pieces $f_j(t)$) at m distinct points in $\mathbf{I}_j$, then for $0 \leq k \leq l \leq m$ we have

$$\| q^{(k)}(t) - f^{(k)}(t) \| \leq \mathcal{C}\, \mathbf{h}^{l-k} \max_{s \in \mathbf{I}_j} \| f^{(l)}(s) \|, \quad t \in \mathbf{I}_j. \tag{A.29}$$

For a proof of (A.29) see [1] or [10].

Some denotation: Difference quotients are written according to (A.1), $\partial_h \eta_{j,\ell} := (\eta_{j,\ell} - \eta_{j,\ell-1})/h_j$, now with the piecewise constant stepsizes $h_{j,\ell} \equiv h_j$. For the error w.r.t. to the IPDeC fixed point (collocation solution) $\tilde{p}^{[C]}$ we write

$$\varepsilon_{j,\ell}^{[\nu]} := \eta_{j,\ell}^{[\nu]} - \tilde{p}^{[C]}(t_{j,\ell}) = p^{[\nu]}(t_{j,\ell}) - \tilde{p}^{[C]}(t_{j,\ell}), \tag{A.30}$$

and

$$e^{[\nu]}(t) := p^{[\nu]}(t) - \tilde{p}^{[C]}(t), \tag{A.31}$$

i.e. the function $\in \mathcal{P}_{\mathbf{h}}^m$ which interpolates the $\varepsilon_{j,\ell}^{[\nu]}$ at the grid points $t_{j,\ell}$. Furthermore we define

$$u^{[\nu]} \in \mathcal{P}_{\mathbf{h}}^{m-1} \quad \ldots \text{ interpolant of } A(t)e^{[\nu]}(t) \text{ at the grid points } t_{j,1}, \ldots, t_{j,m}, \tag{A.32a}$$

$$\tilde{u}^{[\nu]} \in \mathcal{P}_{\mathbf{h}}^{m-1} \quad \ldots \text{ interpolant of } A(t)e^{[\nu]}(t) \text{ at the collocation nodes } \tilde{t}_{j,1}, \ldots, \tilde{t}_{j,m}. \tag{A.32b}$$

To prove Proposition 2 we will now inductively derive the estimates

$$\| \varepsilon_{j,\ell}^{[\nu]} \| = \mathcal{O}(\mathbf{h}^{\nu+1}), \tag{A.33a}$$

$$\| \frac{d^k}{dt^k} e^{[\nu]}(t) \| = \mathcal{O}(\mathbf{h}^{\nu+1}), \qquad k = 0 \dots m, \tag{A.33b}$$

for $\nu = 0, 1, \dots$.

- *Let* $\nu = 0$. Here, (A.33a) holds due to

$$\| \varepsilon_{j,\ell}^{[0]} \| \le \| \eta_{j,\ell}^{[0]} - y^*(t_{j,\ell}) \| + \| \tilde{p}^{[C]}(t_{j,\ell}) - y^*(t_{j,\ell}) \| = \mathcal{O}(\mathbf{h}) + \mathcal{O}(\mathbf{h}^m) = \mathcal{O}(\mathbf{h}), \tag{A.34}$$

where we have made use of the fact that the backward Euler method and polynomial collocation (degree m) have uniform convergence order 1 and m, respectively. For (A.33b) we split $e^{[0]}(t)$ into $e^{[0]} = (p^{[0]} - y^*) - (\tilde{p}^{[C]} - y^*)$. The estimate $\| (d^k/dt^k)(\tilde{p}^{[C]}(t) - y^*(t)) \| = \mathcal{O}(\mathbf{h}^{\min(m,m+1-k)}) \le \mathcal{O}(\mathbf{h})$ is a standard result about collocation methods, cf. [8, Theorem 7.1]. The fact that $\| (d^k/dt^k)(p^{[0]}(t) - y^*(t)) \| = \mathcal{O}(\mathbf{h})$ holds is a standard result for the backward Euler method, which can be proved by means of an asymptotic expansion of its global error. Note that the assumption of a piecewise equidistant grid is essential here.

- *Induction step* $\nu - 1 \to \nu$. We assume that (A.33) holds for $\nu - 1 \ge 0$, i.e.,

$$\| \varepsilon_{j,\ell}^{[\nu-1]} \| = \mathcal{O}(\mathbf{h}^{\nu}), \tag{A.35a}$$

$$\| \frac{d^k}{dt^k} e^{[\nu-1]}(t) \| = \mathcal{O}(\mathbf{h}^{\nu}), \qquad k = 0 \dots m. \tag{A.35b}$$

Using the equations

$$\partial_h \eta_{j,\ell}^{[0]} = A(t_{j,\ell}) \eta_{j,\ell}^{[0]} + g(t_{j,\ell}), \tag{A.36}$$

$$\partial_h \pi_{j,\ell}^{[\nu-1]} = A(t_{j,\ell}) \pi_{j,\ell}^{[\nu-1]} + g(t_{j,\ell}) + \tilde{d}^{[\nu-1]}(t_{j,\ell}), \tag{A.37}$$

and

$$\varepsilon_{j,\ell}^{[\nu]} - \varepsilon_{j,\ell}^{[\nu-1]} = \eta_{j,\ell}^{[0]} - \pi_{j,\ell}^{[\nu-1]}, \qquad \partial_h \varepsilon_{j,\ell}^{[\nu]} - \partial_h \varepsilon_{j,\ell}^{[\nu-1]} = \partial_h \eta_{j,\ell}^{[0]} - \partial_h \pi_{j,\ell}^{[\nu-1]}, \tag{A.38}$$

which hold by definition of the backward Euler and IPDeC schemes, we see that $\varepsilon_{j,\ell}^{[\nu]}$ satisfies the difference equation

$$\partial_h \varepsilon_{j,\ell}^{[\nu]} = A(t_{j,\ell}) \varepsilon_{j,\ell}^{[\nu]} + r_{j,\ell}^{[\nu]}, \tag{A.39}$$

where

$$r_{j,\ell}^{[\nu]} = \partial_h \varepsilon_{j,\ell}^{[\nu-1]} - A(t_{j,\ell}) \varepsilon_{j,\ell}^{[\nu-1]} - \tilde{d}^{[\nu-1]}(t_{j,\ell}). \tag{A.40}$$

We shall now derive an estimate for the $r_{j,\ell}^{[\nu]}$. To this end, let $\tilde{q}^{[\nu-1]}$ and $\tilde{q}^{[C]} \in \mathcal{P}_{\mathbf{h}}^{m-1}$ interpolate the functions $A(t)p^{[\nu-1]}(t) + g(t)$ and $A(t)\tilde{p}^{[C]}(t) + g(t)$ at the collocation nodes $\tilde{t}_{j,\kappa}$. By definition of the interpolated defect $\tilde{d}^{[\nu-1]} \in \mathcal{P}_{\mathbf{h}}^{m-1}$ and of the collocation polynomial $\tilde{p}^{[C]} \in \mathcal{P}_{\mathbf{h}}^{m}$ (cf. Subsection 4.1) there holds

$$\tilde{d}^{[\nu-1]}(t) = \frac{d}{dt} p^{[\nu-1]}(t) - \tilde{q}^{[\nu-1]}(t), \quad \text{and} \quad 0 = \frac{d}{dt} \tilde{p}^{[C]}(t) - \tilde{q}^{[C]}(t). \tag{A.41}$$

Together with $\tilde{q}^{[\nu-1]} - \tilde{q}^{[C]} = \tilde{u}^{[\nu-1]}$ (cf. definition (A.32b)) this yields

$$\tilde{d}^{[\nu-1]}(t_{j,\ell}) = \frac{d}{dt} e^{[\nu-1]}(t_{j,\ell}) - \tilde{u}^{[\nu-1]}(t_{j,\ell}). \tag{A.42}$$

Inserting (A.42) into (A.40) yields

$$r_{j,\ell}^{[\nu]} = \underbrace{\Big(\partial_h e^{[\nu-1]}(t_{j,\ell}) - \frac{d}{dt}\, e^{[\nu-1]}(t_{j,\ell})\Big)}_{=:\, \overset{*}{r}{}_{j,\ell}^{[\nu]}} - \underbrace{\Big(A(t_{j,\ell}) e^{[\nu-1]}(t_{j,\ell}) - \tilde{u}^{[\nu-1]}(t_{j,\ell})\Big)}_{=:\, \overset{**}{r}{}_{j,\ell}^{[\nu]}}. \tag{A.43}$$

The contributions $\overset{*}{r}{}_{j,\ell}^{[\nu]}$ and $\overset{**}{r}{}_{j,\ell}^{[\nu]}$ are now estimated separately.

(*) First, the $\overset{*}{r}{}_{j,\ell}^{[\nu]}$ are the grid values of the function $\overset{*}{r}{}^{[\nu]}(t) := \partial_h e^{[\nu-1]}(t) - \frac{d}{dt}\, e^{[\nu-1]}(t) \in \mathcal{P}_{\mathbf{h}}^{m-1}$. Expressed in Taylor form, this reads

$$\overset{*}{r}{}_j^{[\nu]}(t) = \sum_{i=2}^{m} \frac{(-1)^{i-1} h_j^{i-1}}{i!}\, \frac{d^i}{dt^i}\, e_j^{[\nu-1]}(t), \tag{A.44}$$

from which with the help of (A.35b) we obtain

$$\Big\| \frac{d^k}{dt^k}\, \overset{*}{r}{}^{[\nu]}(t) \Big\| = \mathcal{O}(\mathbf{h}^{\nu+1}), \quad k = 0 \ldots m-1. \tag{A.45}$$

In particular, we have $\overset{*}{r}{}_{j,\ell}^{[\nu]} = \mathcal{O}(\mathbf{h}^{\nu+1})$.

(**) Furthermore, the function $\overset{**}{r}{}^{[\nu]} \in \mathcal{P}_{\mathbf{h}}^{m-1}$ which interpolates the $\overset{**}{r}{}_{j,\ell}^{[\nu]}$ at the grid points $t_{j,1}, \ldots, t_{j,m}$ is given by $\overset{**}{r}{}^{[\nu]} = u^{[\nu-1]} - \tilde{u}^{[\nu-1]}$ (cf. (A.32)). We write this as a difference of interpolation errors,

$$\overset{**}{r}{}^{[\nu]}(t) = \big(u^{[\nu-1]}(t) - A(t) e^{[\nu-1]}(t)\big) - \big(\tilde{u}^{[\nu-1]}(t) - A(t) e^{[\nu-1]}(t)\big) \tag{A.46}$$

and apply (A.29) twice with $l = k+1$ to estimate its derivatives:

$$\begin{aligned} \Big\| \frac{d^k}{dt^k}\, \overset{**}{r}{}^{[\nu]}(t) \Big\| \;&\le\; \Big\| \frac{d^k}{dt^k} \big(u^{[\nu-1]}(t) - A(t) e^{[\nu-1]}(t)\big) \Big\| + \Big\| \frac{d^k}{dt^k} \big(\tilde{u}^{[\nu-1]}(t) - A(t) e^{[\nu-1]}(t)\big) \Big\| \\ &\le\; \mathcal{C}\, \mathbf{h} \max_{t \in [t_0, t_{end}]} \Big\| \frac{d^{k+1}}{dt^{k+1}} \big(A(t) e^{[\nu-1]}(t)\big) \Big\| \;=\; \mathcal{O}(\mathbf{h}^{\nu+1}), \qquad k = 0 \ldots m-1. \end{aligned} \tag{A.47}$$

Here we have used (A.35b) and suitable smoothness assumptions for $A(t)$.

Altogether, for the inhomogeneous part in (A.39) we obtain

$$r_{j,\ell}^{[\nu]} \;=\; \mathcal{O}(\mathbf{h}^{\nu+1}), \tag{A.48a}$$

$$\Big\| \frac{d^k}{dt^k}\, r^{[\nu]}(t) \Big\| \;=\; \mathcal{O}(\mathbf{h}^{\nu+1}), \quad k = 0 \ldots m-1, \tag{A.48b}$$

where $r^{[\nu]} \in \mathcal{P}_{\mathbf{h}}^{m-1}$ interpolates the $r_{j,\ell}^{[\nu]}$ at the grid points $t_{j,1}, \ldots, t_{j,m}$.

Finally, from (A.39) together with (A.48a) and using a conventional stability argument for the backward Euler scheme we obtain (A.33a) and (A.33b, $k = 0$). To prove (A.33b) inductively for $k = 1 \ldots m$, we consider the function $\partial_h e^{[\nu]} \in \mathcal{P}_{\mathbf{h}}^{m-1}$ interpolating the $\partial_h \varepsilon_{j,\ell}^{[\nu]} = A(t_{j,\ell}) \varepsilon_{j,\ell}^{[\nu]} + r_{j,\ell}^{[\nu]}$ at the grid points $t_{j,1}, \ldots, t_{j,m}$. This function satisfies the identity $\partial_h e^{[\nu]}(t) = u^{[\nu]}(t) + r^{[\nu]}(t)$. Now we apply (A.27) to obtain

$$\| \frac{d^k}{dt^k} e^{[\nu]}(t) \| \le \mathcal{C} \max_{t\in[t_0,t_{end}]} \| \frac{d^{k-1}}{dt^{k-1}} u^{[\nu]}(t) \| + \mathcal{C} \max_{t\in[t_0,t_{end}]} \| \frac{d^{k-1}}{dt^{k-1}} r^{[\nu]}(t) \|. \tag{A.49}$$

Due to (A.48b), the second term on the right hand side of (A.49) is $\mathcal{O}(\mathbf{h}^{\nu+1})$ for $k = 1 \ldots m$. Using (A.32a) and (A.29) with $l = k$, the first term can be estimated by

$$\begin{aligned} \| \frac{d^{k-1}}{dt^{k-1}} u^{[\nu]}(t) \| &\le \| \frac{d^{k-1}}{dt^{k-1}} A(t)e^{[\nu]}(t) \| + \| \frac{d^{k-1}}{dt^{k-1}} u^{[\nu]}(t) - \frac{d^{k-1}}{dt^{k-1}} A(t)e^{[\nu]}(t) \| \\ &\le \mathcal{C} \max_{t\in[t_0,t_{end}]} \| \frac{d^{k-1}}{dt^{k-1}} A(t)e^{[\nu]}(t) \|. \end{aligned} \tag{A.50}$$

Now we use an induction argument w.r.t. k. Assume that (A.33b) has already been shown for all $k' < k$; then,

$$\| \frac{d^{k-1}}{dt^{k-1}} A(t)e^{[\nu]}(t) \| = \mathcal{O}(\mathbf{h}^{\nu+1}) \tag{A.51}$$

easily follows using suitable smoothness assumptions for $A(t)$. Due to (A.49)–(A.51), (A.33b) is indeed valid for $k = 1 \ldots m$.

This completes the proof of Proposition 2. □

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