SIAM J. NUMER. ANAL.
Vol. 27, No. 1, pp. 67-104, February 1990

005

ASYMPTOTIC ERROR EXPANSIONS FOR STIFF EQUATIONS: THE IMPLICIT EULER SCHEME*

W. AUZINGER,† R. FRANK,† AND F. MACSEK†

Abstract. This paper discusses the existence of an asymptotic expansion for the global error of the implicit Euler scheme applied to stiff nonlinear systems of ordinary differential equations. It is shown that in strongly stiff situations, a full asymptotic expansion exists at all gridpoints. For the mildly stiff case it is shown that the full order of the remainder term, which inevitably breaks down at the first gridpoints after a stepsize change, reappears at the subsequent gridpoints. Our analysis is based on singular perturbation techniques.

Key words. stiff problems, asymptotic error expansions, singular perturbations

AMS(MOS) subject classifications. 65L05, 65B05

1. Introduction. During recent years there has been considerable progress in the theory of discretizations of nonlinear stiff initial value problems. For a large class of methods, global error bounds have been derived that are independent of the Lipschitz constant L of the right-hand side of the differential equation; they depend only on the one-sided Lipschitz constant m and on some other moderate-sized quantities characterizing the given stiff problem. Concerning the relevant notions, for instance, "one-sided Lipschitz continuity," "G-stability," "B-stability," "B-convergence," see the respective literature, e.g., [4], [5], [7]-[10]. In particular, it will be assumed in the following that the reader is sufficiently familiar with the concept of B-convergence (cf. [8]).

It is natural, as a next goal, to strive for results concerning the *structure* of the global discretization error, i.e., we are aiming at asymptotic error expansions of the form

$$\zeta_\nu - z(t_\nu) = h^p e_p(t_\nu) + \cdots + h^q e_q(t_\nu) + R_\nu, \tag{1.1}$$

where ζ_ν denotes the numerical approximation to the exact solution $z(t)$ of the given problem at the gridpoint t_ν, obtained by a method of order p, and h is the stepsize of the grid used. The functions $e_i(t)$ are solutions of the so-called "variational equations." For any given method the derivation of these variational equations is a purely formal procedure based on simple Taylor expansions and arrangement in powers of h. The crucial point is to estimate the remainder term R_ν, i.e., to show

$$R_\nu = O(h^{q+1}). \tag{1.2}$$

For large classes of methods, estimates of this type are well known in the "classical sense," where R_ν depends on the Lipschitz constant L. For stiff problems the question arises of whether (1.2) remains valid in the "B-sense," where we require that the O-constant not depend on the classical Lipschitz constant L, but is only allowed to depend on the one-sided Lipschitz constant m and some other, moderate-sized parameters characterizing the underlying problem.

In the present paper we discuss the existence of an asymptotic error expansion for the *implicit Euler scheme* applied to a certain class of nonlinear stiff systems where the degree of stiffness is characterized by a small parameter ε ($1/\varepsilon$ is the order of

* Received by the editors May 11, 1987; accepted for publication (in revised form) November 16, 1988.
† Institut für Angewandte und Numerische Mathematik, Technical University of Vienna, Wiedner Hauptstrasse 6-10, A-1040 Vienna, Austria.

magnitude of the stiff eigenvalues). Our results are valid for those parts of the integration interval where the true solution $z(t)$ is smooth. That is, we do not describe error structures within a transient phase. However, for the case where a transient phase occurs its effect on the error structure *after* the transients is discussed to some extent.

It is shown that in "strongly stiff" situations (that is, if ε is sufficiently small compared with the stepsize h) a full asymptotic expansion exists at all gridpoints (Theorem 4.1). If ε is not sufficiently smaller than h ("mildly stiff" case) then there holds a weaker result:

(i) Order defects of the remainder term R_ν inevitably occur at the first gridpoints after a change of the stepsize. However, our analysis will show that these order defects are rapidly damped away at the subsequent gridpoints such that the full order $q+1$ reappears.

(ii) The precise form of the global error, i.e., $\zeta_\nu - z(t_\nu) = he_1(t_\nu) + \cdots + h^q e_q(t_\nu) + R_\nu$ (where R_ν is rapidly decaying according to (i) above) is proved in Theorem 5.1 for $q=4$ under certain assumptions. For the general case we prove a slightly weaker result (Theorems 5.6 and 5.7).

The results presented in this paper were first proved in 1986 by Auzinger, Frank, and Macsek (cf. [1], [2]). Whereas the proofs in [1] and [2] are rather technical, we now present a more concise and transparent argumentation.

There also exist three earlier papers dealing with asymptotic error expansions for stiff problems. In particular, the implicit midpoint rule is discussed by Dahlquist and Lindberg [6] and by Van Veldhuizen [13]. The implicit trapezoidal rule and a certain semi-implicit two-step method are considered in [6] and in Bader and Deuflhard [3], respectively. The results presented in these papers, however, are not rigorous in our sense. In particular, the estimates for the remainder term presented in [3] depend in an implicit way on the Lipschitz constant L.

The present paper is subdivided into six sections. In § 2 we derive the variational equations of the implicit Euler scheme and a difference equation for the remainder term. All this material is, of course, state of the art; we have included it for convenience and to introduce some denotation to be used later. In § 3 we study smooth solutions of the variational equations, providing the basis for our general theory presented in § 4 (strongly stiff case) and § 5 (mildly stiff case).

2. The variational equations and a difference equation for the remainder term. Consider the initial value problem

$$y'(t) = f(t, y(t)), \quad f: G \to \mathbb{R}^n, \quad G \subset \{[0, T] \times \mathbb{R}^n\}, \qquad y(0) = z_0, \quad z_0 \in \mathbb{R}^n, \tag{2.1}$$

with the exact solution $z(t)$. The implicit Euler discretization of (2.1),

$$\frac{1}{h}(\zeta_\nu - \zeta_{\nu-1}) = f(t_\nu, \zeta_\nu), \qquad \zeta_0 = z_0, \tag{2.2}$$

yields approximations ζ_ν for $z(t_\nu)$ at the gridpoints $t_\nu = \nu h$.

Let $z_h(t)$ be a smooth function interpolating the ζ_ν. We use the ansatz

$$z_h(t) = z(t) + he_1(t) + h^2 e_2(t) + r_h(t). \tag{2.3}$$

For the moment we restrict our considerations to an expansion up to h^2 and an $O(h^3)$-remainder term. The extension to an arbitrary order is obvious.

Since $z_h(t_\nu) = \zeta_\nu$, (2.2) can be written as

$$\frac{1}{h}(z_h(t_\nu) - z_h(t_{\nu-1})) - f(t_\nu, z_h(t_\nu)) = 0. \tag{2.4}$$

To establish the asymptotic expansion for the global error we insert the ansatz (2.3) into (2.4). Taylor expansion and rearrangement in powers of h will yield the desired equations.

Consider the Taylor expansions

$$z(t_\nu - h) = z(t_\nu) - hz'(t_\nu) + \frac{h^2}{2} z''(t_\nu) - \frac{h^3}{6} z'''(t_\nu) + I_{\nu,0},$$

(2.5a)

$$I_{\nu,0} = \frac{h^4}{6} \int_0^1 z^{(4)}(t_\nu - \sigma h)(1-\sigma)^3 \, d\sigma,$$

$$e_1(t_\nu - h) = e_1(t_\nu) - he_1'(t_\nu) + \frac{h^2}{2} e_1''(t_\nu) + I_{\nu,1},$$

(2.5b)

$$I_{\nu,1} = -\frac{h^3}{2} \int_0^1 e_1'''(t_\nu - \sigma h)(1-\sigma)^2 \, d\sigma,$$

$$e_2(t_\nu - h) = e_2(t_\nu) - he_2'(t_\nu) + I_{\nu,2},$$

(2.5c)

$$I_{\nu,2} = h^2 \int_0^1 e_2''(t_\nu - \sigma h)(1-\sigma) \, d\sigma.$$

Thus,

(2.6)

$$\frac{1}{h}(z_h(t_\nu) - z_h(t_\nu - h)) = z'(t_\nu) - \frac{h}{2} z''(t_\nu) + \frac{h^2}{6} z'''(t_\nu) - \frac{1}{h} I_{\nu,0} + he_1'(t_\nu) - \frac{h^2}{2} e_1''(t_\nu)$$
$$- I_{\nu,1} + h^2 e_2'(t_\nu) - hI_{\nu,2} + \frac{1}{h}(r_h(t_\nu) - r_h(t_\nu - h)).$$

$f(t_\nu, z_h(t_\nu))$ is expanded into

(2.7a)

$$f(t_\nu, z_h(t_\nu)) = f(t_\nu, z(t_\nu) + he_1(t_\nu) + h^2 e_2(t_\nu) + r_h(t_\nu))$$
$$= f(t_\nu, z(t_\nu) + he_1(t_\nu) + h^2 e_2(t_\nu)) + \hat{J}(r_h(t_\nu)) \cdot r_h(t_\nu),$$
$$\hat{J}(r_h(t_\nu)) := \int_0^1 f_y(t_\nu, z(t_\nu) + he_1(t_\nu) + h^2 e_2(t_\nu) + \sigma r_h(t_\nu)) \, d\sigma.$$

Furthermore,

(2.7b)

$$f(t_\nu, z(t_\nu) + he_1(t_\nu) + h^2 e_2(t_\nu)) = f(t_\nu, z(t_\nu)) + f_y(t_\nu, z(t_\nu))[he_1(t_\nu) + h^2 e_2(t_\nu)]$$
$$+ \frac{1}{2} f_{yy}(t_\nu, z(t_\nu))[he_1(t_\nu) + h^2 e_2(t_\nu)]^2$$
$$+ \frac{1}{2} \int_0^1 f_{yyy}(t_\nu, z(t_\nu) + \sigma(he_1(t_\nu) + h^2 e_2(t_\nu)))(1-\sigma)^2 \, d\sigma$$
$$\cdot [he_1(t_\nu) + h^2 e_2(t_\nu)]^3.$$

Inserting (2.6) and (2.7) into (2.4) and rearranging in powers of h results in

$$
\begin{aligned}
0 &= \frac{1}{h}(z_h(t_\nu) - z_h(t_\nu - h)) - f(t_\nu, z_h(t_\nu)) \\
&= z'(t_\nu) - f(t_\nu, z(t_\nu)) + h\left[e_1'(t_\nu) - f_y(t_\nu, z(t_\nu))e_1(t_\nu) - \frac{1}{2}z''(t_\nu)\right] \\
&\quad + h^2\left[e_2'(t_\nu) - f_y(t_\nu, z(t_\nu))e_2(t_\nu) + \frac{1}{6}z'''(t_\nu) - \frac{1}{2}e_1''(t_\nu) - \frac{1}{2}f_{yy}(t_\nu, z(t_\nu))e_1^2(t_\nu)\right] \\
&\quad + \frac{1}{h}(r_h(t_\nu) - r_h(t_\nu - h)) - \hat{J}(r_h(t_\nu)) \cdot r_h(t_\nu) - b_\nu + c_\nu,
\end{aligned} \tag{2.8}
$$

where

$$c_\nu := -\frac{1}{h} I_{\nu,0} - I_{\nu,1} - hI_{\nu,2}, \tag{2.9}$$

and b_ν is the collection of all terms of order greater or equal to 3 from (2.7b).

Since $z(t)$ is the solution of the original problem (2.1), (2.8) is satisfied if $e_1(t)$, $e_2(t)$ are solutions of the linear differential equations

$$e_1'(t) = f_y(t, z(t))e_1(t) + \tfrac{1}{2}z''(t), \tag{2.10}$$

$$e_2'(t) = f_y(t, z(t))e_2(t) - \tfrac{1}{6}z'''(t) + \tfrac{1}{2}e_1''(t) + \tfrac{1}{2}f_{yy}(t, z(t))e_1^2(t), \tag{2.11}$$

and if the discrete remainder term $R_\nu = r_h(t_\nu)$ satisfies the *nonlinear* difference equation

$$\frac{1}{h}(R_\nu - R_{\nu-1}) = \hat{J}(R_\nu) \cdot R_\nu + b_\nu - c_\nu. \tag{2.12}$$

Equations (2.10), (2.11) are called *variational equations.* They are defined in a recursive way: the inhomogeneous term in (2.11) depends on $e_1(t)$. (In general, the ith variational equation depends on all $e_k(t)$, $k < i$.) Similarly, the b_ν and c_ν in the remainder term equation (2.12) are defined in terms of the solutions of the variational equations.

The starting values of the $e_i(t)$ and of R_ν remain to be fixed. For $t = t_\nu$, (2.3) reads

$$\zeta_\nu - z(t_\nu) = he_1(t_\nu) + h^2 e_2(t_\nu) + R_\nu. \tag{2.13}$$

In particular, for $\nu = 0$,

$$0 = he_1(0) + h^2 e_2(0) + R_0, \tag{2.14}$$

since $\zeta_0 = z(0)$. A natural choice to satisfy (2.14) is

$$e_1(0) = e_2(0) = R_0 = 0; \tag{2.15}$$

in the context of stiff equations, however, it will turn out that *another* choice for the $e_i(0)$ and R_0 (satisfying (2.14)) is preferable.

Up to now we have considered only equidistant grids. Of course, nonequidistant grids are unavoidable in the context of stiff problems. In the transients it is necessary to use a very small stepsize to obtain a good level of accuracy. When the transients have died away, large stepsizes (adjusted to the smoothness of the solution) can be used. We assume that our particular nonequidistant grid is a member of a so-called *coherent grid sequence* in the sense of Stetter [12], where the subintervals with constant stepsize are kept fixed during the asymptotic process of the stepsize refinement, i.e., the points where stepsize changes occur remain fixed. In this sense, the interval $[0, T]$ has to be interpreted as one of these subintervals with constant stepsize. Thus, $\zeta_0 \neq z(0)$ must be admitted in contrast to (2.2). The quantity $\zeta_0 - z(0)$ is the accumulated error from the preceding intervals.

Therefore, (2.14) must be replaced by

$$\zeta_0 - z(0) = \tilde{h}\tilde{e}_1(0) + \tilde{h}^2\tilde{e}_2(0) + \tilde{R}_N = he_1(0) + h^2 e_2(0) + R_0, \tag{2.16}$$

where the left-hand side is the accumulated global error at $t=0$; we have assumed that $\tilde{h}$ is the stepsize used in the subinterval that precedes $[0, T]$. $\tilde{e}_1(0)$, $\tilde{e}_2(0)$ are the solution values of the variational equations at the endpoint $t=0$ of that interval, and $\tilde{R}_N$ is the remainder term at this point. A natural choice for the new starting values would be[1]

$$e_1(0) = \frac{\tilde{h}}{h}\tilde{e}_1(0), \quad e_2(0) = \left(\frac{\tilde{h}}{h}\right)^2 \tilde{e}_2(0), \quad R_0 = \tilde{R}_N, \tag{2.17}$$

but again another choice is preferable within the context of stiff equations.

The nonlinear difference equation (2.12) for R_ν is of the same type as the implicit Euler equation (2.2), and so it is obvious that R_ν can be estimated in the spirit of the B-theory (cf. Frank, Schneid, and Ueberhuber [8]). Using the inequality

$$\|[I - h\hat{J}(R_\nu)]^{-1}\| \leqq \frac{1}{1-hm} \tag{2.18}$$

(where m denotes a one-sided Lipschitz constant of f that is also a bound for the respectively logarithmic norms of f_y and $\hat{J}$), it can easily be shown that

$$\|R_\nu\| \leqq \begin{cases} \|R_0\| + t_\nu \max\limits_{\mu=1,\cdots,\nu} \|b_\mu - c_\mu\|, & m = 0, \\ \left(\dfrac{1}{1-hm}\right)^\nu \|R_0\| + \dfrac{1}{m}\left[\left(\dfrac{1}{1-hm}\right)^\nu - 1\right] \max\limits_{\mu=1,\cdots,\nu} \|b_\mu - c_\mu\|, & m \neq 0 \end{cases} \tag{2.19}$$

(with the usual stepsize restriction $hm \leqq \rho < 1$ if $m > 0$). This would yield the desired result $R_\nu = O(h^{q+1})$ if

$$R_0 = O(h^{q+1}), \quad b_\nu = O(h^{q+1}), \quad c_\nu = O(h^{q+1}). \tag{2.20}$$

By definition, the c_ν depend on certain derivatives of the $e_i(t)$, $i \leqq q$ (e.g., on $(d^{q+1}/dt^{q+1})e_1(t)$). The $e_i(t)$ are solutions of the variational equations that are, of course, stiff since their Jacobians coincide with that of the original problem. Thus it must be expected that the $e_i(t)$ (with the starting values (2.15) or (2.17)) are *not* smooth solutions of the stiff differential equations (2.10), (2.11), respectively, i.e., their derivatives are influenced by powers of the Lipschitz constant L. If, for instance, $e_1(t)$ is not a smooth solution of (2.10), then it typically contains a transient solution component of the form $e^{\lambda t}/\lambda$ (where λ is a stiff eigenvalue, that is, $\mathrm{Re}(\lambda) \ll 0$), and $(d^{q+1}/dt^{q+1})e_1(t)$ contains a term of the form $\lambda^q e^{\lambda t}$. Thus, since $L \simeq |\lambda|$, any bound for $(d^{q+1}/dt^{q+1})e_1(t)$ with respect to $[0, T]$ is of magnitude $O(L^q)$. As a simple consequence, $c_\nu = O(h^{q+1})$ is not true in general. Simple models show that in the mildly stiff case (i.e., for $h \approx 1/|\lambda|$) only $c_\nu = O(h)$ holds at the first gridpoints.

In [3], Bader and Deuflhard, discussing a semi-implicit two-step method, derive estimates for the remainder term that are analogous to (2.19). But they completely ignore the crucial point, namely, the fact that c_ν is not $O(h^{q+1})$ in general. For the implicit midpoint and trapezoidal schemes, Dahlquist and Lindberg [6] use another approach: to obtain smooth solutions of the variational equations, the starting value ζ_0 is modified after each change of the stepsize.

---

[1] Note that for coherent grid sequences the relation between the different stepsizes remains fixed during the stepsize refinement, i.e., $(\tilde{h}/h)^i$ is always constant, and so the starting values $e_i(0)$ (and consequently the quantities $e_i(t)$) are independent of the stepsize parameter.

In the present paper we adopt this idea—namely, to work with smooth solutions of the variational equations—for our analysis of the implicit Euler scheme. This ensures that the inhomogeneity $b_\nu - c_\nu$ is at $O(h^{q+1})$-level for all gridpoints t_ν. But special starting values $e_i^{sm}(0)$ must be chosen to obtain smooth solutions. The starting value R_0 is then fixed by

$$R_0 = \zeta_0 - z(0) - he_1^{sm}(0) - \cdots - h^q e_q^{sm}(0) = O(h), \tag{2.21}$$

but $R_0 = O(h^{q+1})$—as required in (2.20)—is *not* satisfied in this case.[2] As a consequence, order reductions must be expected within R_ν. Our analysis will show that:

(i) Actually, these order reductions do not appear in strongly stiff situations;

(ii) In mildly stiff situations R_ν shows a reduced order at the first gridpoints; however, the full order $R_\nu = O(h^{q+1})$ reappears eventually.

In the scalar case, where $m \ll 0$ for stiff problems, it is immediately clear that order reduction effects at the beginning are *quickly damped away* by the factor $(1/(1-hm))^\nu$ (cf. (2.19)). However, it is important to note that for nonscalar problems this damping effect cannot be concluded from (2.19)—in the vector case there will usually be nonstiff eigenvalues of moderate size besides the stiff eigenvalues. Since the one-sided Lipschitz constant m is influenced by these nonstiff eigenvalues (note that $m \geqq \max \operatorname{Re}(\lambda_i)$, $\lambda_i \cdots$ eigenvalues of f_y), the $(1/(1-hm))^\nu$-factors (cf. (2.19)) have no damping power: in many cases, $m > 0$ and then $(1/(1-hm))^\nu$ *increases* with ν; but also if $m < 0$ (but not $m \ll 0$), $h|m|$ is small and the quantity $(1/(1-hm))^\nu$ decreases very slowly with ν, such that at a fixed $t = t_\nu \in [0, T]$, $R_\nu = O(h^{q+1})$ cannot be guaranteed. Nevertheless, § 5 will show that similar damping properties hold as in the scalar case. Our analysis is based on techniques completely different from B-convergence estimates, namely, on *discrete singular perturbation techniques* on the basis of smooth solutions of the variational equations.

Note that the damping of order reduction effects within R_ν cannot be expected for methods that are not strongly B-stable, such as the implicit midpoint and trapezoidal schemes. In [6], Dahlquist and Lindberg have tried to overcome this problem by algorithmically fitting the starting values (after each change of the stepsize) such that $R_0 = O(h^{q+1})$. However, this causes considerable difficulties in practice.

3. Smooth solutions of the variational equations. We now proceed towards a general theory of asymptotic error expansions for the implicit Euler scheme. In the following we assume that $z(t)$ is a smooth solution of (2.1) in the spirit of the B-theory. Furthermore, we assume that

$$f_y(t, z(t)) = T(t)\Lambda(t)T^{-1}(t), \tag{3.1a}$$

with

$$\Lambda(t) = \begin{pmatrix} c_1(t) & 0 \\ 0 & -c_2(t)/\varepsilon \end{pmatrix}. \tag{3.1b}$$

We require that the complex-valued quantities $c_1(t)$, $c_2(t)$, $T(t)$, and $T^{-1}(t)$ are bounded, smooth functions (i.e., the derivatives which appear in the following are assumed to exist and to be of moderate size). Furthermore, we assume that $\operatorname{Re}(c_2(t)) \geqq \kappa > 0$. $\varepsilon > 0$ is a small real parameter. According to (3.1a, b), $f_y(t, z(t)) = O(\varepsilon^{-1})$. With

[2] If we had chosen the starting values as in (2.17), then the quantity $\zeta_0 - z(0)$ would be compensated up to $O(h^{q+1})$-level due to (2.16), i.e., $R_0 = O(h^{q+1})$ would be true. In (2.21), however, such a compensation cannot be expected.

respect to the higher derivatives of f we assume that they are smooth, i.e.,

$$f_{yy}(t, y) = O(\varepsilon^0),$$
$$f_{yyy}(t, y) = O(\varepsilon^0), \tag{3.1c}$$
$$\vdots$$

Our analysis will refer to the two-dimensional case (i.e., $c_1(t)$ and $c_2(t)$ are scalar functions). However, all considerations can also be understood in the sense that the $c_k(t)$ are vector-valued functions. We do not discuss the more general case of more than two clusters of eigenvalues which would require a multiparameter $(\varepsilon_1, \cdots, \varepsilon_\kappa)$-theory.

The first step in our analysis is to study smooth solutions of the variational equations (VEs). The general form of the VEs is (cf. (2.10), (2.11))

$$e_i'(t) = f_y(t, z(t))e_i(t) + g_i(t), \tag{3.2}$$

where $g_i(t)$ depends in a recursive way on certain derivatives of the solutions of the preceding VEs. To enable the application of singular perturbation techniques we transform the VEs (3.2), using the matrix $T(t)$ which diagonalizes $f_y(t, z(t))$ (cf. (3.1a, b)). With

$$\bar{e}_i(t) = T^{-1}(t)e_i(t), \tag{3.3}$$

equation (3.2) transforms into

$$\bar{e}_i'(t) = \Lambda(t)\bar{e}_i(t) + A(t)\bar{e}_i(t) + \bar{g}_i(t), \tag{3.4a}$$

where

$$A(t) := -T^{-1}(t)T'(t), \qquad \bar{g}_i(t) := T^{-1}(t)g_i(t). \tag{3.4b}$$

Using the notation

$$\bar{e}_i(t) = \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix}, \quad \bar{g}_i(t) = \begin{pmatrix} r_i(t) \\ s_i(t) \end{pmatrix}, \tag{3.5}$$

and multiplying the second component of (3.4a) by ε, we obtain

$$\begin{aligned} x_i'(t) &= c_1(t)x_i(t) + a_{1,1}(t)x_i(t) + a_{1,2}(t)y_i(t) + r_i(t), \\ \varepsilon y_i'(t) &= -c_2(t)y_i(t) + \varepsilon a_{2,1}(t)x_i(t) + \varepsilon a_{2,2}(t)y_i(t) + \varepsilon s_i(t). \end{aligned} \tag{3.6}$$

This is a special (linear) case of an equation of the type

$$x'(t) = u(t, x, y, \varepsilon), \qquad \varepsilon y'(t) = v(t, x, y, \varepsilon), \tag{3.7}$$

which is usually considered in the singular perturbation theory (cf., e.g., O'Malley [11]). Note that the singular perturbation theory applies to (3.6) because we have assumed that $\operatorname{Re}(c_2(t)) \geqq \kappa > 0$.

Note that we have *not* assumed that the *original* problem (2.1) is of the singularly perturbed model type (3.7). This would be a fundamental loss of generality and would significantly simplify the analysis.[3]

[3] In this special case it would be natural to relax the smoothness assumption (3.1c). The second component of f, which corresponds to $\varepsilon^{-1}v(t, x, y, \varepsilon)$ of (3.7), could then be admitted to have derivatives that are $O(\varepsilon^{-1})$.

We have supposed that the $c_k(t)$ and $a_{k,l}(t)$ do not depend on the parameter ε. On the other hand, for many models with a stiff eigenvalue λ (where ε is identified with $-\mathrm{Re}\,(1/\lambda)$) smooth solutions $z(t)$ depend on λ (i.e., on ε) in a harmless way, that is, on nonnegative powers of ε. Consequently, $f_y(t, z(t))$ depends on ε, too. Therefore, in (3.6) we would have to work with expansions $c_k(t) = c_{k,0}(t) + \varepsilon c_{k,1}(t) + \cdots$, $A(t) = A_0(t) + \varepsilon A_1(t) + \cdots$. We have conveniently restricted ourselves to ε-independent data since this causes no substantial differences in our singular perturbation analysis. All the material presented below could easily be rewritten for the slightly more general case of ε-dependent data functions. The *really* essential thing is the ε-dependence of the $e_i(t)$ (and not of $z(t)$).

We will now construct (sufficiently) smooth solutions of the VEs by means of singular perturbation techniques. Our particular goal is to ensure that the inhomogeneity of the remainder term equation (2.12) satisfies

$$b_\nu - c_\nu = O(h^{q+1}). \tag{3.8}$$

The general form of solutions of (3.6) is not smooth enough for our purpose; it is given by

$$x(t) = X(t, \varepsilon) + \varepsilon m(\tau, \varepsilon), \qquad y(t) = Y(t, \varepsilon) + n(\tau, \varepsilon) \tag{3.9}$$

(cf., e.g., O'Malley [11]). Here, $X(t, \varepsilon)$ and $Y(t, \varepsilon)$ are the components of the so-called "outer solution"; they are smooth functions, i.e., their derivatives with respect to t are of moderate size. On the other hand, there are also "inner solution" terms $m(\tau, \varepsilon)$, $n(\tau, \varepsilon)$, which depend on the "stretched time variable"

$$\tau := t/\varepsilon. \tag{3.10}$$

$m(\tau, \varepsilon)$ and $n(\tau, \varepsilon)$ are of "boundary layer type" and rapidly decay towards zero away from $t = 0$. In particular, $m(\tau, \varepsilon)$ and $n(\tau, \varepsilon)$ consist (essentially) of terms of the type

$$\varepsilon^r p(\tau)\, e^{-c_2(0)\tau} \tag{3.11}$$

with certain polynomials $p(\tau)$. (The latter can be shown by a singular perturbation analysis of (3.6) but is of no relevance for our further argumentation.)

Inner solution terms $\varepsilon^r p(\tau)\, e^{-c_2(0)\tau}$ are $O(\varepsilon^r)$ since functions of the form $p(\tau)\, e^{-c_2(0)\tau}$ are bounded for $\tau \in [0, \infty)$. But their derivatives with respect to the slow time variable $t = \varepsilon\tau$ satisfy only

$$\max_{t \in [0,T]} \left| \frac{d^k}{dt^k}\, \varepsilon^r p\left(\frac{t}{\varepsilon}\right) e^{-c_2(0)t/\varepsilon} \right| = O(\varepsilon^{r-k}), \qquad k = 0, 1, \cdots. \tag{3.12}$$

We now show how to *choose special starting values* for (3.6) such that no inner solution terms appear up to a certain ε^p-level, ensuring that the derivatives up to the order p remain bounded uniformly for $\varepsilon \to 0$.

LEMMA 3.1 (Smooth solutions). *Consider an equation*

$$\begin{aligned} x'(t) &= c_1(t)x(t) + a_{1,1}x(t) + a_{1,2}y(t) + r(t), \\ \varepsilon y'(t) &= -c_2(t)y(t) + \varepsilon a_{2,1}x(t) + \varepsilon a_{2,2}y(t) + \varepsilon s(t) \end{aligned} \tag{3.13}$$

(*cf.* (3.6)), *where* $r(t)$, $s(t)$ *have an* ε-*expansion of the form*

$$\begin{aligned} r(t) &= R_0(t) + \varepsilon R_1(t) + \cdots + \varepsilon^p R_p(t) + R_{\mathrm{rem}}(t, \varepsilon), \\ \varepsilon s(t) &= \varepsilon S_0(t) + \cdots + \varepsilon^p S_{p-1}(t) + S_{\mathrm{rem}}(t, \varepsilon), \end{aligned} \tag{3.14a}$$

with smooth terms $R_j(t)$, $S_j(t)$ and remainder terms satisfying

$$\left\|\frac{d^k}{dt^k}\begin{pmatrix}R_{\text{rem}}(t,\varepsilon)\\ S_{\text{rem}}(t,\varepsilon)\end{pmatrix}\right\|_\infty = \max_{t\in[0,T]}\left\|\frac{d^k}{dt^k}\begin{pmatrix}R_{\text{rem}}(t,\varepsilon)\\ S_{\text{rem}}(t,\varepsilon)\end{pmatrix}\right\| = O(\varepsilon^{p+1-k}),\qquad k=0,1,\cdots. \tag{3.14b}$$

Consider starting values of the form

$$\begin{pmatrix}x(0)\\ y(0)\end{pmatrix}=\begin{pmatrix}x_0+\varepsilon x_1+\cdots+\varepsilon^p x_p+x_{\text{rem}}\\ \varepsilon y_1+\cdots+\varepsilon^p y_p+y_{\text{rem}}\end{pmatrix}, \tag{3.15}$$

where $x_{\text{rem}}=O(\varepsilon^{p+1})$, $y_{\text{rem}}=O(\varepsilon^{p+1})$. Choose the $x_j=O(1)$ arbitrarily. Then there exists a unique set of starting values $y_1,\cdots,y_p$, where each y_j is uniquely determined by $x_0,\cdots,x_{j-1}$ and $y_1,\cdots,y_{j-1}$, such that the solution of (3.13) *attains the form*

$$\begin{aligned} x(t)&=X_0(t)+\varepsilon X_1(t)+\cdots+\varepsilon^p X_p(t)+X_{\text{rem}}(t,\varepsilon),\\ y(t)&=\varepsilon Y_1(t)+\cdots+\varepsilon^p Y_p(t)+Y_{\text{rem}}(t,\varepsilon),\end{aligned} \tag{3.16a}$$

with smooth functions $X_j(t)$, $Y_j(t)$ and remainder terms satisfying

$$\begin{aligned} X_{\text{rem}}(t,\varepsilon)&=O(\varepsilon^{p+1}),\quad \frac{d^k}{dt^k}X_{\text{rem}}(t,\varepsilon)=O(\varepsilon^{p+2-k}),\quad k=1,2,\cdots,\\ Y_{\text{rem}}(t,\varepsilon)&=O(\varepsilon^{p+1}),\quad \frac{d^k}{dt^k}Y_{\text{rem}}(t,\varepsilon)=O(\varepsilon^{p+1-k}),\quad k=1,2,\cdots.\end{aligned} \tag{3.16b}$$

Proof. Inserting the ansatz (3.16a) into (3.13) we obtain

$$\begin{aligned} x'(t)&=X_0'(t)+\varepsilon X_1'(t)+\varepsilon^2X_2'(t)+\cdots+\varepsilon^pX_p'(t)+X_{\text{rem}}'(t,\varepsilon)\\ &=[(c_1(t)+a_{1,1}(t))X_0(t)+R_0(t)]\\ &\quad+\varepsilon[(c_1(t)+a_{1,1}(t))X_1(t)+a_{1,2}(t)Y_1(t)+R_1(t)]\\ &\quad+\varepsilon^2[(c_1(t)+a_{1,1}(t))X_2(t)+a_{1,2}(t)Y_2(t)+R_2(t)]+\cdots\\ &\quad+\varepsilon^p[(c_1(t)+a_{1,1}(t))X_p(t)+a_{1,2}(t)Y_p(t)+R_p(t)]\\ &\quad+[(c_1(t)+a_{1,1}(t))X_{\text{rem}}(t,\varepsilon)+a_{1,2}(t)Y_{\text{rem}}(t,\varepsilon)+R_{\text{rem}}(t,\varepsilon)],\end{aligned} \tag{3.17a}$$

$$\begin{aligned} \varepsilon y'(t)&=\varepsilon^2Y_1'(t)+\varepsilon^3Y_2'(t)+\cdots+\varepsilon^{p+1}Y_p'(t)+\varepsilon Y_{\text{rem}}'(t,\varepsilon)\\ &=\varepsilon[-c_2(t)Y_1(t)+a_{2,1}(t)X_0(t)+S_0(t)]\\ &\quad+\varepsilon^2[-c_2(t)Y_2(t)+a_{2,1}(t)X_1(t)+a_{2,2}Y_1(t)+S_1(t)]+\cdots\\ &\quad+\varepsilon^p[-c_2(t)Y_p(t)+a_{2,1}(t)X_{p-1}(t)+a_{2,2}Y_{p-1}(t)+S_{p-1}(t)]+\cdots\\ &\quad+\varepsilon^{p+1}[a_{2,1}(t)X_p(t)+a_{2,2}Y_p(t)]\\ &\quad+[-c_2(t)Y_{\text{rem}}(t,\varepsilon)+\varepsilon a_{2,1}(t)X_{\text{rem}}(t,\varepsilon)+\varepsilon a_{2,2}(t)Y_{\text{rem}}(t,\varepsilon)+S_{\text{rem}}(t,\varepsilon)].\end{aligned} \tag{3.17b}$$

Equating coefficients of ε^j, $j\geqq 0$, yields the following results.

Coefficients of ε^0.

$$X_0'(t)=(c_1(t)+a_{1,1}(t))X_0(t)+R_0(t). \tag{3.18}$$

Each solution $X_0(t)$ of (3.18) satisfying an *arbitrary* initial condition $X_0(0)=x_0$ with $x_0=O(1)$ is a smooth function (according to our smoothness assumptions with respect to $c_1(t)$, $a_{1,1}(t)$ and $R_0(t)$).

Coefficients of ε^1.

$$X_1'(t) = (c_1(t) + a_{1,1}(t))X_1(t) + a_{1,2}(t)Y_1(t) + R_1(t),$$
$$0 = -c_2(t)Y_1(t) + a_{2,1}(t)X_0(t) + S_0(t). \tag{3.19}$$

The second equation within (3.19) is purely algebraic and uniquely defines $Y_1(t)$ (depending on $X_0(t)$). Thus, no starting value can be prescribed for $Y_1(t)$, but $Y_1(0)$ is *fixed* by

$$y_1 := Y_1(0) = \frac{1}{c_2(0)}[a_{2,1}(0)x_0 + S_0(0)]. \tag{3.20}$$

That is, y_1 is uniquely determined by x_0. Again, an *arbitrary* $x_1 = O(1)$ defines a smooth solution $X_1(t)$ of the first equation within (3.19) with $X_1(0) = x_1$.

Coefficients of ε^2.

$$X_2'(t) = (c_1(t) + a_{1,1}(t))X_2(t) + a_{1,2}(t)Y_2(t) + R_2(t),$$
$$0 = -c_2(t)Y_2(t) + a_{2,1}(t)X_1(t) + a_{2,2}(t)Y_1(t) + S_1(t) - Y_1'(t). \tag{3.21}$$

Again, $Y_2(0)$ is fixed by

$$y_2 := Y_2(0) = \frac{1}{c_2(0)}[a_{2,1}(0)x_1 + a_{2,2}(0)y_1 + S_1(0) - Y_1'(0)]. \tag{3.22a}$$

From the second relation within (3.19) and from (3.18), we have

$$Y_1'(0) = \frac{1}{c_2(0)}\{-c_2'(0)y_1 + a_{2,1}'(0)x_0 + a_{2,1}(0)[(c_1(0) + a_{1,1}(0))x_0 + R_0(0)] + S_0'(0)\}. \tag{3.22b}$$

Thus, y_2 depends recursively on x_0, x_1, and y_1. Furthermore, an *arbitrary* $x_2 = O(1)$ defines a smooth solution $X_2(t)$ of the first equation within (3.21) with $X_2(0) = x_2$.

This procedure continues in an obvious way up to the ε^p-level.

Estimation of the ε^{p+1}*-remainder terms.* Within the equation for the remainder terms $X_{\text{rem}}(t, \varepsilon)$ and $Y_{\text{rem}}(t, \varepsilon)$, the second component is rescaled to its original form:

$$X_{\text{rem}}'(t, \varepsilon) = (c_1(t) + a_{1,1}(t))X_{\text{rem}}(t, \varepsilon) + a_{1,2}(t)Y_{\text{rem}}(t, \varepsilon) + R_{\text{rem}}(t, \varepsilon),$$
$$Y_{\text{rem}}'(t, \varepsilon) = -\frac{c_2(t)}{\varepsilon}Y_{\text{rem}}(t, \varepsilon) + a_{2,1}(t)X_{\text{rem}}(t, \varepsilon) + a_{2,2}(t)Y_{\text{rem}}(t, \varepsilon) + \varepsilon^p[a_{2,1}(t)X_p(t) + a_{2,2}(t)Y_p(t)] + \frac{1}{\varepsilon}S_{\text{rem}}(t, \varepsilon). \tag{3.23}$$

Let $\hat{m}$ denote a one-sided Lipschitz bound for the right-hand side of the differential equation (3.23). Then the well-known growth estimates yield (cf., for instance, Frank, Schneid, and Ueberhuber [8]):

$$\left\| \begin{pmatrix} X_{\text{rem}}(t, \varepsilon) \\ Y_{\text{rem}}(t, \varepsilon) \end{pmatrix} \right\| \leq e^{\hat{m}t}\delta_1 + \frac{e^{\hat{m}t} - 1}{\hat{m}}\delta_2 \tag{3.24a}$$

with (cf. (3.15))

$$\delta_1 := \left\| \begin{pmatrix} X_{\text{rem}}(0, \varepsilon) \\ Y_{\text{rem}}(0, \varepsilon) \end{pmatrix} \right\| = \left\| \begin{pmatrix} x_{\text{rem}} \\ y_{\text{rem}} \end{pmatrix} \right\| = O(\varepsilon^{p+1}) \tag{3.24b}$$

and (cf. (3.14b))

$$\delta_2 := \max_{t\in[0,T]} \left\| \begin{pmatrix} R_{\text{rem}}(t,\varepsilon) \\ \varepsilon^p[a_{2,1}(t)X_p(t)+a_{2,2}(t)Y_p(t)]+(1/\varepsilon)S_{\text{rem}}(t,\varepsilon) \end{pmatrix} \right\| = O(\varepsilon^p). \tag{3.24c}$$

So far we have shown

$$\left\| \begin{pmatrix} X_{\text{rem}}(t,\varepsilon) \\ Y_{\text{rem}}(t,\varepsilon) \end{pmatrix} \right\| = O(\varepsilon^p); \tag{3.25}$$

but to ensure (3.16b) we need one further power of ε. Such a strengthened estimate can be obtained by the following reasoning. Due to (3.25), the second equation within (3.23) can be considered as a scalar differential equation of the form

$$Y'_{\text{rem}}(t,\varepsilon) = \frac{-c_2(t)}{\varepsilon} Y_{\text{rem}}(t,\varepsilon) + O(\varepsilon^p) \tag{3.26}$$

(where $a_{2,1}(t)X_{\text{rem}}(t,\varepsilon)$ and $a_{2,2}(t)Y_{\text{rem}}(t,\varepsilon)$ are now considered as inhomogeneous terms at $O(\varepsilon^p)$-level). The right-hand side of (3.26) has a one-sided Lipschitz constant m^* with

$$m^* \ll 0, \qquad |m^*| = O\left(\frac{1}{\varepsilon}\right). \tag{3.27}$$

Now we can estimate $Y_{\text{rem}}(t,\varepsilon)$ as in (3.24a) (with m^* instead of $\hat{m}$):

$$|Y_{\text{rem}}(t,\varepsilon)| \leqq e^{m^*t}|y_{\text{rem}}| + \frac{e^{m^*t}-1}{m^*} \cdot O(\varepsilon^p) = O(\varepsilon^{p+1}), \tag{3.28a}$$

because the starting value y_{rem} is $O(\varepsilon^{p+1})$ and $(e^{m^*t}-1)/m^* = O(\varepsilon)$.

Considering the first equation within (3.23) and taking account of the sharper estimate (3.28a) for $Y_{\text{rem}}(t,\varepsilon)$ and $R_{\text{rem}}(t,\varepsilon) = O(\varepsilon^{p+1})$, we further conclude

$$|X_{\text{rem}}(t,\varepsilon)| = O(\varepsilon^{p+1}). \tag{3.28b}$$

Formulae (3.28a, b) yield (3.16b) for $k=0$. For $k=1$, assertion (3.16b) follows directly from (3.23) and (3.28). For $k \geqq 2$, the result can be concluded recursively by successive differentiation of (3.23), taking account of assumption (3.14b). □

PROPOSITION 3.2. *Consider starting values for the ith* VE*s, $i=1,\cdots,q$, with an ε-expansion of the form*

$$\bar{e}_i(0) = \begin{pmatrix} x_i(0) \\ y_i(0) \end{pmatrix} = \begin{pmatrix} x_{0,i}+\varepsilon x_{1,i}+\cdots+\varepsilon^{q+1-i}x_{q+1-i,i}+\cdots \\ \varepsilon y_{1,i}+\cdots+\varepsilon^{q+1-i}y_{q+1-i,i}+\cdots \end{pmatrix}. \tag{3.29}$$

For arbitrarily chosen $x_{k,i} = O(1)$ there exists a unique set of values $y_{k,i} = O(1)$, $k = 1,\cdots,q+1-i$, $i=1,\cdots,q$, such that the $\bar{e}_i(t)$ are sufficiently smooth to ensure that (3.8) *is satisfied.*

Proof. Recall that, according to the definition of $b_\nu - c_\nu$ (cf. § 2), derivatives of the $e_i(t)$ appear only within c_ν:[4]

$$c_\nu = -\frac{1}{h} I_{\nu,0} - I_{\nu,1} - \cdots - h^{q-1} I_{\nu,q}. \tag{3.30}$$

[4] To ensure $b_\nu = O(h^{q+1})$ it is required only that the $e_i(t)$ are bounded.

For $i \geqq 1$, $I_{\nu,i}$ is an integral remainder term of a Taylor expansion of $e_i(t)$ involving the $(q+2-i)$th derivative of $e_i(t)$ (cf. (2.5)). If this derivative is $O(1)$, we have $h^{i-1}I_{\nu,i} = O(h^{q+1})$.

Consider the first transformed VE (cf. (3.4a)):

$$\bar{e}_1'(t) = \Lambda(t)\bar{e}_1(t) + A(t)\bar{e}_1(t) + \tfrac{1}{2}T^{-1}(t)z''(t). \tag{3.31}$$

By our assumptions the inhomogeneity $T^{-1}(t)z''(t)$ is smooth. Apply Lemma 3.1 with $p=q$. For arbitrarily chosen $x_{k,1}$ the $y_{k,1}$, $k=1,\cdots,q$, are fixed such that (3.16) is valid for $\bar{e}_1(t)$ (with $p=q$):

$$\bar{e}_1(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} X_{0,1}(t) + \varepsilon X_{1,1}(t) + \cdots + \varepsilon^q X_{q,1}(t) + X_{\mathrm{rem},1}(t,\varepsilon) \\ \varepsilon Y_{1,1}(t) + \cdots + \varepsilon^q Y_{q,1}(t) + Y_{\mathrm{rem},1}(t,\varepsilon) \end{pmatrix}, \tag{3.32a}$$

where

$$\begin{aligned} &X_{\mathrm{rem},1}(t,\varepsilon) = O(\varepsilon^{q+1}), \quad \frac{d^k}{dt^k} X_{\mathrm{rem},1}(t,\varepsilon) = O(\varepsilon^{q+2-k}), \quad k \geqq 1, \\ &Y_{\mathrm{rem},1}(t,\varepsilon) = O(\varepsilon^{q+1}), \quad \frac{d^k}{dt^k} Y_{\mathrm{rem},1}(t,\varepsilon) = O(\varepsilon^{q+1-k}), \quad k \geqq 1. \end{aligned} \tag{3.32b}$$

Hence it is obvious that

$$\frac{d^k}{dt^k}\bar{e}_1(t) = O(1), \qquad k = 0,\cdots,q+1, \tag{3.33a}$$

and therefore

$$\begin{aligned} \frac{d^{q+1}}{dt^{q+1}} e_1(t) &= \frac{d^{q+1}}{dt^{q+1}}(T(t)\bar{e}_1(t)) \\ &= T(t)\frac{d^{q+1}}{dt^{q+1}}\bar{e}_1(t) + (q+1)T'(t)\frac{d^q}{dt^q}\bar{e}_1(t) + \cdots = O(1) \end{aligned} \tag{3.33b}$$

due to our smoothness assumptions with respect to $T(t)$. Thus,

$$I_{\nu,1} = O(h^{q+1}). \tag{3.33c}$$

The second transformed VE reads

$$\begin{aligned} \bar{e}_2'(t) &= \Lambda(t)\bar{e}_2(t) + A(t)\bar{e}_2(t) + \tfrac{1}{2}T^{-1}(t)(T(t)\bar{e}_1(t))'' - \tfrac{1}{6}T^{-1}(t)z'''(t) \\ &\quad + \tfrac{1}{2}T^{-1}(t)f_{yy}(t,z(t))(T(t)\bar{e}_1(t))^2. \end{aligned} \tag{3.34a}$$

From (3.32) we obtain an ε-expansion for the inhomogeneity of (3.34a):

$$\begin{aligned} &\tfrac{1}{2}T^{-1}(t)(T(t)\bar{e}_1(t))'' - \tfrac{1}{6}T^{-1}(t)z'''(t) + \tfrac{1}{2}T^{-1}(t)f_{yy}(t,z(t))(T(t)\bar{e}_1(t))^2 \\ &= \tfrac{1}{2}\bar{e}_1''(t) + T^{-1}(t)T'(t)\bar{e}_1'(t) + \tfrac{1}{2}T^{-1}(t)T''(t)\bar{e}_1(t) - \tfrac{1}{6}T^{-1}(t)z'''(t) \\ &\quad + \tfrac{1}{2}T^{-1}(t)f_{yy}(t,z(t))(T(t)\bar{e}_1(t))^2 \\ &=: \begin{pmatrix} r_2(t) \\ s_2(t) \end{pmatrix} = \begin{pmatrix} R_{0,2}(t) + \varepsilon R_{1,2}(t) + \cdots + \varepsilon^{q-1}R_{q-1,2}(t) + R_{\mathrm{rem},2}(t,\varepsilon) \\ S_{0,2}(t) + \varepsilon S_{1,2}(t) + \cdots + \varepsilon^{q-2}S_{q-2,2}(t) + (1/\varepsilon)S_{\mathrm{rem},2}(t,\varepsilon) \end{pmatrix} \end{aligned} \tag{3.34b}$$

with

$$\left\| \frac{d^k}{dt^k}\begin{pmatrix} R_{\mathrm{rem},2}(t,\varepsilon) \\ S_{\mathrm{rem},2}(t,\varepsilon) \end{pmatrix} \right\|_\infty = O(\varepsilon^{q-k}), \qquad k = 0,1,\cdots. \tag{3.34c}$$

Hence (3.34a) satisfies the assumptions of Lemma 3.1 for $p=q-1$, such that for suitable values $y_{k,2}$, $k=1,\cdots,q-1$ (depending on the choice for the $x_{k,2}$), $\bar{e}_2(t)$ is sufficiently smooth to ensure

$$hI_{\nu,2}=O(h^{q+1}). \tag{3.35}$$

These inductive arguments can be continued up to $i=q$ in an obvious way, establishing the assertion. □

4. The strongly stiff case. It is a well-known fact that for B-convergent methods and, in particular, for the implicit Euler scheme, the stepsize h can be adjusted to the smoothness of the solution being sought independently of the magnitude of the stiff eigenvalues. Therefore in many practical situations—if the accuracy requirements are not too severe—h is significantly larger than ε (in our notation, the stiff eigenvalue is $-c_2(t)/\varepsilon$). In the sequel we will refer to such situations as *strongly stiff situations.* More precisely, "strongly stiff" means

$$\text{“}\varepsilon \text{ is at least of the magnitude } h^q\text{,”} \quad \text{i.e., } \varepsilon=O(h^q). \tag{4.1}$$

The subdomian $\varepsilon \leqq Ch^q$ of the "ε-h-plane" is singled out in this section because it allows a particularly easy and transparent argumentation. "Mildly stiff" situations—where (4.1) is violated—will be discussed at length in § 5 (see the survey at the beginning of § 5; Fig. 1 illustrates the various cases).

THEOREM 4.1 (The strongly stiff case). *Assume that our smoothness assumptions* (3.1a-c) *are satisfied and that* ε *satisfies* (4.1). *Consider a subinterval* $[0, T]$ *of the whole integration interval with constant stepsize h and assume inductively that the accumulated global error* $\zeta_0-z(0)$ *at* $t=0$ *satisfies*

$$T^{-1}(0)(\zeta_0-z(0))=h\begin{pmatrix}\hat{x}_{0,1}+O(\varepsilon)\\ O(\varepsilon)\end{pmatrix}+\cdots+h^q\begin{pmatrix}\hat{x}_{0,q}+O(\varepsilon)\\ O(\varepsilon)\end{pmatrix}+O(h^{q+1}) \tag{4.2a}$$

with certain $\hat{x}_{0,i}$ *that are independent of h and* ε. *Then the discretization error of the implicit Euler scheme on* $[0, T]$ *admits a full asymptotic expansion*

$$\zeta_\nu-z(t_\nu)=he_1(t_\nu)+\cdots+h^qe_q(t_\nu)+R_\nu \tag{4.2b}$$

with smooth, h-independent functions $e_i(t)$ *that are solutions of the* VEs (*cf.* (2.10), (2.11), (3.2)); *furthermore,*

$$R_\nu=O(h^{q+1}) \tag{4.2c}$$

at all gridpoints $t_\nu=\nu h$, $\nu=0, 1,\cdots, T/h$.

Remark. The use of the symbol $O(h^{q+1})$ in conjunction with our assumption $\varepsilon=O(h^q)$ (i.e., $\varepsilon\leqq Ch^q$ with moderate C) deserves some explanation. Usually, $\|x\|=O(h^{q+1})$ means that $\|x\|\leqq Ch^{q+1}$ is true for $h\to 0$. But for a particular problem under consideration ε is a fixed parameter (characterizing the given problem) and therefore $h\to 0$ is not compatible with $\varepsilon\leqq Ch^q$. Therefore we use the symbol $O(h^{q+1})$ in a somewhat weaker sense: $\|x\|=O(h^{q+1})$ means that $\|x\|\leqq Ch^{q+1}$ is true for $h\in[(\varepsilon/C)^{1/q}, h_0]$ where h_0 is the largest admissible stepsize.

Proof. For $\nu=0$, (4.2b) reads

$$\zeta_0-z(0)=he_1(0)+\cdots+h^qe_q(0)+R_0; \tag{4.3a}$$

after transformation (3.3) we therefore have the following initial condition for the remainder term:

$$\rho_0:=T^{-1}(0)R_0=T^{-1}(0)(\zeta_0-z(0))-h\bar{e}_1(0)-\cdots-h^q\bar{e}_q(0). \tag{4.3b}$$

Now we apply Proposition 3.2 to construct smooth solutions $\bar{e}_i(t)$ on $[0, T]$ such that $b_\nu - c_\nu = O(h^{q+1})$. In particular, we use the degree of freedom within the first components of the starting values in the following way. At ε^0-level we set (cf. (3.29))

$$x_{0,i} := \hat{x}_{0,i}, \qquad i = 1, \cdots, q \tag{4.4}$$

with the $\hat{x}_{0,i}$ from (4.2a). (At ε^k-level, $k \geqq 1$, the $x_{k,i}$ from (3.29) can be chosen arbitrarily at $O(1)$-level.) Then ρ_0 satisfies (cf. (4.1), (4.2a), (4.3b)):

$$\begin{aligned} \rho_0 &= h\begin{pmatrix} \hat{x}_{0,1} - x_{0,1} + O(\varepsilon) \\ O(\varepsilon) \end{pmatrix} + \cdots + h^q \begin{pmatrix} \hat{x}_{0,q} - x_{0,q} + O(\varepsilon) \\ O(\varepsilon) \end{pmatrix} + O(h^{q+1}) \\ &= O(h\varepsilon) + O(h^{q+1}) = O(h^{q+1}). \end{aligned} \tag{4.5}$$

Together with (3.8) and (2.19) the desired estimate (4.2c) immediately follows. □

By construction we now have smooth solutions $\bar{e}_i(t)$ of the transformed VEs that are of the form (3.16) (in particular, the second component is $O(\varepsilon)$). The transformed global error at $t_\nu \in [0, T]$ therefore reads

$$\begin{aligned} T^{-1}(t_\nu)(\zeta_\nu - z(t_\nu)) = h\begin{pmatrix} X_{0,1}(t_\nu) + O(\varepsilon) \\ O(\varepsilon) \end{pmatrix} + \cdots + h^q \begin{pmatrix} X_{0,q}(t_\nu) + O(\varepsilon) \\ O(\varepsilon) \end{pmatrix} \\ + O(h^{q+1}). \end{aligned} \tag{4.6a}$$

At the endpoint $t_\nu = T$ of the current subinterval we thus again have the same structure as assumed in (4.2a). Therefore our argumentation can be extended in an inductive way to the subinterval that follows $[0, T]$. With $\hat{h}$ denoting the stepsize in that subinterval we have a new initial value for the transformed global error, namely,

$$\hat{h}\left(\frac{h}{\hat{h}}\right)\begin{pmatrix} X_{0,1}(T) + O(\varepsilon) \\ O(\varepsilon) \end{pmatrix} + \cdots + \hat{h}^q \left(\frac{h}{\hat{h}}\right)^q \begin{pmatrix} X_{0,q}(T) + O(\varepsilon) \\ O(\varepsilon) \end{pmatrix} + O(\hat{h}^{q+1}), \tag{4.6b}$$

which is again of the form (4.2a) with h-independent quantities $(h/\hat{h})^i X_{0,i}(T)$ (because $h/\hat{h}$ is constant for coherent grid sequences).

The beginning of this induction is trivial if, on the very first of these subintervals, we have started the integration on a smooth solution $z(t)$ (for $\zeta_0 - z(0) = 0$, (4.2a) is trivially satisfied). However, we usually have to reckon with a transient phase where $z(t)$ is not smooth. Our argumentation based on smooth solutions does, of course, break down in the transient phase. (The inhomogeneities of the VEs contain certain derivatives of $z(t)$ that are not smooth in the transients, and therefore smooth $e_i(t)$ do not exist there.) But immediately after the transients (4.2a) is again justified. To see this, we consider the well-known B-convergence estimate for the implicit Euler scheme:

$$\|\zeta_\nu - z(t_\nu)\| \leqq \begin{cases} \|\zeta_0 - z(t_0)\| + t_\nu \dfrac{h}{2} M_2, & m = 0, \\ \left(\dfrac{1}{1-hm}\right)^\nu \|\zeta_0 - z(t_0)\| + \dfrac{1}{m}\left[\left(\dfrac{1}{1-hm}\right)^\nu - 1\right]\dfrac{h}{2} M_2, & m \neq 0 \\ M_2 \cdots \text{bound for } \|z''(t)\| \end{cases} \tag{4.7}$$

(with the usual stepsize restriction $hm \leqq \rho < 1$ if $m > 0$), and apply it inductively on subintervals $[t_0, t_N]$ (with constant stepsizes) of the transient phase. Since the length of the transient phase is always $O(\varepsilon)$ (which can easily be shown by a singular perturbation analysis of the original problem), we conclude that the error bound immediately after the transients contains a factor ε. If we assume further that the stepsizes in the transients are adjusted to the gradually improving smoothness of the solution such that the product hM_2 is approximately equidistributed (where, for each subinterval with constant stepsize, M_2 is a local bound for $\|z''(t)\|$), then the error immediately after the transients is $O(\varepsilon h)$ (where h now denotes the stepsize *after* the transient phase). Due to assumption (4.1) we therefore have an initial value $O(\varepsilon h) = O(h^{q+1})$ after the transients, and (4.2a) can again be assumed.

5. The mildly stiff case. We will now discuss situations where ε is small but, in contrast to assumption (4.1), not at the h^q-level. Such situations will be called "mildly stiff." It will turn out that in the mildly stiff case $R_\nu = O(h^{q+1})$ is not true at the first gridpoints after a change of the stepsize. Our analysis will show that, however, these order reduction effects are *damped away* with increasing ν such that the full order reappears at the later gridpoints.

In the mildly stiff case we further distinguish two cases:

(i) $\varepsilon \leqq Ch$,

(ii) $h \leqq C\varepsilon$,

where C denotes some constant of moderate size. To motivate this, let us have a look at the "ε-h-plane," which divides into several subdomains representing the various cases (cf. Fig. 1). Note that we have characterized the classical case by "ε not small." This appears not to be in accordance with usual definitions of stiffness distinguishing between "nonstiff" and "stiff" via "hL is small" and "hL is large," respectively. (L is the conventional Lipschitz constant, i.e., $L = O(1/\varepsilon)$). With such a definition of stiffness, subdomain D (where $hL \approx h/\varepsilon \leqq C$ is small) would be considered as the "nonstiff region" of the ε-h-plane, and we could expect that no further analysis would need to be done to cover D. In our opinion, however, only in the transient phase—where we usually work with small stepsizes satisfying $h \leqq C\varepsilon$—is it justified to consider the problem as nonstiff. In the transients, small stepsizes are necessary to obtain small local errors in spite of the large derivatives of $z(t)$. But after the transients (where $z(t)$ is smooth), a very small stepsize $h \leqq C\varepsilon$ is necessary only under very strong accuracy requirements. Such an extremely good level of accuracy cannot, however, be guaranteed

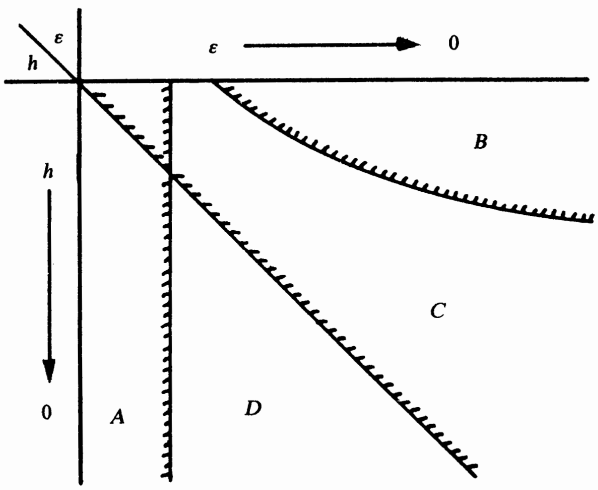

FIG. 1. A: *Subdomain of the ε-h-plane for which the problem is not stiff (classical case: ε is not a small parameter).* B: *Subdomain of the ε-h-plane for which $\varepsilon \leqq Ch^q$ (strongly stiff case).* C\B: *Subdomain of the ε-h-plane for which $\varepsilon \leqq Ch$ but not $\varepsilon \leqq Ch^q$ (mildly stiff, case* (i): *"moderately stiff").* D: *Subdomain of the ε-h-plane for which $h \leqq C\varepsilon$ (mildly stiff, case* (ii): *"weakly stiff").*

by means of a classical convergence estimate based on L:

$$\|\zeta_\nu - z(t_\nu)\| \leqq \|\zeta_0 - z(0)\| e^{Lt_\nu} + \frac{M_2}{2} h \frac{e^{Lt_\nu} - 1}{L} \quad \text{where } \|z''(t)\| \leqq M_2 \quad \text{for } t \in [0, T].$$

If $L \gg 0$, e^{Lt_ν} is large also for stepsizes h which are so small that $hL \ll 1$. Therefore the above convergence bound is not competitive with a B-convergence bound based on the one-sided Lipschitz constant m. Similarly, classical estimates for R_ν based on L are not appropriate in subdomain D. We have to investigate R_ν whenever L is large, i.e., in the case where ε is small, and $h \leqq C\varepsilon$ must also be analyzed.

Our results for the cases (i) ($\varepsilon \leqq Ch$) and (ii) ($h \leqq C\varepsilon$) are given in §§ 5.2 and 5.3, respectively.

5.1. The transformed remainder term equation. Due to (2.7a), the nonlinear term $\hat{J}(R_\nu) \cdot R_\nu$ within the remainder term equation (2.12) can be expanded into

$$\begin{aligned} \hat{J}(R_\nu) \cdot R_\nu &= f(t_\nu, z(t_\nu) + v_q(t_\nu) + R_\nu) - f(t_\nu, z(t_\nu) + v_q(t_\nu)) \\ &= f(t_\nu, z(t_\nu) + v_q(t_\nu)) + f_y(t_\nu, z(t_\nu) + v_q(t_\nu)) \cdot R_\nu \\ &\quad + \int_0^1 f_{yy}(t_\nu, z(t_\nu) + v_q(t_\nu) + \sigma R_\nu)(1-\sigma)\, d\sigma \cdot R_\nu^2 - f(t_\nu, z(t_\nu) + v_q(t_\nu)) \\ &= f_y(t_\nu, z(t_\nu)) \cdot R_\nu + \int_0^1 f_{yy}(t_\nu, z(t_\nu) + \sigma v_q(t_\nu))\, d\sigma \cdot v_q(t_\nu) \cdot R_\nu \\ &\quad + \int_0^1 f_{yy}(t_\nu, z(t_\nu) + v_q(t_\nu) + \sigma R_\nu)(1-\sigma)\, d\sigma \cdot R_\nu^2; \end{aligned} \tag{5.1}$$

here we have used the abbreviation

$$v_q(t) := he_1(t) + \cdots + h^q e_q(t). \tag{5.2a}$$

Similarly, we denote

$$\bar{v}_q(t) := h\bar{e}_1(t) + \cdots + h^q \bar{e}_q(t). \tag{5.2b}$$

Using the notation

$$\begin{aligned} G_\nu(R_\nu) \equiv G(t_\nu, R_\nu) &:= \int_0^1 f_{yy}(t_\nu, z(t_\nu) + \sigma v_q(t_\nu))\, d\sigma \cdot v_q(t_\nu) \cdot R_\nu \\ &\quad + \int_0^1 f_{yy}(t_\nu, z(t_\nu) + v_q(t_\nu) + \sigma R_\nu)(1-\sigma)\, d\sigma \cdot R_\nu^2, \end{aligned} \tag{5.3}$$

we rewrite (2.12):

$$\frac{1}{h}(R_\nu - R_{\nu-1}) = f_y(t_\nu, z(t_\nu)) \cdot R_\nu + G(t_\nu, R_\nu) + b_\nu - c_\nu. \tag{5.4}$$

Due to our smoothness assumptions concerning f_{yy} (cf. (3.1c)), $G_\nu(R)$ is Lipschitz continuous with a moderate Lipschitz bound; let L_G denote a common Lipschitz bound for the G_ν uniformly in ν. We have

$$G_\nu(0) = 0, \qquad \|G_\nu(R)\| \leqq L_G \|R\|. \tag{5.5}$$

Now we transform (5.4) analogously as in § 3, using the transformation

$$T_\nu := T(t_\nu) \tag{5.6a}$$

that diagonalizes $f_y(t_\nu, z(t_\nu))$ (cf. (3.1a), (3.1b)):

$$f_y(t_\nu, z(t_\nu)) = T_\nu \Lambda_\nu T_\nu^{-1} \tag{5.6b}$$

with

$$\Lambda_\nu := \Lambda(t_\nu) = \begin{pmatrix} c_1(t_\nu) & 0 \\ 0 & -c_2(t_\nu)/\varepsilon \end{pmatrix}. \tag{5.6c}$$

Denoting

$$\begin{aligned} &\rho_\nu := T_\nu^{-1} R_\nu, \qquad \bar{v}_q(t_\nu) := T_\nu^{-1} v_q(t_\nu), \\ &\Theta_\nu := -T_\nu^{-1} \frac{1}{h}(T_\nu - T_{\nu-1}) = \frac{1}{h}(T_\nu^{-1} - T_{\nu-1}^{-1}) T_{\nu-1}, \\ &\delta_\nu := T_\nu^{-1}(b_\nu - c_\nu), \end{aligned} \tag{5.7}$$

and premultiplying all terms in (5.4) by T_ν^{-1}, we obtain

$$\begin{aligned} T_\nu^{-1} \frac{1}{h}(R_\nu - R_{\nu-1}) &= \frac{1}{h}[T_\nu^{-1} R_\nu - T_{\nu-1}^{-1} R_{\nu-1} + (T_{\nu-1}^{-1} - T_\nu^{-1}) R_{\nu-1}] \\ &= \frac{1}{h}(\rho_\nu - \rho_{\nu-1}) - \Theta_\nu \rho_{\nu-1}, \end{aligned} \tag{5.8a}$$

$$T_\nu^{-1} f_y(t_\nu, z(t_\nu)) T_\nu T_\nu^{-1} R_\nu = \Lambda_\nu \rho_\nu, \tag{5.8b}$$

$$\begin{aligned} \Gamma_\nu(\rho_\nu) &:= T_\nu^{-1} G_\nu(T_\nu \rho_\nu) \\ &= T_\nu^{-1} \int_0^1 f_{yy}(t_\nu, z(t_\nu) + \sigma T_\nu \bar{v}_q(t_\nu))\, d\sigma \cdot T_\nu \bar{v}_q(t_\nu) \cdot T_\nu \rho_\nu \\ &\quad + T_\nu^{-1} \int_0^1 f_{yy}(t_\nu, z(t_\nu) + T_\nu \bar{v}_q(t_\nu) + \sigma T_\nu \rho_\nu)(1-\sigma)\, d\sigma \cdot (T_\nu \rho_\nu)^2. \end{aligned} \tag{5.8c}$$

We end up with the transformed remainder term equation

$$\frac{1}{h}(\rho_\nu - \rho_{\nu-1}) = \Lambda_\nu \rho_\nu + \Theta_\nu \rho_{\nu-1} + \Gamma_\nu(\rho_\nu) + \delta_\nu. \tag{5.9}$$

Note the analogy between (5.9) and the transformed VEs (cf. (3.4)):

$$\bar{e}'(t) = \Lambda(t)\bar{e}(t) + A(t)\bar{e}(t) + \bar{g}(t)$$

(where $A(t) = -T^{-1}(t) T'(t)$). In contrast to the VEs, the difference equation (5.9) is nonlinear. Using the notation

$$\begin{aligned} &\rho_\nu = \begin{pmatrix} x_\nu \\ y_\nu \end{pmatrix}, \qquad \delta_\nu = \begin{pmatrix} \delta_{\nu;1} \\ \delta_{\nu;2} \end{pmatrix}, \\ &\Theta_\nu = \begin{pmatrix} \vartheta_{1,1}(t_\nu) & \vartheta_{1,2}(t_\nu) \\ \vartheta_{2,1}(t_\nu) & \vartheta_{2,2}(t_\nu) \end{pmatrix}, \qquad \Gamma_\nu(\rho_\nu) = \begin{pmatrix} \gamma_1(t_\nu, x_\nu, y_\nu) \\ \gamma_2(t_\nu, x_\nu, y_\nu) \end{pmatrix}, \end{aligned} \tag{5.10}$$

we obtain from (5.9) after multiplying the second component by ε:

$$\begin{aligned} \frac{1}{h}(x_\nu - x_{\nu-1}) &= c_1(t_\nu) x_\nu + \vartheta_{1,1}(t_\nu) x_{\nu-1} + \vartheta_{1,2}(t_\nu) y_{\nu-1} + \gamma_1(t_\nu, x_\nu, y_\nu) + \delta_{\nu;1}, \\ \frac{\varepsilon}{h}(y_\nu - y_{\nu-1}) &= -c_2(t_\nu) y_\nu + \varepsilon \vartheta_{2,1}(t_\nu) x_{\nu-1} + \varepsilon \vartheta_{2,2}(t_\nu) y_{\nu-1} + \varepsilon \gamma_2(t_\nu, x_\nu, y_\nu) + \varepsilon \delta_{\nu;2}. \end{aligned} \tag{5.11}$$

5.2. Discrete singular perturbation analysis: the case $\varepsilon \leqq Ch$. In Theorem 5.1 we establish the existence of an asymptotic error expansion for $q=4$. The general case will be discussed in Theorem 5.6, where we prove the existence of an asymptotic expansion in a slightly weaker sense. The assertion about the structure of the global error in Theorem 5.6 is, however, sufficient for many applications.

THEOREM 5.1 (The mildly stiff case, $\varepsilon \leqq Ch$, $q=4$). *Assume that our smoothness assumptions* (3.1a–c) *are satisfied and that $\varepsilon \leqq Ch$. Consider a subinterval* $[0, T]$ *of the whole integration interval with constant stepsize h and assume inductively that the accumulated global error $\zeta_0 - z(0)$ at $t=0$ satisfies*

$$T^{-1}(0)(\zeta_0 - z(0)) = h\begin{pmatrix} \hat{x}_{0,1} + \varepsilon \hat{x}_{1,1} + \cdots + \varepsilon^3 \hat{x}_{3,1} + O(\varepsilon^4) \\ \varepsilon \hat{y}_{1,1} + \cdots + \varepsilon^3 \hat{y}_{3,1} + O(\varepsilon^4) \end{pmatrix} + \cdots + h^4 \begin{pmatrix} \hat{x}_{0,4} + O(\varepsilon) \\ O(\varepsilon) \end{pmatrix} + O(h^5), \tag{5.12a}$$

with certain $\hat{x}_{0,i}$ and $\hat{y}_{0,i}$ independent of h and ε. Then the discretization error of the implicit Euler scheme on $[0, T]$ *admits an asymptotic expansion*

$$\zeta_\nu - z(t_\nu) = he_1(t_\nu) + \cdots + h^4 e_4(t_\nu) + R_\nu \tag{5.12b}$$

with smooth, h-independent functions $e_i(t)$ that are solutions of the VEs (*cf.* (2.10), (2.11), (3.2)), *and $R_\nu = T_\nu \rho_\nu$, where*

$$\rho_\nu = \begin{pmatrix} x_\nu \\ y_\nu \end{pmatrix} = \begin{pmatrix} h^3 \xi_\nu^{(1)} + h^4 \xi_\nu^{(2)} \\ h^2 \eta_\nu^{(1)} + h^3 \eta_\nu^{(2)} + h^4 \eta_\nu^{(3)} \end{pmatrix} + O(h^5). \tag{5.12c}$$

The terms $\xi_\nu^{(j)}$, $\eta_\nu^{(j)}$ are bounded by

$$\frac{\varepsilon}{h} C(\varepsilon, h) p(\nu) |Q|^{-\nu}, \tag{5.12d}$$

which is rapidly decaying with increasing ν. $C(\varepsilon, h)$ denotes some quantity of moderate size uniformly for $\varepsilon \leqq Ch$. $p(\nu)$ denotes some polynomial of low degree, and

$$Q := 1 + \frac{c_2(0)h}{\varepsilon}. \tag{5.12e}$$

Remark. For $\nu \to \infty$, (5.12d) tends to zero. Moreover, (5.12d) is uniformly bounded for all ν and $\varepsilon \leqq Ch$. The particular behaviour of (5.12d) depends on the size of

$$|Q| = \left| 1 + \frac{c_2(0)h}{\varepsilon} \right| \geqq 1 + \frac{|c_2(0)|}{C}. \tag{5.12f}$$

For moderate C, $1 + |c_2(0)|/C$ and therefore $|Q|$ is bounded away from 1. Hence (5.12d) is indeed rapidly decaying and achieves the $O(h^5)$-level after a few steps. The (low) degree of the polynomials within (5.12d) can be seen from the proof below.

Remark. Analogously as in Theorem 4.1, $x = O(h^5)$ (cf. (5.12c)) now means—under the assumption $\varepsilon \leqq Ch$—that $\|x\| \leqq \text{const.}\, h^5$ is true for $h \in [\varepsilon/C, h_0]$.

For the proof of Theorem 5.1 some technical preliminaries are required. In the sequel we will use a generic notation for polynomials: $p(\cdot)$ denotes some polynomial (degree not specified); $p_k(\cdot)$ denotes some polynomial of degree $\leqq k$. Multiple occurrences of $p(\cdot)$ or $p_k(\cdot)$ within a formula usually indicate different polynomials.

LEMMA 5.2. *The solution of a difference equation of the form*

$$\frac{\varepsilon}{h}(\eta_\nu - \eta_{\nu-1}) = -c_2(0)\eta_\nu + j_\nu \tag{5.13a}$$

with starting value η_0 *is*

$$\eta_\nu = Q^{-\nu}\eta_0 + \frac{h}{\varepsilon}\sum_{\mu=1}^{\nu} Q^{\mu-1-\nu} j_\mu, \qquad \nu \geqq 0 \tag{5.13b}$$

with Q from (5.12e).

Proof. The proof is by induction with respect to ν.

LEMMA 5.3. *Let* $\Sigma_k^{(\nu)} := \sum_{\mu=1}^{\nu} \mu^k Q^{-\mu}$. *Then the following recursive representation holds*:

$$\Sigma_0^{(\nu)} = \frac{1}{1-Q}(Q^{-\nu}-1), \tag{5.14a}$$

$$\Sigma_k^{(\nu)} = \frac{1}{1-Q}\nu^k Q^{-\nu} + \frac{Q}{1-Q}\sum_{l=0}^{k-1}(-1)^{k-l}\binom{k}{l}\Sigma_l^{(\nu)}, \qquad k \geqq 1. \tag{5.14b}$$

Proof. The proof is by summation by parts. □

LEMMA 5.4. *Let* $p_k(\cdot)$ *denote some polynomial of degree* $\leqq k$ *with coefficients independent of h and* ε. *Consider the difference equation* (5.13a) *with an inhomogeneity of the form*

$$j_\nu = p_k(\nu) Q^{-\nu}. \tag{5.15a}$$

Then,

$$\eta_\nu = Q^{-\nu}\eta_0 + \frac{h}{\varepsilon}\nu p_k(\nu) Q^{-\nu-1}, \qquad \nu \geqq 0, \tag{5.15b}$$

with some other polynomial $p_k(\cdot)$ *of degree* $\leqq k$ *with coefficients independent of h and* ε.

Proof. From (5.13b),

$$\eta_\nu = Q^{-\nu}\eta_0 + \frac{h}{\varepsilon}\sum_{\mu=1}^{\nu} Q^{\mu-1-\nu} p_k(\mu) Q^{-\mu} = Q^{-\nu}\eta_0 + \frac{h}{\varepsilon} Q^{-\nu-1}\sum_{\mu=1}^{\nu} p_k(\mu),$$

which implies (5.15b). (The fact that $\sum_{\mu=1}^{\nu} p_k(\mu)$ is a polynomial of degree $\leqq k+1$ in ν with a nonvanishing constant coefficient can easily be shown by induction.) □

LEMMA 5.5. *Let the* $p_k(\cdot)$ *denote generic polynomials of degree* $\leqq k$ *with coefficients independent of h and* ε. *Then,*

$$\sum_{\mu=1}^{\nu} p_k(\mu) Q^{-\mu} = \frac{\varepsilon}{h}\left[p_k\left(\frac{\varepsilon}{h}\right) p_k(\nu) Q^{-\nu} + p_k\left(\frac{\varepsilon}{h}\right)\right]. \tag{5.16}$$

Proof. The proof is by induction with respect to k, using Lemma 5.3. First we note that for Q from (5.12e),

$$\frac{1}{1-Q} = -\frac{\varepsilon}{c_2(0)h}, \qquad \frac{Q}{1-Q} = -\left(1+\frac{\varepsilon}{c_2(0)h}\right). \tag{5.17}$$

$k=0$. From (5.14a) and (5.17),

$$\sum_{\mu=1}^{\nu} Q^{-\mu} = -\frac{\varepsilon}{c_2(0)h}[Q^{-\nu}-1] = \frac{\varepsilon}{h}\left[p_0\left(\frac{\varepsilon}{h}\right) p_0(\nu) Q^{-\nu} + p_0\left(\frac{\varepsilon}{h}\right)\right].$$

$k \geqq 1$. Assume (5.16) for $l = 1, \cdots, k-1$. From (5.14b) and (5.17),

$$\sum_{\mu=1}^{\nu} \mu^k Q^{-\mu} = -\frac{\varepsilon}{c_2(0)h}\nu^k Q^{-\nu} - \left(1+\frac{\varepsilon}{c_2(0)h}\right)\sum_{l=0}^{k-1} C_{kl}\frac{\varepsilon}{h}\left[p_l\left(\frac{\varepsilon}{h}\right) p_l(\nu) Q^{-\nu} + p_l\left(\frac{\varepsilon}{h}\right)\right],$$

which implies (5.16). (For $l \leqq k-1$, $(1+\varepsilon/(c_2(0)h))p_l(\varepsilon/h)$ is a polynomial of degree $\leqq k$ in ε/h.) □

Proof of Theorem 5.1. For the solution $\rho_\nu = \binom{x_\nu}{y_\nu}$ of the transformed remainder term equation (5.11) we make a *singular perturbation ansatz in powers of the stepsize h*:[5]

$$
(5.18)\qquad
\begin{aligned}
x_\nu &= hX_\nu^{(0)} + h^2X_\nu^{(1)} + \cdots + h^5X_\nu^{(4)} + h^2\xi_\nu^{(0)} + \cdots + h^5\xi_\nu^{(3)} + x_\nu^{(\mathrm{rem})},\\
y_\nu &= hY_\nu^{(0)} + h^2Y_\nu^{(1)} + \cdots + h^5Y_\nu^{(4)} + h\eta_\nu^{(0)} + h^2\eta_\nu^{(1)} + \cdots + h^5\eta_\nu^{(4)} + y_\nu^{(\mathrm{rem})}.
\end{aligned}
$$

The main idea of the proof is to show that there exists a set of smooth solutions $\bar{e}_i(t)$ of the transformed VEs in the sense of § 3 such that all "discrete outer solution terms" $X_\nu^{(j)}$, $Y_\nu^{(j)}$ vanish up to the h^5-level. The "discrete inner solution terms" $\xi_\nu^{(j)}$, $\eta_\nu^{(j)}$ will turn out to be bounded by (5.12d) (moreover, $\xi_\nu^{(0)} \equiv \eta_\nu^{(0)} \equiv 0$).

Recall that for *any* set of smooth solutions in the sense of Proposition 3.2 we have

$$
(5.19a)\qquad \delta_\nu = \begin{pmatrix}\delta_{\nu;1}\\ \delta_{\nu;2}\end{pmatrix} = T_\nu^{-1}(b_\nu - c_\nu) = O(h^5);
$$

in the sequel we will write

$$
(5.19b)\qquad \delta_\nu = \begin{pmatrix}\delta_{\nu;1}\\ \delta_{\nu;2}\end{pmatrix} = \begin{pmatrix}h^5\hat{\delta}_{\nu;1}\\ h^5\hat{\delta}_{\nu;2}\end{pmatrix},
$$

with $\hat{\delta}_{\nu;j} = O(1)$.

For $\nu = 0$, (5.12b) reads

$$
(5.20a)\qquad \zeta_0 - z(0) = he_1(0) + \cdots + h^q e_q(0) + R_0;
$$

after transformation (3.3) we therefore have the following initial condition for the remainder term:

$$
(5.20b)\qquad \rho_0 := T^{-1}(0)R_0 = T^{-1}(0)(\zeta_0 - z(0)) - h\bar{e}_1(0) - \cdots - h^q\bar{e}_q(0).
$$

The starting values for smooth solutions $\bar{e}_i(t)$ are of the form

$$
(5.21)\qquad \bar{e}_i(0) = \begin{pmatrix} x_{0,i} + \varepsilon x_{1,i} + \cdots + \varepsilon^{5-i}x_{5-i,i} + O(\varepsilon^{6-i})\\ \varepsilon y_{1,i} + \cdots + \varepsilon^{5-i}y_{5-i,i} + O(\varepsilon^{6-i})\end{pmatrix}, \qquad i = 1, \cdots, 4
$$

where the $x_{k,i}$ can be chosen arbitrarily and the $y_{k,i}$ $(k \leqq 5-i)$ are then fixed by $x_{0,i} \cdots x_{k-1,i}$, $y_{0,i} \cdots y_{k-1,i}$ (cf. (3.29)).

We have (inductively) assumed that at the end of the preceding subinterval the (transformed) global error satisfies (5.12a). From this and with (5.20b) and (5.21) we obtain

$$
(5.22)\qquad
\begin{aligned}
\rho_0 = {} & h\begin{pmatrix}\hat{x}_{0,1} - x_{0,1}\\ 0\end{pmatrix} + h^2\begin{pmatrix}\hat{x}_{0,2} - x_{0,2} + \dfrac{\varepsilon}{h}(\hat{x}_{1,1} - x_{1,1})\\ \dfrac{\varepsilon}{h}(\hat{y}_{1,1} - y_{1,1})\end{pmatrix}\\
& + h^3\begin{pmatrix}\hat{x}_{0,3} - x_{0,3} + \dfrac{\varepsilon}{h}(\hat{x}_{1,2} - x_{1,2}) + \left(\dfrac{\varepsilon}{h}\right)^2(\hat{x}_{2,1} - x_{2,1})\\ \dfrac{\varepsilon}{h}(\hat{y}_{1,2} - y_{1,2}) + \left(\dfrac{\varepsilon}{h}\right)^2(\hat{y}_{2,1} - y_{2,1})\end{pmatrix}
\end{aligned}
$$

[5] Due to the h^5-remainder term in (5.12c) we could expect that a shorter ansatz (without the terms $h^5X_\nu^{(4)}$, $h^5Y_\nu^{(4)}$, $h^5\xi_\nu^{(3)}$, $h^5\eta_\nu^{(4)}$) would be adequate. These additional terms are introduced for technical reasons; they enable a convenient estimation of $x_\nu^{(\mathrm{rem})}$ and $y_\nu^{(\mathrm{rem})}$ at h^5-level.

$$+h^4\begin{pmatrix} \hat{x}_{0,4}-x_{0,4}+\frac{\varepsilon}{h}(\hat{x}_{1,3}-x_{1,3})+\left(\frac{\varepsilon}{h}\right)^2(\hat{x}_{2,2}-x_{2,2})+\left(\frac{\varepsilon}{h}\right)^3(\hat{x}_{3,1}-x_{3,1}) \\ \frac{\varepsilon}{h}(\hat{y}_{1,3}-y_{1,3})+\left(\frac{\varepsilon}{h}\right)^2(\hat{y}_{2,2}-y_{2,2})+\left(\frac{\varepsilon}{h}\right)^3(\hat{y}_{3,1}-y_{3,1}) \end{pmatrix}+O(h^5).$$

Here we have rearranged with respect to powers of h according to

$$h^i\varepsilon^j=h^{i+j}\left(\frac{\varepsilon}{h}\right)^j \tag{5.23}$$

with $(\varepsilon/h)^j\leqq C^j$ due to assumption $\varepsilon\leqq Ch$.

By (5.22) we have the following initial conditions for the quantities $X_\nu^{(j)}$, $Y_\nu^{(j)}$, $\xi_\nu^{(j)}$, $\eta_\nu^{(j)}$, and $x_\nu^{(\mathrm{rem})}$, $y_\nu^{(\mathrm{rem})}$ (cf. (5.18)):

$$\begin{aligned}
&h^1\text{-level:} && X_0^{(0)}=\hat{x}_{0,1}-x_{0,1},\\
& && Y_0^{(0)}+\eta_0^{(0)}=0,\\
&h^2\text{-level:} && X_0^{(1)}+\xi_0^{(0)}=\hat{x}_{0,2}-x_{0,2}+\frac{\varepsilon}{h}(\hat{x}_{1,1}-x_{1,1}),\\
& && Y_0^{(1)}+\eta_0^{(1)}=\frac{\varepsilon}{h}(\hat{y}_{1,1}-y_{1,1}),\\
&h^3\text{-level:} && X_0^{(2)}+\xi_0^{(1)}=\hat{x}_{0,3}-x_{0,3}+\frac{\varepsilon}{h}(\hat{x}_{1,2}-x_{1,2})+\left(\frac{\varepsilon}{h}\right)^2(\hat{x}_{2,1}-x_{2,1}),\\
& && Y_0^{(2)}+\eta_0^{(2)}=\frac{\varepsilon}{h}(\hat{y}_{1,2}-y_{1,2})+\left(\frac{\varepsilon}{h}\right)^2(\hat{y}_{2,1}-y_{2,1}),\\
&h^4\text{-level:} && X_0^{(3)}+\xi_0^{(2)}=\hat{x}_{0,4}-x_{0,4}+\frac{\varepsilon}{h}(\hat{x}_{1,3}-x_{1,3})+\cdots+\left(\frac{\varepsilon}{h}\right)^3(\hat{x}_{3,1}-x_{3,1}),\\
& && Y_0^{(3)}+\eta_0^{(3)}=\frac{\varepsilon}{h}(\hat{y}_{1,3}-y_{1,3})+\cdots+\left(\frac{\varepsilon}{h}\right)^3(\hat{y}_{3,1}-y_{3,1}),\\
&h^5\text{-level:} && h^5(X_0^{(4)}+\xi_0^{(3)})+x_0^{(\mathrm{rem})}=O(h^5),\\
& && h^5(Y_0^{(4)}+\eta_0^{(3)})+y_0^{(\mathrm{rem})}=O(h^5).
\end{aligned} \tag{5.24a}$$

Concerning the freedom at the h^5-level, it will turn out to be appropriate (for technical reasons) to choose

$$\begin{aligned} (X_0^{(4)}+\xi_0^{(3)})&=0, \qquad x_0^{(\mathrm{rem})}=O(h^5),\\ (Y_0^{(4)}+\eta_0^{(4)})&=0, \qquad y_0^{(\mathrm{rem})}=O(h^5). \end{aligned} \tag{5.24b}$$

As usual in the singular perturbation theory we will use expansions of the data functions $c_k(t)$, $\vartheta_{k,l}(t)$ and of $\gamma_k(t,x,y)$ (cf. (3.1b), (5.7), (5.8c), (5.10)) around $t=0$:

$$c_k(t_\nu)=c_k(0)+h\nu c_k'(0)+h^2\frac{\nu^2}{2}c_k''(0)+\cdots. \tag{5.25}$$

Due to the definition of Θ_ν (cf. (5.7)) the $\vartheta_{k,l}(t)$ are h-dependent functions. Θ_ν is expanded into

$$\begin{aligned}\Theta_\nu = \Theta(t_\nu) &= -\frac{1}{h} T^{-1}(t_\nu)[T(t_\nu) - T(t_\nu - h)] \\ &= -T^{-1}(t_\nu)T'(t_\nu) + h\frac{1}{2} T^{-1}(t_\nu)T''(t_\nu) + \cdots \\ &= A(t_\nu) + h\bar{A}(t_\nu) + \cdots = A(0) + t_\nu A'(0) + \cdots + h\bar{A}(0) + \cdots \\ &= A(0) + h(\nu A'(0) + \bar{A}(0)) + \cdots . \end{aligned} \tag{5.26a}$$

(Here we have used the notation $A(t) := -T^{-1}(t)T'(t)$ (cf. (3.4b)), $\bar{A}(t) := \frac{1}{2}T^{-1}(t)T''(t)$.) According to (5.26a) we have

$$\vartheta_{k,l}(t_\nu) = a_{k,l}(0) + h(\nu a'_{k,l}(0) + \bar{a}_{k,l}(0)) + h^2 p_2(\nu) + \cdots , \tag{5.26b}$$

with a generic polynomial $p_2(\nu)$ of degree less than or equal to 2.

At first sight, these expansions at $t_\nu = \nu h$ around $t = 0$ seem to be inadequate since for fixed $t \equiv t_\nu$, $\nu \to \infty$ as $h \to 0$. However, (5.25) and (5.26b) will only be used in conjunction with inner solution terms $\xi_\nu^{(j)}$ and $\eta_\nu^{(j)}$. It will turn out in the sequel that the $\xi_\nu^{(j)}$ and $\eta_\nu^{(j)}$ always contain a factor $Q^{-\nu}$; the use of (5.25), (5.26b) is then justified by the fact that $\nu^k Q^{-\nu}$ is bounded for $\varepsilon \leqq Ch$ and rapidly decaying. ($\nu^k Q^{-\nu}$ is a discrete analogue of $\tau^k e^{-c_2(0)\tau} = (t/\varepsilon)^k e^{-c_2(0)t/\varepsilon}$—cf. (3.11).)

The *nonlinear* terms $\gamma_k(t_\nu, x_\nu, y_\nu)$ within (5.11) satisfy[6]

$$\gamma_k(t_\nu, 0, 0) = 0, \tag{5.27a}$$

$$\gamma_{kx}(t_\nu, 0, 0) = O(h), \qquad \gamma_{ky}(t_\nu, 0, 0) = O(h). \tag{5.27b}$$

(All higher derivatives are $O(1)$ due to our smoothness assumptions.) On the basis of the ansatz (5.18) we expand the $\gamma_k(t_\nu, x_\nu, y_\nu)$ into

$$\begin{aligned}\gamma_k(t_\nu, x_\nu, y_\nu) &= \gamma_k(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)} + h\eta_\nu^{(0)}) + h^2\gamma_{kx}(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)} + h\eta_\nu^{(0)})(X_\nu^{(1)} + \xi_\nu^{(0)}) \\ &\quad + h^2\gamma_{ky}(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)} + h\eta_\nu^{(0)})(Y_\nu^{(1)} + \eta_\nu^{(1)}) + \cdots \\ &= \gamma_k(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)}) \\ &\quad + [\gamma_k(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)} + h\eta_\nu^{(0)}) - \gamma_k(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)})] + O(h^2),\end{aligned} \tag{5.28a}$$

where, according to (5.27a),

$$\gamma_k(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)}) = \gamma_k(t_\nu, 0, 0) + O(h^2) = O(h^2), \tag{5.28b}$$

and according to (5.27a, b),

$$\begin{aligned}&[\gamma_k(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)} + h\eta_\nu^{(0)}) - \gamma_k(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)})] \\ &\quad = \gamma_{ky}(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)})h\eta_\nu^{(0)} + O(h^2) \\ &\quad = [\gamma_{ky}(t_\nu, 0, 0) + O(h)]h\eta_\nu^{(0)} + O(h^2) = O(h^2).\end{aligned} \tag{5.28c}$$

[6] Equation (5.27a) follows immediately from the definition of Γ_ν (cf. (5.8c)). With the notation

$$W \equiv \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix} := T_\nu^{-1} \int_0^1 f_{yy}(t_\nu, z(t_\nu) + \sigma T_\nu \bar{v}_4(t_\nu))\, d\sigma \cdot T_\nu \bar{v}_4(t_\nu) \cdot T_\nu$$

for the matrix appearing in the first term of (5.8c), we have

$$\frac{\partial}{\partial x}(W\cdot\rho) = \frac{\partial}{\partial x}\left(W\cdot\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} w_{1,1} \\ w_{2,1} \end{pmatrix}, \qquad \frac{\partial}{\partial y}(W\cdot\rho) = \frac{\partial}{\partial y}\left(W\cdot\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} w_{1,2} \\ w_{2,2} \end{pmatrix};$$

note that $W = O(h)$ due to the factor $\bar{v}_4(t_\nu) = O(h)$. The partial derivatives of the second term of (5.8c) (with the bilinear operator) obviously vanish for $\rho = 0$. Thus (5.27b) holds.

Expansions of the terms (5.28b, c) around $t=0$ will also be used but are not needed for the moment.

After these preliminaries we now present the essential part of the proof. We insert the ansatz (5.18) into the difference equation (5.11) and equate coefficients of h^1, $h^2, \cdots$. It will turn out that there exist starting values for smooth solutions $\bar{e}_i(t)$ of the (transformed) VEs that are compatible with vanishing outer solution terms $X_\nu^{(j)}$, $Y_\nu^{(j)}$ in (5.18).

As a first step we equate coefficients of h^1. According to (5.28a–c), only $\gamma_1(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)})$ is at h^1-level. Therefore the leading terms of the difference equation (5.11) read:

$$h\frac{1}{h}(X_\nu^{(0)}-X_{\nu-1}^{(0)})+\cdots+h(\xi_\nu^{(0)}-\xi_{\nu-1}^{(0)})+\cdots$$

$$=hc_1(t_\nu)X_\nu^{(0)}+\cdots+h\vartheta_{1,1}(t_\nu)X_{\nu-1}^{(0)}+\cdots+h\vartheta_{1,2}(t_\nu)Y_{\nu-1}^{(0)}+\cdots$$
$$+h[a_{1,2}(0)+\cdots]\eta_{\nu-1}^{(0)}+\cdots+\gamma_1(t_\nu, hX_\nu^{(0)}, hY_\nu^{(0)})+\cdots, \tag{5.29a}$$

$$h^2\frac{\varepsilon}{h}\frac{1}{h}(Y_\nu^{(0)}-Y_{\nu-1}^{(0)})+\cdots+h\frac{\varepsilon}{h}(\eta_\nu^{(0)}-\eta_{\nu-1}^{(0)})+\cdots$$
$$=-hc_2(t_\nu)Y_\nu^{(0)}+\cdots-h[c_2(0)+\cdots]\eta_\nu^{(0)}+\cdots. \tag{5.29b}$$

According to (5.23) the quantities $h\varepsilon\vartheta_{2,1}(t_\nu)X_{\nu-1}^{(0)}=h^2(\varepsilon/h)\vartheta_{2,1}(t_\nu)X_{\nu-1}^{(0)},\cdots$ are at h^2-level and have therefore been omitted within (5.29b). The difference quotients $1/h(\cdots)$ and $\varepsilon/h(\cdots)$ are the discrete analogues of the derivatives d/dt and $d/d\tau$ ($\tau=t/\varepsilon$). Note that, again according to (5.23), the first term on the left-hand side of (5.29b) appears already at h^2-level and does not influence the equations at h^1-level. Hence $Y_\nu^{(0)}$ will be fixed at h^1-level by an algebraic equation (and not by a difference equation as is the case for $X_\nu^{(0)}$). This situation is analogous to one that occurs for singular perturbation theory for differential equations (cf., for instance, (3.19), where the equation for $Y_1(t)$ is algebraic; the derivative $Y_1'(t)$ appears as an inhomogeneous term in (3.21)).

Coefficients of h^1.

$$\frac{1}{h}(X_\nu^{(0)}-X_{\nu-1}^{(0)})=c_1(t_\nu)X_\nu^{(0)}+\vartheta_{1,1}(t_\nu)X_{\nu-1}^{(0)}+\vartheta_{1,2}(t_\nu)Y_{\nu-1}^{(0)},$$
$$0=-c_2(t_\nu)Y_\nu^{(0)}, \tag{5.30a}$$

$$\xi_\nu^{(0)}-\xi_{\nu-1}^{(0)}=a_{1,2}(0)\eta_{\nu-1}^{(0)}, \qquad \frac{\varepsilon}{h}(\eta_\nu^{(0)}-\eta_{\nu-1}^{(0)})=-c_2(0)\eta_\nu^{(0)}. \tag{5.30b}$$

From (5.30a), $Y_\nu^{(0)}\equiv 0$. Due to (5.27a) this implies $X_\nu^{(0)}\equiv 0$ if $X_0^{(0)}=0$. The latter can be achieved by the choice

$$x_{0,1}:=\hat{x}_{0,1} \tag{5.31a}$$

(cf. (5.24a) and cf. (5.12a), (5.20), (5.21) for the meaning of $x_{0,1}$, $\hat{x}_{0,1}$). Here we have made use of the degree of freedom within the first component of the starting value for $\bar{e}_1(t)$ at ε^0-level (cf. Proposition 3.2). Thus,

$$X_\nu^{(0)}\equiv Y_\nu^{(0)}\equiv 0. \tag{5.31b}$$

Furthermore, (5.24a) and $Y_0^{(0)}=0$ imply $\eta_0^{(0)}=0$. Thus, $\eta_\nu^{(0)}\equiv 0$ by (5.30b). Moreover, (5.30b) and the usual requirement $\lim_{\nu\to\infty}\xi_\nu^{(0)}=0$ imply $\xi_\nu^{(0)}\equiv \text{const.}=0$. Thus,

$$\xi_\nu^{(0)}\equiv \eta_\nu^{(0)}\equiv 0. \tag{5.31c}$$

Due to (5.31b, c) the term $[\cdots]$ within (5.28a) vanishes, and (5.28a) reduces to

$$\gamma_k(t_\nu, x_\nu, y_\nu) = \gamma_k(t_\nu, 0, 0) + h^2\gamma_{kx}(t_\nu, 0, 0)X_\nu^{(1)} + h^2\gamma_{ky}(t_\nu, 0, 0)(Y_\nu^{(1)} + \eta_\nu^{(1)}) + \cdots$$

$$= 0 + h^2\gamma_{kx}(t_\nu, 0, 0)X_\nu^{(1)} + h^2\gamma_{ky}(t_\nu, 0, 0)(Y_\nu^{(1)} + \eta_\nu^{(1)}) + \cdots = O(h^3) \tag{5.32}$$

(cf. (5.27a, b)).

Due to (5.19b), (5.25), (5.26), (5.31b, c) and (5.32), the difference equation (5.11) together with the ansatz (5.18) now attains the following form:

(5.33a)

$$\begin{aligned}
&h^2\frac{1}{h}(X_\nu^{(1)} - X_{\nu-1}^{(1)}) + \cdots + h^5\frac{1}{h}(X_\nu^{(4)} - X_{\nu-1}^{(4)})\\
&\quad + h^2(\xi_\nu^{(1)} - \xi_{\nu-1}^{(1)}) + \cdots + h^4(\xi_\nu^{(3)} - \xi_{\nu-1}^{(3)}) + \frac{1}{h}(x_\nu^{(\mathrm{rem})} - x_{\nu-1}^{(\mathrm{rem})})\\
&= c_1(t_\nu)[h^2X_\nu^{(1)} + \cdots + h^5X_\nu^{(4)}] + (c_1(0) + h\nu c_1'(0) + \cdots)[h^3\xi_\nu^{(1)} + \cdots + h^5\xi_\nu^{(3)}]\\
&\quad + \vartheta_{1,1}(t_\nu)[h^2X_{\nu-1}^{(1)} + \cdots + h^5X_{\nu-1}^{(4)}]\\
&\quad + (a_{1,1}(0) + h(\nu a_{1,1}'(0) + \bar{a}_{1,1}(0)) + \cdots)[h^3\xi_{\nu-1}^{(1)} + \cdots + h^5\xi_{\nu-1}^{(3)}]\\
&\quad + \vartheta_{1,2}(t_\nu)[h^2Y_{\nu-1}^{(1)} + \cdots + h^5Y_{\nu-1}^{(4)}]\\
&\quad + (a_{1,2}(0) + h(\nu a_{1,2}'(0) + \bar{a}_{1,2}(0)) + \cdots)[h^2\eta_{\nu-1}^{(1)} + \cdots + h^5\eta_{\nu-1}^{(4)}]\\
&\quad + [c_1(t_\nu)x_\nu^{(\mathrm{rem})} + \vartheta_{1,1}(t_\nu)x_{\nu-1}^{(\mathrm{rem})} + \vartheta_{1,2}(t_\nu)y_{\nu-1}^{(\mathrm{rem})}]\\
&\quad + hh^{-1}\gamma_{1x}(t_\nu, 0, 0)[h^2X_\nu^{(1)} + \cdots + h^5X_\nu^{(4)} + h^3\xi_\nu^{(1)} + \cdots + h^5\xi_\nu^{(3)} + x_\nu^{(\mathrm{rem})}]\\
&\quad + hh^{-1}\gamma_{1y}(t_\nu, 0, 0)[h^2Y_\nu^{(1)} + \cdots + h^5Y_\nu^{(4)} + h^2\eta_\nu^{(1)} + \cdots + h^5\eta_\nu^{(4)} + y_\nu^{(\mathrm{rem})}]\\
&\quad + \text{terms with higher } \gamma_1\text{-derivatives} + h^5\hat{\delta}_{\nu;1},
\end{aligned}$$

(5.33b)

$$\begin{aligned}
&h^3\frac{\varepsilon}{h}\frac{1}{h}(Y_\nu^{(1)} - Y_{\nu-1}^{(1)}) + \cdots + h^6\frac{\varepsilon}{h}\frac{1}{h}(Y_\nu^{(4)} - Y_{\nu-1}^{(4)})\\
&\quad + h^2\frac{\varepsilon}{h}(\eta_\nu^{(1)} - \eta_{\nu-1}^{(1)}) + \cdots + h^5\frac{\varepsilon}{h}(\eta_\nu^{(4)} - \eta_{\nu-1}^{(4)}) + \frac{\varepsilon}{h}(y_\nu^{(\mathrm{rem})} - y_{\nu-1}^{(\mathrm{rem})})\\
&= -c_2(t_\nu)[h^2Y_\nu^{(1)} + \cdots + h^5Y_\nu^{(4)}] + (-c_2(0) - h\nu c_2'(0) - \cdots)[h^2\eta_\nu^{(1)} + \cdots + h^5\eta_\nu^{(4)}]\\
&\quad + h\frac{\varepsilon}{h}\vartheta_{2,1}(t_\nu)[h^2X_{\nu-1}^{(1)} + \cdots + h^5X_{\nu-1}^{(4)}]\\
&\quad + h\frac{\varepsilon}{h}(a_{2,1}(0) + h(\nu a_{2,1}'(0) + \bar{a}_{2,1}(0)) + \cdots)[h^3\xi_{\nu-1}^{(1)} + \cdots + h^5\xi_{\nu-1}^{(3)}]\\
&\quad + h\frac{\varepsilon}{h}\vartheta_{2,2}(t_\nu)[h^2Y_{\nu-1}^{(1)} + \cdots + h^5Y_{\nu-1}^{(4)}]\\
&\quad + h\frac{\varepsilon}{h}(a_{2,2}(0) + h(\nu a_{2,2}'(0) + \bar{a}_{2,2}(0)) + \cdots)[h^2\eta_{\nu-1}^{(1)} + \cdots + h^5\eta_{\nu-1}^{(4)}]\\
&\quad + \left[-c_2(t_\nu)y_\nu^{(\mathrm{rem})} + h\frac{\varepsilon}{h}\vartheta_{2,1}(t_\nu)x_{\nu-1}^{(\mathrm{rem})} + h\frac{\varepsilon}{h}\vartheta_{2,2}(t_\nu)y_{\nu-1}^{(\mathrm{rem})}\right]\\
&\quad + h^2\frac{\varepsilon}{h}h^{-1}\gamma_{2x}(t_\nu, 0, 0)[h^2X_\nu^{(1)} + \cdots + h^5X_\nu^{(4)} + h^3\xi_\nu^{(1)} + \cdots + h^5\xi_\nu^{(3)} + x_\nu^{(\mathrm{rem})}]\\
&\quad + h^2\frac{\varepsilon}{h}h^{-1}\gamma_{2y}(t_\nu, 0, 0)[h^2Y_\nu^{(1)} + \cdots + h^5Y_\nu^{(4)} + h^2\eta_\nu^{(1)} + \cdots + h^5\eta_\nu^{(4)} + y_\nu^{(\mathrm{rem})}]
\end{aligned}$$

$$+\text{terms with higher } \gamma_2\text{-derivatives} + h^6 \frac{\varepsilon}{h} \hat{\delta}_{\nu;2}.$$

Coefficients of h^2.

(5.34a)
$$\frac{1}{h}(X_\nu^{(1)} - X_{\nu-1}^{(1)}) = c_1(t_\nu) X_\nu^{(1)} + \vartheta_{1,1}(t_\nu) X_{\nu-1}^{(1)} + \vartheta_{1,2}(t_\nu) Y_{\nu-1}^{(1)},$$
$$0 = -c_2(t_\nu) Y_\nu^{(1)},$$

(5.34b)
$$\xi_\nu^{(1)} - \xi_{\nu-1}^{(1)} = a_{1,2}(0)\eta_{\nu-1}^{(1)}, \qquad \frac{\varepsilon}{h}(\eta_\nu^{(1)} - \eta_{\nu-1}^{(1)}) = -c_2(0)\eta_\nu^{(1)}.$$

From (5.34a), $Y_\nu^{(1)} \equiv 0$. Furthermore, $X_\nu^{(1)} \equiv 0$ if $X_0^{(1)} = 0$. Due to (5.24a) and $\xi_0^{(0)} = 0$ (cf. (5.31c)), the latter can be achieved by the choice:

(5.35a)
$$x_{0,2} := \hat{x}_{0,2} \text{ at } \left(\frac{\varepsilon}{h}\right)^0\text{-level}, \qquad x_{1,1} := \hat{x}_{1,1} \text{ at } \left(\frac{\varepsilon}{h}\right)^1\text{-level}.$$

Thus,

(5.35b)
$$X_\nu^{(1)} \equiv Y_\nu^{(1)} \equiv 0.$$

Moreover, (5.24a) and $Y_0^{(1)} = 0$ imply

(5.36a)
$$\eta_0^{(1)} = \frac{\varepsilon}{h}(\hat{y}_{1,1} - y_{1,1}) = \frac{\varepsilon}{h} p_0\left(\frac{\varepsilon}{h}\right).$$

(Recall that $y_{1,1}$ is fixed by the choice (5.31a) for $x_{0,1}$; cf. (3.20).) From (5.34b) and Lemma 5.4 (with $j_\nu \equiv 0$),

(5.36b)
$$\eta_\nu^{(1)} = Q^{-\nu}\eta_0^{(1)} = \frac{\varepsilon}{h} p_0\left(\frac{\varepsilon}{h}\right) Q^{-\nu}.$$

From (5.34b) and (5.36b),

(5.37a)
$$\xi_\nu^{(1)} = \xi_0^{(1)} + a_{1,2}(0) \sum_{\mu=1}^{\nu} \eta_{\mu-1}^{(1)} = \xi_0^{(1)} + a_{1,2}(0) \frac{\varepsilon}{h} p_0\left(\frac{\varepsilon}{h}\right) Q \sum_{\mu=1}^{\nu} Q^{-\mu}.$$

With $a_{1,2}(0)(\varepsilon/h)p_0(\varepsilon/h)Q = p_1(\varepsilon/h)$ and applying Lemma 5.5, we obtain

(5.37b)
$$\xi_\nu^{(1)} = \xi_0^{(1)} + \frac{\varepsilon}{h} p_1\left(\frac{\varepsilon}{h}\right)\left[p_0\left(\frac{\varepsilon}{h}\right) p_0(\nu) Q^{-\nu} + p_0\left(\frac{\varepsilon}{h}\right)\right]$$
$$= \xi_0^{(1)} + \frac{\varepsilon}{h} p_1\left(\frac{\varepsilon}{h}\right) p_0(\nu) Q^{-\nu} + \frac{\varepsilon}{h} p_1\left(\frac{\varepsilon}{h}\right).$$

The requirement $\lim_{\nu\to\infty} \xi_\nu^{(1)} = 0$ yields

(5.38a)
$$\xi_0^{(1)} = \frac{\varepsilon}{h} p_1\left(\frac{\varepsilon}{h}\right),$$

(5.38b)
$$\xi_\nu^{(1)} = \frac{\varepsilon}{h} p_1\left(\frac{\varepsilon}{h}\right) p_0(\nu) Q^{-\nu}.$$

Due to (5.36b) and (5.38b), $\xi_\nu^{(1)}$ and $\eta_\nu^{(1)}$ can obviously be estimated by the bound (5.12d).

Within the equations at h^3-level, terms originating from the nonlinear functions $\gamma_k(t, x, y)$ appear for the first time. In particular, there is one nonlinear h^3-term, namely,

$$h^{-1}\gamma_{1y}(t_\nu, 0, 0)\eta_\nu^{(1)} \tag{5.39a}$$

(cf. (5.33a) and recall that $X_\nu^{(1)} \equiv Y_\nu^{(1)} \equiv 0$). Note that, by definition of Γ_ν (cf. (5.8c)), $\gamma_{1y}(t_\nu, 0, 0)$ depends on $\bar{v}_4(t_\nu)$ (cf. (5.2b)) and therefore on h and ε. When equating coefficients of h^3, we will see that the starting values $x_{0,3}$, $x_{1,2}$, and $x_{2,1}$ are influenced by the nonlinear term (5.39a). Since we require that the $x_{k,i}$ are independent of h and ε (to ensure a pure asymptotic expansion with h-independent functions $\bar{e}_i(t)$), it is therefore necessary to expand $\gamma_{1y}(t_\nu, 0, 0)$ in the following way:

$$\begin{aligned}
h^{-1}\gamma_{1y}(t_\nu, 0, 0) &\equiv h^{-1}\gamma_{1y}(t_\nu, 0, 0; \bar{v}_{4,1}(t_\nu), \bar{v}_{4,2}(t_\nu)) \\
&= h^{-1}\gamma_{1y}(t_\nu, 0, 0; 0, 0) + h^{-1}\gamma_{1yv_1}(t_\nu, 0, 0; 0, 0)\bar{v}_{4,1}(t_\nu) \\
&\quad + h^{-1}\gamma_{1yv_2}(t_\nu, 0, 0; 0, 0)\bar{v}_{4,2}(t_\nu) + O(h) \\
&= 0 + h^{-1}\gamma_{1yv_1}(t_\nu, 0, 0; 0, 0)[hX_{0,1}(t_\nu) + \cdots] \\
&\quad + h^{-1}\gamma_{1yv_2}(t_\nu, 0, 0; 0, 0)\left[h^2\frac{\varepsilon}{h}Y_{1,1}(t_\nu) + \cdots\right] + O(h) \\
&= \gamma_{1yv_1}(t_\nu, 0, 0; 0, 0)X_{0,1}(t_\nu) + O(h).
\end{aligned} \tag{5.39b}$$

(Here, $\bar{v}_{4,1}(t)$ and $\bar{v}_{4,2}(t)$ denote the compenents of $\bar{v}_4(t) = h\bar{e}_1(t) + O(h^2)$; we have $\bar{v}_{4,1}(t) = hX_{0,1}(t) + O(h^2)$, $\bar{v}_{4,2}(t) = h^2(\varepsilon/h)Y_{1,1}(t) + O(h^3)$; cf. (3.32a). Note that $\gamma_{1y}(t_\nu, 0, 0; 0, 0) = 0$ due to the definition of Γ_ν; cf. (5.8c).) Furthermore, since $\gamma_{1y}(t_\nu, 0, 0)$ appears within (5.39a) in conjunction with the inner solution term $\eta_\nu^{(1)}$, we expand around $t = 0$ (analogously as in (5.25), (5.26b)):

$$\gamma_{1yv_1}(t_\nu, 0, 0; 0, 0)X_{0,1}(t_\nu) =: g(t_\nu) = g(0) + h\nu g'(0) + \cdots, \tag{5.39c}$$

with $g(0) = \gamma_{1yv_1}(0, 0, 0; 0, 0)X_{0,1}(0)$ independent of h and ε.

Coefficients of h^3.

$$\begin{aligned}
&\frac{1}{h}(X_\nu^{(2)} - X_{\nu-1}^{(2)}) = c_1(t_\nu)X_\nu^{(2)} + \vartheta_{1,1}(t_\nu)X_{\nu-1}^{(2)} + \vartheta_{1,2}(t_\nu)Y_{\nu-1}^{(2)}, \\
&0 = -c_2(t_\nu)Y_\nu^{(2)},
\end{aligned} \tag{5.40a}$$

$$\xi_\nu^{(2)} - \xi_{\nu-1}^{(2)} = a_{1,2}(0)\eta_{\nu-1}^{(2)} + i_\nu^{(2)}, \qquad \frac{\varepsilon}{h}(\eta_\nu^{(2)} - \eta_{\nu-1}^{(2)}) = -c_2(0)\eta_\nu^{(2)} + j_\nu^{(2)}, \tag{5.40b}$$

with

$$\begin{aligned}
&i_\nu^{(2)} := c_1(0)\xi_\nu^{(1)} + a_{1,1}(0)\xi_{\nu-1}^{(1)} + (\nu a_{1,2}'(0) + \bar{a}_{1,2}(0))\eta_{\nu-1}^{(1)} + g(0)\eta_\nu^{(1)}, \\
&j_\nu^{(2)} := -\nu c_2'(0)\eta_\nu^{(1)} + \frac{\varepsilon}{h}a_{2,2}(0)\eta_{\nu-1}^{(1)}.
\end{aligned} \tag{5.40c}$$

From (5.40a), $Y_\nu^{(2)} \equiv 0$. Furthermore, $X_\nu^{(2)} \equiv 0$ if $X_0^{(2)} = 0$. Due to (5.24a) and since $\xi_0^{(1)}$ is of the form (cf. (5.38a))

$$\xi_0^{(1)} = \frac{\varepsilon}{h}p_1\left(\frac{\varepsilon}{h}\right) = C_1\frac{\varepsilon}{h} + C_2\left(\frac{\varepsilon}{h}\right)^2 \tag{5.41}$$

(with certain constants C_1 and C_2), $X_0^{(2)}=0$ can be achieved by the choice:

(5.42a)
$$x_{0,3} := \hat{x}_{0,3} \quad \text{at } \left(\frac{\varepsilon}{h}\right)^0\text{-level,}$$
$$x_{1,2} := \hat{x}_{1,2} - C_1 \quad \text{at } \left(\frac{\varepsilon}{h}\right)^1\text{-level,}$$
$$x_{2,1} := \hat{x}_{2,1} - C_2 \quad \text{at } \left(\frac{\varepsilon}{h}\right)^2\text{-level.}$$

Thus,

(5.42b)
$$X_\nu^{(2)} \equiv Y_\nu^{(2)} \equiv 0.$$

Moreover, (5.24a) and $Y_0^{(2)}=0$ imply

(5.43a)
$$\eta_0^{(2)} = \frac{\varepsilon}{h}(\hat{y}_{1,2}-y_{1,2}) + \left(\frac{\varepsilon}{h}\right)^2(\hat{y}_{2,1}-y_{2,1}) = \frac{\varepsilon}{h}p_1\left(\frac{\varepsilon}{h}\right).$$

Applying Lemma 5.4 with $j_\nu = j_\nu^{(2)}$ from (5.40c), which can (due to (5.36b)) be written in the form

(5.43b)
$$j_\nu^{(2)} = \frac{\varepsilon}{h}p_0\left(\frac{\varepsilon}{h}\right)p_1(\nu)Q^{-\nu} + \left(\frac{\varepsilon}{h}\right)^2 Qp_0(\nu)Q^{-\nu},$$

we obtain together with (5.43a)

(5.43c)
$$\begin{aligned}\eta_\nu^{(2)} &= Q^{-\nu}\eta_0^{(2)} + p_0\left(\frac{\varepsilon}{h}\right)\nu p_1(\nu)Q^{-\nu-1} + \frac{\varepsilon}{h}p_0\left(\frac{\varepsilon}{h}\right)\nu p_0(\nu)Q^{-\nu}\\ &= \frac{\varepsilon}{h}p_1\left(\frac{\varepsilon}{h}\right)p_1(\nu)Q^{-\nu} + p_0\left(\frac{\varepsilon}{h}\right)p_2(\nu)Q^{-\nu-1}.\end{aligned}$$

To compute $\xi_\nu^{(2)}$ it is convenient to introduce

(5.44)
$$\hat{\xi}_\nu^{(2)} := \xi_\nu^{(2)} + \frac{\varepsilon}{h}\frac{a_{1,2}(0)}{c_2(0)}\eta_\nu^{(2)},$$

which satisfies the difference equation (cf. (5.40b, c))

(5.45a)
$$\begin{aligned}\hat{\xi}_\nu^{(2)} - \hat{\xi}_{\nu-1}^{(2)} &= -a_{1,2}(0)(\eta_\nu^{(2)} - \eta_{\nu-1}^{(2)}) + i_\nu^{(2)} + \frac{a_{1,2}(0)}{c_2(0)}j_\nu^{(2)}\\ &= -a_{1,2}(0)(\eta_\nu^{(2)} - \eta_{\nu-1}^{(2)}) + C_3\xi_\nu^{(1)} + C_4\xi_{\nu-1}^{(1)}\\ &\quad + (C_5 + C_6\nu)\eta_\nu^{(1)} + (C_7 + C_8\nu)\eta_{\nu-1}^{(1)} + \frac{\varepsilon}{h}C_9\eta_{\nu-1}^{(1)}\end{aligned}$$

(with certain constants C_k); this can be recombined into

(5.45b)
$$\begin{aligned}\hat{\xi}_\nu^{(2)} - \hat{\xi}_{\nu-1}^{(2)} &= -a_{1,2}(0)(\eta_\nu^{(2)} - \eta_{\nu-1}^{(2)}) + C_{10}(\xi_\nu^{(1)} - \xi_{\nu-1}^{(1)}) + C_{11}\xi_\nu^{(1)}\\ &\quad + [(C_{12} + C_{13}\nu)\eta_\nu^{(1)} - (C_{12} + C_{13}(\nu-1))\eta_{\nu-1}^{(1)}] + (C_{14} + C_{15}\nu)\eta_\nu^{(1)}\\ &\quad + \frac{\varepsilon}{h}C_{16}(\eta_\nu^{(1)} - \eta_{\nu-1}^{(1)}) + \frac{\varepsilon}{h}C_{17}\eta_\nu^{(1)}.\end{aligned}$$

From (5.45a, b) we see that

$$\xi_\nu^{(2)} = \hat{\xi}_\nu^{(2)} - \frac{\varepsilon}{h}\frac{a_{1,2}(0)}{c_2(0)}\eta_\nu^{(2)}$$

$$(5.46) \qquad = -a_{1,2}(0)\eta_\nu^{(2)} - \frac{\varepsilon}{h}\frac{a_{1,2}(0)}{c_2(0)}\eta_\nu^{(2)} + C_{10}\xi_\nu^{(1)} + C_{11}\sum_{\mu=1}^{\nu}\xi_\mu^{(1)}$$

$$+(C_{12}+C_{13}\nu)\eta_\nu^{(1)} + \sum_{\mu=1}^{\nu}(C_{14}+C_{15}\mu)\eta_\mu^{(1)} + \frac{\varepsilon}{h}C_{16}\eta_\nu^{(1)} + \frac{\varepsilon}{h}C_{17}\sum_{\mu=1}^{\nu}\eta_\mu^{(1)} + C_S$$

with a summation constant C_S.

Now we use the representations (5.38b) and (5.36b) for $\xi_\nu^{(1)}$ and $\eta_\nu^{(1)}$ and apply Lemma 5.5. Thus, for instance,

$$(5.47) \quad \sum_{\mu=1}^{\nu}\xi_\mu^{(1)} = \frac{\varepsilon}{h}p_1\left(\frac{\varepsilon}{h}\right)\sum_{\mu=1}^{\nu}p_0(\mu)Q^{-\mu} = \left(\frac{\varepsilon}{h}\right)^2 p_1\left(\frac{\varepsilon}{h}\right)\left[p_0\left(\frac{\varepsilon}{h}\right)p_0(\nu)Q^{-\nu} + p_0\left(\frac{\varepsilon}{h}\right)\right].$$

The other sums within (5.46) are treated analogously. Furthermore, the summation constant C_S is set such that $\lim_{\nu\to\infty}\xi_\nu^{(2)} = 0$; this determines the starting value $\xi_0^{(2)}$. All this finally results in

$$(5.48a) \qquad \xi_0^{(2)} = \frac{\varepsilon}{h}p_2\left(\frac{\varepsilon}{h}\right),$$

$$(5.48b) \qquad \xi_\nu^{(2)} = \frac{\varepsilon}{h}p_2\left(\frac{\varepsilon}{h}\right)p_1(\nu)Q^{-\nu} + p_1\left(\frac{\varepsilon}{h}\right)p_2(\nu)Q^{-\nu-1}.$$

Due to (5.43c) and (5.48b), $\xi_\nu^{(2)}$ and $\eta_\nu^{(2)}$ are of the form

$$(5.49a) \qquad \frac{\varepsilon}{h}p\left(\frac{\varepsilon}{h}\right)p(\nu)Q^{-\nu} + p\left(\frac{\varepsilon}{h}\right)p(\nu)Q^{-\nu-1} = \frac{\varepsilon}{h}p\left(\frac{\varepsilon}{h}\right)p(\nu)\left[1 + \frac{h}{\varepsilon}Q^{-1}\right]Q^{-\nu},$$

that can be estimated by (5.12d) because

$$(5.49b) \qquad \left|\frac{h}{\varepsilon}Q^{-1}\right| = \left|\left(\frac{\varepsilon}{h} + c_2(0)\right)^{-1}\right| \leqq |c_2(0)|^{-1} \leqq \kappa^{-1}.$$

Coefficients of h^4.

$$(5.50a) \qquad \begin{aligned} &\frac{1}{h}(X_\nu^{(3)} - X_{\nu-1}^{(3)}) = c_1(t_\nu)X_\nu^{(3)} + \vartheta_{1,1}(t_\nu)X_{\nu-1}^{(3)} + \vartheta_{1,2}(t_\nu)Y_{\nu-1}^{(3)}, \\ &0 = -c_2(t_\nu)Y_\nu^{(3)}, \end{aligned}$$

$$(5.50b) \qquad \xi_\nu^{(3)} - \xi_{\nu-1}^{(3)} = a_{1,2}(0)\eta_{\nu-1}^{(3)} + i_\nu^{(3)}, \qquad \frac{\varepsilon}{h}(\eta_\nu^{(3)} - \eta_{\nu-1}^{(3)}) = -c_2(0)\eta_\nu^{(3)} + j_\nu^{(3)},$$

where, due to (5.33a, b), $i_\nu^{(3)}$ and $j_\nu^{(3)}$ are of the form

$$i_\nu^{(3)} = p_1(\nu)\xi_\nu^{(1)} + p_0(\nu)\xi_\nu^{(2)} + p_1(\nu)\xi_{\nu-1}^{(1)} + p_0(\nu)\xi_{\nu-1}^{(2)}$$

$$(5.50c) \qquad + p_1(\nu)\eta_\nu^{(1)} + p_0(\nu)\eta_\nu^{(2)} + p_2(\nu)\eta_{\nu-1}^{(1)} + p_1(\nu)\eta_{\nu-1}^{(2)} + p_0(\nu)[\eta_\nu^{(1)}]^2,$$

$$j_\nu^{(3)} = p_2(\nu)\eta_\nu^{(1)} + p_1(\nu)\eta_\nu^{(2)} + \frac{\varepsilon}{h}[p_0(\nu)\xi_{\nu-1}^{(1)} + p_1(\nu)\eta_{\nu-1}^{(1)} + p_0(\nu)\eta_{\nu-1}^{(2)} + p_0(\nu)\eta_\nu^{(1)}].$$

Here we have tacitly assumed that the terms originating from the nonlinear functions $\gamma_k(t, x, y)$ are expanded around $t=0$ in the spirit of (5.39c). Note, in particular, the presence of the *quadratic* term $p_0(\nu)[\eta_\nu^{(1)}]^2$ within $i_\nu^{(3)}$ (originating from the second derivative of $\gamma_1(t, x, y)$).

From (5.50a), $Y_\nu^{(3)} \equiv 0$. Furthermore, $X_\nu^{(3)} \equiv 0$ if $X_0^{(3)} = 0$. Due to (5.24a) and since $\xi_0^{(2)}$ is of the form (cf. (5.48a))

(5.51) $$\xi_0^{(2)} = \frac{\varepsilon}{h} p_2\left(\frac{\varepsilon}{h}\right) = C_1 \frac{\varepsilon}{h} + C_2\left(\frac{\varepsilon}{h}\right)^2 + C_3\left(\frac{\varepsilon}{h}\right)^3$$

(with certain constants C_1, C_2, C_3), $X_0^{(3)} = 0$ can be achieved by the choice:

(5.52a) $$\begin{aligned} x_{0,4} &:= \hat{x}_{0,4} && \text{at } \left(\frac{\varepsilon}{h}\right)^0\text{-level,} \\ x_{1,3} &:= \hat{x}_{1,3} - C_1 && \text{at } \left(\frac{\varepsilon}{h}\right)^1\text{-level,} \\ x_{2,2} &:= \hat{x}_{2,2} - C_2 && \text{at } \left(\frac{\varepsilon}{h}\right)^2\text{-level,} \\ x_{3,1} &:= \hat{x}_{3,1} - C_3 && \text{at } \left(\frac{\varepsilon}{h}\right)^3\text{-level.} \end{aligned}$$

Thus,

(5.52b) $$X_\nu^{(3)} \equiv Y_\nu^{(3)} \equiv 0.$$

Note that we are considering the case $q=4$. To this end, it is now sufficient to prove that the solutions $\xi_\nu^{(3)}$, $\eta_\nu^{(3)}$ of (5.50b, c) can be estimated by (5.12d). The detailed structure of $\xi_\nu^{(3)}$, $\eta_\nu^{(3)}$ is of no importance. To extend Theorem 5.1 to the general case $q>4$, however, it would be necessary to discuss the $\xi_\nu^{(j)}$, $\eta_\nu^{(j)}$, $j \geqq 3$, in the same way as for $j=1, 2$. In particular, a relation $\xi_0^{(3)} = (\varepsilon/h)p_3(\varepsilon/h)$ would be required for $q=5$. Indeed, such a relation *cannot* be expected for $\xi_0^{(3)}$. This is due to the occurrence of the quadratic term

(5.53a) $$\text{const.}\,[\eta_\nu^{(1)}]^2 = \text{const.}\left(\frac{\varepsilon}{h}\right)^2 Q^{-2\nu}$$

(cf. (5.36b)). The starting value $\xi_0^{(3)}$ (that is again fixed by $\lim_{\nu\to\infty} \xi_\nu^{(3)} = 0$) will be influenced by

(5.53b) $$\begin{aligned} \text{const.} \sum_{\mu=1}^{\infty} [\eta_\mu^{(1)}]^2 &= \text{const.}\left(\frac{\varepsilon}{h}\right)^2 \frac{Q^{-2}}{1-Q^{-2}} \\ &= \text{const.}\left(\frac{\varepsilon}{h}\right)^2 1\Big/\left(2\frac{c_2(0)h}{\varepsilon} + \left(\frac{c_2(0)h}{\varepsilon}\right)^2\right), \end{aligned}$$

a *rational* expression in ε/h and not a polynomial. The consequence is that it will not be possible to find pure (i.e., h- and ε-independent) starting values $x_{k,i}$ in the spirit of (5.52a). Hence our proof of Theorem 5.1 (with h-independent functions $e_i(t)$) cannot be extended to the general case $q \geqq 5$. However, (5.53b) is bounded and depends in a moderate way on h for $\varepsilon \leqq Ch$. This results in a weaker formulation of the theorem for general q (cf. Theorem 5.6 below).

To continue the proof of Theorem 5.1 we show that $\xi_\nu^{(3)}$ and $\eta_\nu^{(3)}$ can be estimated by (5.12d). To conclude the latter for $\eta_\nu^{(3)}$ we recall that the $\xi_\nu^{(j)}$, $\eta_\nu^{(j)}$, $j=1, 2$, are of the form

$$C(\varepsilon, h)\frac{\varepsilon}{h}p\left(\frac{\varepsilon}{h}\right)p(\nu)Q^{-\nu} \tag{5.54}$$

with certain quantities $C(\varepsilon, h)$ that are uniformly bounded for $\varepsilon \leqq Ch$ (cf. (5.36b), (5.38b), (5.49a, b)). Therefore terms of the following types appear within $j_\nu^{(3)}$ (cf. (5.50c)):

$$\begin{aligned} &\text{(i)} \quad C(\varepsilon, h)\frac{\varepsilon}{h}p\left(\frac{\varepsilon}{h}\right)p(\nu)Q^{-\nu}, \\ &\text{(ii)} \quad \frac{\varepsilon}{h}C(\varepsilon, h)\frac{\varepsilon}{h}p\left(\frac{\varepsilon}{h}\right)p(\nu)Q^{-\nu}, \\ &\text{(iii)} \quad \frac{\varepsilon}{h}C(\varepsilon, h)\frac{\varepsilon}{h}p\left(\frac{\varepsilon}{h}\right)p(\nu)Q^{-(\nu-1)}. \end{aligned} \tag{5.55}$$

By Lemma 5.4 we see that the respective terms within $\eta_\nu^{(3)}$ read:

$$\begin{aligned} &\text{(i)} \quad C(\varepsilon, h)\frac{h}{\varepsilon}Q^{-1}\frac{\varepsilon}{h}p\left(\frac{\varepsilon}{h}\right)p(\nu)Q^{-\nu}, \\ &\text{(ii)} \quad C(\varepsilon, h)Q^{-1}\frac{\varepsilon}{h}p\left(\frac{\varepsilon}{h}\right)p(\nu)Q^{-\nu}, \\ &\text{(iii)} \quad C(\varepsilon, h)Q^{-1}\frac{\varepsilon}{h}p\left(\frac{\varepsilon}{h}\right)p(\nu)Q^{-(\nu-1)}, \end{aligned} \tag{5.56}$$

which are again of the form (5.54) and can therefore be estimated by (5.12d). (Here the essential point is that those terms within $j_\nu^{(3)}$ that involve $\xi_{\nu-1}^{(j)}$ or $\eta_{\nu-1}^{(j)}$ contain an additional factor ε/h.) Moreover, the starting value for $\eta_\nu^{(3)}$ is given in (5.24a). Together with $Y_0^{(3)}=0$ we have

$$\eta_0^{(3)}=\frac{\varepsilon}{h}p_2\left(\frac{\varepsilon}{h}\right); \tag{5.57}$$

therefore the term $Q^{-\nu}\eta_0^{(3)}$ (cf. (5.15b)) can again be estimated by (5.12d).

Similar arguments can be given for $\xi_\nu^{(3)}$ on the basis of Lemma 5.5; the result is that $\xi_\nu^{(3)}$ can again be written in the form (5.54) and can be estimated by (5.12d). (The quadratic term $p_0(\nu)[\eta_\nu^{(1)}]^2$ within $i_\nu^{(3)}$ causes no problem because $|Q|^{-2} \leqq |Q|^{-1}$.) The starting value $\xi_0^{(3)}$ (that is again fixed by the requirement $\lim_{\nu\to\infty} \xi_\nu^{(3)}=0$) contains a factor ε/h and is uniformly bounded for $\varepsilon \leqq Ch$. (See, however, (5.53a, b) above.)

Coefficients of h^5.

$$\begin{aligned} &\frac{1}{h}(X_\nu^{(4)}-X_{\nu-1}^{(4)})=c_1(t_\nu)X_\nu^{(4)}+\vartheta_{1,1}(t_\nu)X_{\nu-1}^{(4)}+\vartheta_{1,2}(t_\nu)Y_{\nu-1}^{(4)}+\hat{\delta}_{\nu;1}, \\ &0=-c_2(t_\nu)Y_\nu^{(4)}, \end{aligned} \tag{5.58a}$$

with $\hat{\delta}_{\nu;1}=O(1)$ from (5.19b). For the inner solution at h^5-level there is only an equation defining $\eta_\nu^{(4)}$ (a quantity $\xi_\nu^{(4)}$ does not appear in the ansatz (5.18)):

$$\frac{\varepsilon}{h}(\eta_\nu^{(4)}-\eta_{\nu-1}^{(4)})=-c_2(0)\eta_\nu^{(4)}+j_\nu^{(4)}, \tag{5.58b}$$

where $j_\nu^{(4)}$ is the collection of recursively known quantities (similar to $j_\nu^{(3)}$ within (5.50c)).

From (5.58a), $Y_\nu^{(4)} \equiv 0$. Furthermore, $X_\nu^{(4)}$ is the solution of a simple difference equation with smooth coefficient functions and a smooth inhomogeneity $\hat{\delta}_{\nu;1} = O(1)$. Due to (5.24b) the starting value for $X_\nu^{(4)}$ is given by

$$X_0^{(4)} = 0 - \xi_0^{(3)} = O(1). \tag{5.59}$$

By a classical stability estimate, we immediately obtain

$$X_\nu^{(4)} = O(1). \tag{5.60}$$

For the solution $\eta_\nu^{(4)}$ of (5.58b) it can be shown by the same inductive arguments as for $\eta_\nu^{(3)}$ above that it can again be estimated by (5.12d). (Note that, according to (5.24b), $\eta_0^{(4)} = 0 - Y_0^{(4)} = 0$.)

Remainder terms $x_\nu^{(\text{rem})}$, $y_\nu^{(\text{rem})}$ *of the ansatz* (5.18). Collecting all terms within (5.33) that have not yet appeared so far, we obtain a difference equation for the remainder term

$$\rho_\nu^{(\text{rem})} = \begin{pmatrix} x_\nu^{(\text{rem})} \\ y_\nu^{(\text{rem})} \end{pmatrix}$$

of the ansatz (5.18). This difference equation is rescaled to its original form, that is, the factor ε by which the second equation has been multiplied (cf. (5.11)) is now omitted. This results in an equation which is essentially of the same type as (5.9):

$$\frac{1}{h}(\rho_\nu^{(\text{rem})} - \rho_{\nu-1}^{(\text{rem})}) = \Lambda_\nu \rho_\nu^{(\text{rem})} + \Theta_\nu \rho_{\nu-1}^{(\text{rem})} + V_\nu(\rho_\nu^{(\text{rem})}) + \delta_\nu^{(\text{rem})}. \tag{5.61a}$$

Here, $V_\nu(\rho)$ is defined as

$$V_\nu(\rho) \equiv V_\nu(x, y) \equiv \begin{pmatrix} V_{\nu;1}(x, y) \\ V_{\nu;2}(x, y) \end{pmatrix}, \tag{5.61b}$$

with

$$\begin{aligned} V_{\nu;j}(x, y) := {} & \gamma_j(t_\nu, h^3\xi_\nu^{(1)} + \cdots + h^5(\xi_\nu^{(3)} + X_\nu^{(4)}) + x, h^2\eta_\nu^{(1)} + \cdots + h^5\eta_\nu^{(4)} + y) \\ & - \gamma_j(t_\nu, h^3\xi_\nu^{(1)} + \cdots + h^5(\xi_\nu^{(3)} + X_\nu^{(4)}), h^2\eta_\nu^{(1)} + \cdots + h^5\eta_\nu^{(4)}). \end{aligned} \tag{5.61c}$$

$V_\nu(\rho)$ is Lipschitz continuous with a Lipschitz constant L_V of moderate size:

$$\| V_\nu(\rho) - V_\nu(\bar{\rho}) \| \leqq L_V \| \rho - \bar{\rho} \|. \tag{5.61d}$$

In particular, due to (5.61c),

$$V_\nu(0) = 0, \qquad \| V_\nu(\rho) \| \leqq L_V \| \rho \|. \tag{5.61e}$$

To estimate $\rho_\nu^{(\text{rem})}$ in the spirit of the B-theory we introduce the nonlinear function

$$\Lambda_\nu^*(\rho) := \Lambda_\nu \rho + V_\nu(\rho), \tag{5.62}$$

which is one-sided Lipschitz continuous with the (moderate-sized) one-sided Lipschitz constant

$$m^* := \text{lognorm}\,(\Lambda_\nu) + L_V. \tag{5.63}$$

Due to (5.61e),

$$\Lambda_\nu^*(0) = 0. \tag{5.64}$$

According to the smoothness assumptions with respect to $T(t)$, we may assume that Θ_ν is bounded:

$$\| \Theta_\nu \| \leqq \vartheta \quad \text{for all } \nu. \tag{5.65}$$

Again we work with smooth solutions of the VEs; thus the inhomogeneity δ_ν of the difference equation (5.9) satisfies

$$\delta_\nu = O(h^{q+1}) = \left(\frac{h}{\varepsilon}\right)^{q+1} O(\varepsilon^{q+1}). \tag{5.81}$$

Due to (5.79a) we now have an initial condition

$$\begin{aligned}\rho_0 &= T^{-1}(0)(\zeta_0 - z(0)) - h\bar{e}_1(0) - \cdots - h^q\bar{e}_q(0)\\ &= \varepsilon\begin{pmatrix}(h/\varepsilon)(\hat{x}(h,\varepsilon)-x_{0,1})\\ 0\end{pmatrix} + \varepsilon^2\begin{pmatrix}-(h/\varepsilon)x_{1,1}-(h/\varepsilon)^2x_{0,2}\\ (h/\varepsilon)(\hat{y}(h,\varepsilon)-y_{1,1})\end{pmatrix} + \cdots;\end{aligned} \tag{5.82}$$

it is now natural to arrange according to

$$h^i\varepsilon^j = \varepsilon^{i+j}\left(\frac{h}{\varepsilon}\right)^i, \tag{5.83}$$

with $(h/\varepsilon)^i \leqq C^i$ due to assumption $h \leqq C\varepsilon$.

Now we proceed along lines similar to the proofs of Theorems 5.1 and 5.6 and obtain

$$X_\nu^{(j)} \equiv Y_\nu^{(j)} \equiv 0, \qquad j = 0, \cdots, q-1. \tag{5.84}$$

Furthermore, $\xi_\nu^{(0)} \equiv \eta_\nu^{(0)} \equiv 0$; the other inner solution terms are bounded by (5.79d).

In contrast to the case $\varepsilon \leqq Ch$, a new situation arises within the estimation of the remainder term

$$\rho_\nu^{(\mathrm{rem})} = \begin{pmatrix} x_\nu^{(\mathrm{rem})}\\ y_\nu^{(\mathrm{rem})}\end{pmatrix}.$$

It is, of course, easy to show $\rho_\nu^{(\mathrm{rem})} = O(\varepsilon^{q+1})$. But from this the desired estimate (5.79e) cannot be concluded for $h \leqq C\varepsilon$ (where $O(\varepsilon^{q+1})$ is not $O(h^{q+1})$). A more refined argumentation has to be used. Again, the remainder term $\rho_\nu^{(\mathrm{rem})}$ satisfies a difference equation

$$\frac{1}{h}(\rho_\nu^{(\mathrm{rem})} - \rho_{\nu-1}^{(\mathrm{rem})}) = \Lambda_\nu\rho_\nu^{(\mathrm{rem})} + \Theta_\nu\rho_{\nu-1}^{(\mathrm{rem})} + V_\nu(\rho_\nu^{(\mathrm{rem})}) + \text{inhomogeneity}, \tag{5.85}$$

and it can be shown that *there exists a solution* $\rho_\nu^{(\mathrm{rem})}$ *of* (5.85) *that is not only* $O(\varepsilon^{q+1})$ *but achieves the* $O(h^{q+1})$*-level with increasing* ν (and is bounded by (5.79e)). Fo technical reasons, the usual Taylor expansions of the data functions $c_k(t_\nu)$, $\vartheta_{kl}(t_\nu), \cdots$, around $t=0$ must be avoided for this proof. Therefore the inner solution terms decay as

$$\frac{h}{\varepsilon} C(\varepsilon, h)p(\tau_\nu) \prod_{\mu=1}^{\nu} Q_\mu^{-1}, \tag{5.86a}$$

where

$$Q_\mu := 1 + \frac{c_2(t_\mu)h}{\varepsilon}, \qquad |Q_\mu| \geqq 1 + \frac{\kappa h}{\varepsilon}. \tag{5.86b}$$

Note that the definition of starting values for smooth $\bar{e}_i(t)$ is influenced by sums over terms of the form (5.86a), which are, of course, bounded for $h \leqq C\varepsilon$. □

Remark. Theorem 5.7 can be considered as an analogue of Theorem 5.6. Concerning the induction with respect to several subintervals, essentially the same remarks as given at the end of the proof of Theorem 5.6 apply.

6. Conclusion. In the present paper we have investigated asymptotic error expansions for stiff problems. In § 4 the strongly stiff case has been covered successfully. The mildly stiff case has been discussed in § 5. In particular, §§ 5.2 and 5.3 are devoted to the subdomains $\varepsilon \leqq Ch$ and $h \leqq C\varepsilon$ of the ε-h-plane. The considerations from § 5.2 are of course of major practical relevance. Section 5.3 applies only if, for problems with large Lipschitz constants, the accuracy requirements are so strong that stepsizes $h \ll \varepsilon$ are necessary. For stiff problems (ε small) such restrictive accuracy requirements hardly occur in practice.

It should also be noted that our recursion over subintervals with constant stepsizes may be of a "mixed" type: The relation between h and ε may vary such that different considerations (§ 5.2 or § 5.3) have to be applied in different subintervals. But in this case we only can guarantee expansions in the weaker sense (cf. Theorems 5.6 and 5.7): Once that only an expansion in the weaker sense exists on one of the subintervals, this is the case for all following subintervals.

A *pure* asymptotic expansion (with h-independent $e_i(t)$) can be guaranteed:

(i) In the strongly stiff case;

(ii) For $q=4$ and $\varepsilon \leqq Ch$ under certain assumptions with respect to the transient phase and if the subintervals are sufficiently long (such that inner solution terms are always damped down to the h^5-level).

The question arises as to whether similar results on asymptotic error expansions can also be proved for other methods. In a forthcoming paper we will discuss asymptotic error expansions for the implicit midpoint rule and the implicit trapezoidal rule, trying to extend the results of Dahlquist and Lindberg [6]. In contrast to the strongly stable implicit Euler scheme, there is no damping in the difference equations for the midpoint and trapezoidal rules. If, therefore, the order of the remainder term breaks down at the first gridpoints, no damping effects are to be expected; hence the reduced order remains present in the whole integration interval. There is, however, some hope that discrete singular perturbation techniques can be applied to establish a *systematically oscillating behaviour* of the dominant components of the remainder term. Therefore we expect that certain numerical procedures for error estimation, stepsize control, convergence acceleration, etc. can be shown to work successfully if the information at odd or even gridpoints only is used. Another question is whether the full asymptotic expansion exists in the strongly stiff case $\varepsilon \ll h$ (as is the case for the implicit Euler scheme). Simple models show that this is not the case in general but only for certain classes of problems that we will try to characterize. The trapezoidal scheme seems to be more "robust" with respect to this point.

Note added in proof. Parallel and independent work on asymptotic error expansions for the implicit Euler scheme (also for the semi-implicit Euler scheme) has been done by Hairer and Lubich [14] (*Numer. Math.*, 52 (1988), pp. 377–400).

REFERENCES

[1] W. Auzinger, R. Frank, and F. Macsek, *Asymptotic error expansions for stiff equations. Part 1: The strongly stiff case*, Report 67/86, Inst. für Angewandte und Numerische Mathematik, Technical University of Vienna, Vienna, 1986.

[2] ———, *Asymptotic error expansions for stiff equations. Part 2: The mildly stiff case*, Report 68/86, Inst. für Angewandte und Numerische Mathematik, Technical University of Vienna, Vienna, 1986.

[3] G. Bader and P. Deuflhard, *A semi-implicit mid-point rule for stiff systems of ordinary differential equations*, Numer. Math., 41 (1983), pp. 373–398.

[4] J. C. Butcher, *A stability property of implicit Runge-Kutta methods*, BIT, 15 (1975), pp. 358–361.

[5] G. DAHLQUIST, *Error Analysis for a Class of Methods for Stiff Nonlinear Initial Value Problems*, Lecture Notes in Math. 506, G. A. Watson, ed., Springer-Verlag, Berlin, New York, 1976.

[6] G. DAHLQUIST AND B. LINDBERG, *On some implicit one-step methods for stiff differential equations*, Report TRITA-NA-7302, Dept. of Computer Sciences, Royal Institute of Technology, Stockholm, 1973.

[7] K. DEKKER AND J. G. VERWER, *Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations*, North-Holland, Amsterdam, New York, 1984.

[8] R. FRANK, J. SCHNEID, AND C. W. UEBERHUBER, *The concept of B-convergence*, SIAM J. Numer. Anal., 18 (1981), pp. 753–780.

[9] ———, *Stability properties of implicit Runge-Kutta methods*, SIAM J. Numer. Anal., 22 (1985), pp. 497–515.

[10] ———, *Order results for implicit Runge-Kutta methods applied to stiff systems*, SIAM J. Numer. Anal., 22 (1985), pp. 515–534.

[11] R. E. O'MALLEY, JR., *Introduction to Singular Perturbations*, Academic Press, New York, London, 1974.

[12] H. J. STETTER, *Analysis of Discretization Methods for Ordinary Differential Equations*, Springer-Verlag, Berlin, New York, 1973.

[13] M. VAN VELDHUIZEN, *Asymptotic expansions of the global error for the implicit midpoint rule* (*stiff case*), Computing, 33 (1984), pp. 185–192.

[14] E. HAIRER AND CH. LUBICH, *Extrapolation at stiff differential equations*, Numer. Math., 52 (1988), pp. 377–400.