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Sectorial operators and normalized numerical range

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Abstract

We study the notion of sectorial operator in a Hilbert space. According to the classical definition, the numerical range $\mathcal{R}(A)$ of a sectorial operator A is contained in a sector $\mathcal{S}_\sigma = \{z \in \mathbb{C}\colon\ |\operatorname{Arg} z| \leqslant \sigma\}$, and this is equivalent to a certain inverse estimate valid outside $\mathcal{S}_\sigma$. In this paper we show that the validity of the same estimate, but with a factor > 1, is equivalent to the validity of a certain strengthened Cauchy–Schwarz inequality for all pairs w, Aw. This extends the original characterization in terms of $\mathcal{R}(A)$ by a more general characterization based on a normalized numerical range $\mathcal{R}_N(A)$. We also show how $\mathcal{R}_N(A)$ can be computed numerically.

1. Introduction and overview

This paper continues the work from [2] and is concerned with the notion of sectorial operators. Linear operators of sectorial type play a prominent role in the theory of semigroups and evolution equations; see, for instance, [9,16,20]. The term ‘sectorial’ has been assigned slightly different meanings (cf. [6], for a brief discussion). Here we proceed from the classical definition in the Hilbert space setting, associated with the notion of numerical range (cf., e.g., [5,9]).

According to this classical definition, a sectorial operator is a special accretive, generally unbounded operator in a Hilbert space $\boldsymbol{H}$, for which the numerical range is contained in a certain sector $\mathcal{S}_\sigma \subseteq \mathbb{C}$, cf. Definition 1.1. The main purpose of this paper is to characterize the natural generalization of this concept of a sectorial operator in terms of a ‘normalized version’ of the numerical range, cf. (2.1). Our generalization is based on inverse (resolvent) estimates.

For the relevance of sectorial conditions in the numerical analysis of parabolic equations, cf., for instance, [1,11,13]. Note that most of the material in Section 1 is standard, but it is included here for completeness and to help us to prepare and motivate the rest of the paper.

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1.1. Notation and basic definitions

Let $\boldsymbol{H}$ denote a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. The norm of bounded linear operators L acting in $\boldsymbol{H}$ will also simply written as $\|L\|$.

Now assume that A is an operator in $\boldsymbol{H}$, generally unbounded, with domain $\boldsymbol{D}(A) \subseteq \boldsymbol{H}$. The *numerical range* of A is denoted by $\mathcal{R}(A)$:

$$\mathcal{R}(A) := \left\{ \frac{\langle Aw, w\rangle}{\langle w, w\rangle}:\ 0 \neq w \in \boldsymbol{D}(A) \right\}. \tag{1.1}$$

Let $\sigma \in (0, \frac{\pi}{2})$ and let S_σ denote the closed sector

$$\mathcal{S}_\sigma := \{0\} \cup \left\{ z \in \mathbb{C}:\ |\operatorname{Arg} z| \leqslant \sigma \right\}, \tag{1.2}$$

in the right complex half-plane. We also denote

$$\mathcal{T}_\sigma := \mathbb{C} \backslash \mathcal{S}_\sigma = \left\{ z \in \mathbb{C}:\ |\operatorname{Arg} z| > \sigma \right\}, \tag{1.3}$$

and

$$\mathcal{Z}_\sigma := \left\{ z \in \mathbb{C}:\ |\operatorname{Arg} z| = \sigma + \frac{\pi}{2} \right\} \subseteq \mathcal{T}_\sigma, \tag{1.4}$$

cf. Fig. 1.

The ‘classical’ definition of a sectorial operator reads as follows (cf., e.g., [9]):

Definition 1.1. A linear operator A in $\boldsymbol{H}$ is called *sectorial* with vertex γ and half-angle σ if the numerical range $\mathcal{R}(A - \gamma I)$ is contained in $\mathcal{S}_\sigma$.

In the following throughout we consider the case $\gamma = 0$.

For complex numbers written in polar form we shall write

$$z \equiv d e^{i\theta}, \quad \theta \in (-\pi, \pi], \tag{1.5}$$

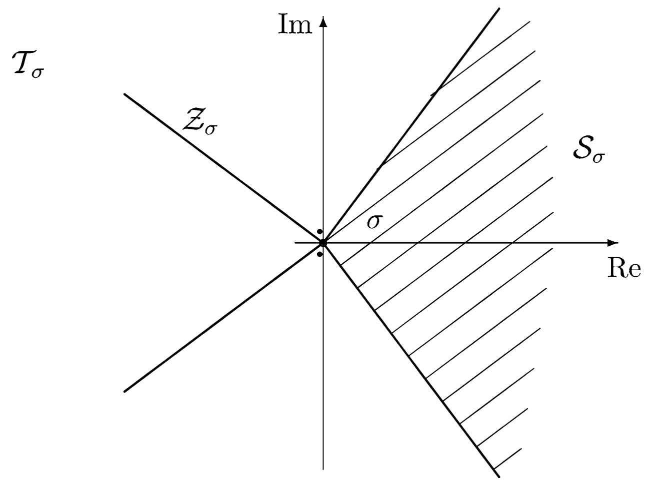

Fig. 1. A sector $\mathcal{S}_\sigma$.

and the symbols d and θ are throughout to be identified with z in the sense of (1.5).

In the same spirit throughout we use the notation

$$\frac{\langle Aw, w\rangle}{\langle w, w\rangle} \equiv re^{i\omega}, \qquad \frac{\|Aw\|}{\|w\|} \equiv s, \tag{1.6}$$

for $0 \neq w \in \boldsymbol{D}(A)$. For $s \neq 0$ (i.e., $\|Aw\| \neq 0$) we will also write

$$a := \frac{r}{s} = \left|\frac{\langle Aw, w\rangle}{\|Aw\|\|w\|}\right| \in [0,1] \quad \text{(Cauchy–Schwarz).} \tag{1.7}$$

We note the basic identity

$$\frac{\|Aw - zw\|^2}{\|w\|^2} = s^2 - 2rd\cos(\theta - \omega) + d^2, \tag{1.8}$$

which follows from

$$\begin{aligned} \|Aw - zw\|^2 &= \langle Aw - zw, Aw - zw\rangle = \|Aw\|^2 - 2\,\mathrm{Re}\big(\bar{z}\langle Aw, w\rangle\big) + |z|^2\|w\|^2 \\ &= \|Aw\|^2 - 2d\,\mathrm{Re}\big(e^{-i\theta}\langle Aw, w\rangle\big) + d^2\|w\|^2. \end{aligned}$$

Let the σ-dependent, continuous function $D_\sigma(\cdot): \mathcal{T}_\sigma \to \mathbb{R}^+$ be defined as the distance

$$D_\sigma(z) := \mathrm{dist}(z, \mathcal{S}_\sigma) = \begin{cases} d\big|\sin(\theta - \sigma)\big|, & |\theta| \in \big(\sigma, \sigma + \frac{\pi}{2}\big), \\ d, & |\theta| \in \big[\sigma + \frac{\pi}{2}, \pi\big]. \end{cases} \tag{1.9}$$

1.2. Preliminary remarks on inverse estimates and resolvent sets

Several assertions to be proved in this paper are formulated as norm bounds for inverse operators of the form

$$\big\|(A - zI)^{-1}\big\| \leqslant K(z), \tag{1.10}$$

where $K(z)$ is an expression depending on $z \in \mathcal{T}_\sigma$ (cf. in particular our main Theorem 2.3). In all these cases, (1.10) is to be interpreted in the following sense:

The operator $A - zI$, with domain $\boldsymbol{D}(A) = \boldsymbol{D}(A - zI)$, is injective, and its inverse exists and is bounded on the domain $\boldsymbol{R}(A - zI)$ (= range of $A - zI$), with $\|(A - zI)^{-1}\| \leqslant K(z)$.

This use of the term 'inverse' and the symbol $(\cdots)^{-1}$ is standard in the literature on functional analysis (cf., e.g., [20]). But strictly speaking, we should not call such an inverse a *resolvent* in general, because by definition, this would mean that z is in the resolvent set of A, i.e., that $(A - zI)^{-1}$ is a bounded linear operator defined on the whole space $\boldsymbol{H}$.

Here the natural question arises under what conditions[1] the values z involved ($z \in \mathcal{T}_\sigma$) belong to the resolvent set of A such that $\boldsymbol{D}((A - zI)^{-1}) = \boldsymbol{H}$ holds. We will briefly discuss this topic in Section 4.

1.3. Basis facts. Characterization of $\mathcal{R}(A) \subseteq \mathcal{S}_\sigma$

Let us start our considerations with an elementary characterization of sectorial operators in terms of norm estimates involving the operators $A - zI$ with $z \in \mathcal{T}_\sigma$.

[1] The case where $\boldsymbol{H}$ has finite dimension is trivial.

Proposition 1.1. *For arbitrary, fixed* $0 \neq w \in \boldsymbol{D}(A)$, $\frac{\langle Aw,w\rangle}{\langle w,w\rangle} \in \mathcal{S}_\sigma$ *implies*

$$\forall z = de^{i\theta} \in \mathcal{T}_\sigma\colon\ \frac{\|Aw - zw\|}{\|w\|} \geqslant D_\sigma(z). \tag{1.11}$$

Proof. Let $0 \neq w \in \boldsymbol{D}(A)$ be given such that $\frac{\langle Aw,w\rangle}{\langle w,w\rangle} = re^{i\omega} \in \mathcal{S}_\sigma$, i.e., with some $\omega \in [-\sigma, \sigma]$.

We now estimate (1.8) from below for arbitrary $z = de^{i\theta} \in \mathcal{T}_\sigma$. By symmetry, we may assume without loss of generality that $\theta > 0$, i.e., $\theta \in (\sigma, \pi]$. For such values of θ, we consider the following cases (i) and (ii) (according to definition (1.9) of $D_\sigma(z)$).

(i) $\theta \in (\sigma, \sigma + \frac{\pi}{2}]$. Here the desired inequality (1.11) is equivalent to

$$s^2 - 2rd\cos(\theta - \omega) + d^2\cos^2(\theta - \sigma) \geqslant 0, \tag{1.12}$$

where

$$\theta - \omega \in \left(\sigma - \omega, \frac{\pi}{2} + \sigma - \omega\right] \subseteq \left[0, \frac{\pi}{2} + 2\sigma\right] \subseteq \left[0, \frac{3\pi}{2}\right),$$
$$\theta - \sigma \in \left(0, \frac{\pi}{2}\right],$$

by assumption on θ.

To prove (1.12), we consider two cases concerning ω:

(ia) $\theta - \omega \in [0, \frac{\pi}{2}]$. Together with $\omega \leqslant \sigma$ we have

$$\left[0, \frac{\pi}{2}\right] \ni \theta - \omega \geqslant \theta - \sigma \in \left[0, \frac{\pi}{2}\right] \implies \cos(\theta - \omega) \leqslant \cos(\theta - \sigma),$$

hence (1.12) can be verified as follows:

$$\begin{aligned} s^2 - 2rd\cos(\theta - \omega) + d^2\cos^2(\theta - \sigma) &\geqslant r^2 - 2rd\cos(\theta - \sigma) + d^2\cos^2(\theta - \sigma) \\ &= \big(r - d\cos(\theta - \sigma)\big)^2 \geqslant 0. \end{aligned}$$

(ib) $\theta - \omega \in (\frac{\pi}{2}, \frac{3\pi}{2}]$. Here, $\cos(\theta - \omega) \leqslant 0$, from which (1.12) immediately follows.

(ii) $\theta \in (\sigma + \frac{\pi}{2}, \pi]$. Here the desired inequality (1.11) is equivalent to

$$s^2 - 2rd\cos(\theta - \omega) \geqslant 0. \tag{1.13}$$

Similarly, as for case (ib), (1.13) follows immediately from $\cos(\theta - \omega) \leqslant 0$ due to $\theta - \omega \in (\frac{\pi}{2}, \sigma + \frac{\pi}{2}] \in (\frac{\pi}{2}, \frac{3\pi}{2})$. □

We now show that the converse of Proposition 1.1 is also true. Moreover, even a weaker form of (1.11) is sufficient for $\frac{\langle Aw,w\rangle}{\langle w,w\rangle} \in \mathcal{S}_\sigma$ to hold:

Proposition 1.2. *For arbitrary, fixed* $0 \neq w \in \boldsymbol{D}(A)$, *the condition*

$$\forall z = de^{i\theta} \in Z_\sigma\colon\ \frac{\|Aw - zw\|}{\|w\|} \geqslant D_\sigma(z), \tag{1.14}$$

implies $\frac{\langle Aw,w\rangle}{\langle w,w\rangle} \in \mathcal{S}_\sigma$.

Proof. For an indirect proof we assume $\frac{\langle Aw,w\rangle}{\langle w,w\rangle} = re^{i\omega} \in \mathcal{T}_\sigma$. By symmetry, we may assume without loss of generality that $\omega > 0$, i.e., $\omega \in (\sigma, \pi]$. We consider points $z = de^{i\theta} \in \mathcal{Z}_\sigma$ with $\theta = \sigma + \frac{\pi}{2}$ and will show that

$$\frac{\|Aw - zw\|}{\|w\|} \geqslant D_\sigma(z), \quad \text{i.e.,} \quad s^2 - 2rd\cos\left(\sigma + \frac{\pi}{2} - \omega\right) \geqslant 0, \tag{1.15}$$

cannot hold for all $d > 0$. Similarly as in the proof of Proposition 1.1 we consider two cases:

(i) $\omega \in (\sigma, \sigma + \frac{\pi}{2}]$. Here we have

$$\sigma + \frac{\pi}{2} - \omega \in \left[0, \frac{\pi}{2}\right) \implies \cos\left(\sigma + \frac{\pi}{2} - \omega\right) > 0, \tag{1.16}$$

thus (1.15) fails to hold for $d \to \infty$.

(ii) $\omega \in (\sigma + \frac{\pi}{2}, \pi]$. Here we have

$$\sigma + \frac{\pi}{2} - \omega \in \left[\sigma - \frac{\pi}{2}, 0\right) \implies \cos\left(\sigma + \frac{\pi}{2} - \omega\right) > 0, \tag{1.17}$$

and again (1.15) fails to hold for $d \to \infty$.

This proves (1.14). □

As a consequence of Propositions 1.1 and 1.2, we have:

Proposition 1.3. *For arbitrary, fixed* $0 \neq w \in \boldsymbol{D}(A)$, *the following assertions are equivalent*:

$$\frac{\langle Aw, w\rangle}{\langle w, w\rangle} \in \mathcal{S}_\sigma, \tag{1.18a}$$

$$\frac{\|Aw - zw\|}{\|w\|} \geqslant D_\sigma(z), \quad \forall z = de^{i\theta} \in \mathcal{T}_\sigma. \tag{1.18b}$$

Theorem 1.4. *The following assertions are equivalent*:

$$\mathcal{R}(A) \subseteq \mathcal{S}_\sigma, \tag{1.19a}$$

$$\left\|(A - zI)^{-1}\right\| \leqslant \frac{1}{D_\sigma(z)}, \quad \forall z \in \mathcal{T}_\sigma, \tag{1.19b}$$

$$\left\|(A - zI)^{-1}\right\| \leqslant \frac{1}{|z|}, \quad \forall z \in \mathcal{Z}_\sigma. \tag{1.19c}$$

1.4. The limiting case $\sigma = \frac{\pi}{2}$

For the limiting case $\sigma = \frac{\pi}{2}$, it is easy to verify that Theorem 1.4 remains valid, and we obtain:

Corollary 1.5. *The following assertions are equivalent*:

$$\operatorname{Re}\langle Aw, w\rangle \geqslant 0, \quad \forall w \in \boldsymbol{D}(A), \tag{1.20a}$$

$$\left\|(A - zI)^{-1}\right\| \leqslant \frac{1}{|\operatorname{Re} z|}, \quad \forall z \in \mathbb{C}\colon \operatorname{Re} z < 0, \tag{1.20b}$$

$$\left\|(A - dI)^{-1}\right\| \leqslant \frac{1}{|d|}, \quad \forall d < 0. \tag{1.20c}$$

This is related to the so-called 'Hille–Yosida' Theorem valid in the general semigroup context (cf., e.g., [7]). See also [5, Lemma 6.1–5].

1.5. The limiting case $\sigma = 0$

So far we have assumed $\sigma > 0$, such that the interior of $\mathcal{S}_\sigma$ is nonempty. However, Proposition 1.1 also holds for $\sigma = 0$, with $\mathcal{S}_\sigma = \mathcal{S}_0 = \mathbb{R}_0^+$, but not Proposition 1.2.

The limiting case $\sigma = 0$ deserves special attention, and we mention it for the sake of completeness. The assertions below easily follow from elementary results given in [5].

Proposition 1.6. *For $\sigma = 0$ we have*:

$$A \text{ is Hermitian nonnegative definite} \iff \mathcal{R}(A) \subseteq \mathbb{R}_0^+. \tag{1.21}$$

Proof. It is a trivial fact that the numerical range of a Hermitian nonnegative definite operator is contained in $\mathbb{R}_0^+$. To prove the converse, however, note that A was not *a priori* assumed to be Hermitian. But this follows from the equivalence

$$\mathcal{R}(A) \subseteq \mathbb{R} \iff A \text{ is Hermitian: } A = A^* \tag{1.22}$$

(cf. [5, Theorem 1.2–2]), which completes the proof of (1.21). □

2. Generalization. A normalized numerical range

In this section we study a more general class of sectorial operators, and give a generalization of Theorem 1.4, (1.19a,b), see Section 2.2. In Section 2.1 we first recall a related result concerning the limiting case $\sigma = \frac{\pi}{2}$.

Our results relate the sectorial condition to a property of the set $\mathcal{R}_N(A)$, which we call the *normalized numerical range* $\mathcal{R}_N(A)$:

$$\mathcal{R}_N(A) := \left\{ \frac{\langle Aw, w\rangle}{\|Aw\|\|w\|}\colon\ w \in \boldsymbol{D}(A),\ Aw \neq 0 \right\}. \tag{2.1}$$

$\mathcal{R}_N(A)$ is a subset of the complex unit circle.

The results of this section show that in a certain context, the set $\mathcal{R}_N(A)$ may provide useful information about an operator A in $\boldsymbol{H}$, generalizing known results formulated in terms of $\mathcal{R}(A)$.

At this point, let us also mention the notion of the *shell* of a Hilbert space operator A, see [3] or [8], Definition 1.8.9. The shell is a three-dimensional generalization of $\mathcal{R}(A)$ which also includes the values $\|Aw\|/\|w\|$. It has, e.g., been useful in the study of certain classes power bounded-operators, see [3,15]. Our results are also related to the concept of pseudospectra (cf., e.g., [17]).

2.1. $\mathcal{R}_N(A)$ and the Kreiss Matrix Theorem

Let us first take another look at Corollary 1.5. Inequality (1.20a) says that A is accretive, and the immediate consequence is

$$\left\|e^{-tA}\right\| \leqslant 1, \quad \forall t \geqslant 0. \tag{2.2}$$

This generalizes as follows. Assume $K \geqslant 1$, and let $\mathcal{M}_K$ denote the following subset of the complex unit circle (cf. Fig. 2):

$$\mathcal{M}_K := \left\{ x + iy \in \mathbb{C}\colon\ x^2 + y^2 \leqslant 1 \text{ and } \left(x \geqslant 0 \text{ or } \frac{x^2}{1-K^{-2}} + y^2 \leqslant 1 \right) \right\}. \tag{2.3}$$

Theorem 2.1 (Kreiss Matrix Theorem, exponential version).

- *For $K \geqslant 1$, the following assertions are equivalent:*[2]

$$\mathcal{R}_N(A) \subseteq \mathcal{M}_K, \tag{2.4a}$$

$$\left\|(A - zI)^{-1}\right\| \leqslant \frac{K}{|\operatorname{Re} z|}, \quad \forall z \in \mathbb{C}\colon\ \operatorname{Re} z < 0. \tag{2.4b}$$

- *If $A \in \mathbb{C}^{n\times n}$ satisfies* (2.4), *then*

$$\left\|e^{-tA}\right\| \leqslant enK, \quad \forall t \geqslant 0. \tag{2.5}$$

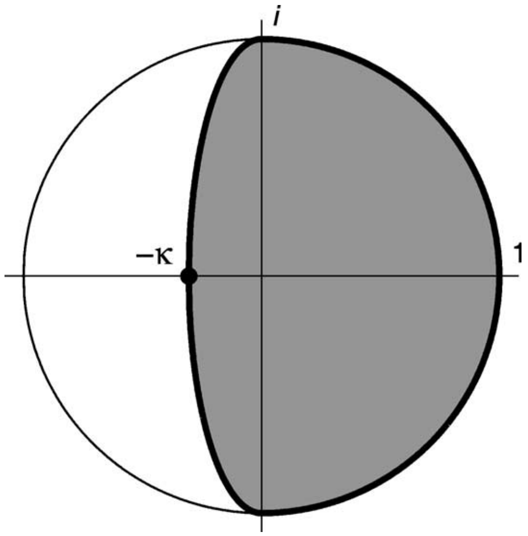

Fig. 2. Moon-shaped domain $\mathcal{M}_K$ ($\kappa = \sqrt{1-K^{-2}}$).

[2] In [2], the result (2.4) is only formulated for $A \in \mathbb{C}^{n\times n}$, but it easily extends to the general Hilbert space context. An analogous result for the matrix power case (classical version of the Kreiss Matrix Theorem) is also given in [2].

This reformulation of the exponential version of the Kreiss Matrix Theorem ([10]; see also [19]) combines results from [2,12,18]. In particular, the strengthened Cauchy–Schwarz inequality (2.4a) was introduced in [2] as the natural generalization of (1.20a). Theorem 2.1 shows that the well-known resolvent condition (2.4b) can be equivalently formulated as a condition directly on A, generalizing the equivalence (1.20a) $\Leftrightarrow$ (1.20b). Note that for $K > 1$, an operator satisfying (2.4) is not accretive in general.

2.2. General characterization of sectorial operators

In Section 1 we have considered sectorial operators according to Definition 1.1, together with certain inverse estimates. Here we propose a more general and useful definition, namely

$$A \text{ is } \sigma\text{-sectorial:} \iff \left\|(A - zI)^{-1}\right\| \leqslant \frac{K}{D_\sigma(z)}, \quad \forall z \in \mathcal{T}_\sigma. \tag{2.6}$$

For $K = 1$ this is equivalent to Definition 1.1; but (2.6) explicitly relies on a bound like (1.19b) but with a factor $K \geqslant 1$. For a motivation, cf., e.g., [1] or [11], where inequality (2.6) is applied in the analysis of numerical integration methods for parabolic equations. Here the essential point is that the particular value of the constant K plays no essential role; everything remains valid when $K > 1$ is admitted.

The goal of this section is to show how this general class can be characterized in a way similar to the equivalence (2.4a) $\Leftrightarrow$ (2.4b). Our main result is Theorem 2.3, which is the analog of (2.4a,b) and the direct generalization of Theorem 1.5, (1.19a,b).

For $K \geqslant 1$ and $\sigma \in (0, \frac{\pi}{2})$, let $\mathcal{A}_{\sigma,K}$ denote the following 'axe-shaped' subset of the complex unit circle (cf. Fig. 3):

$$\mathcal{A}_{\sigma,K} := \left\{ae^{i\omega} \in \mathbb{C}\colon\ a \leqslant a_{\sigma,K}(\omega)\right\}, \tag{2.7a}$$

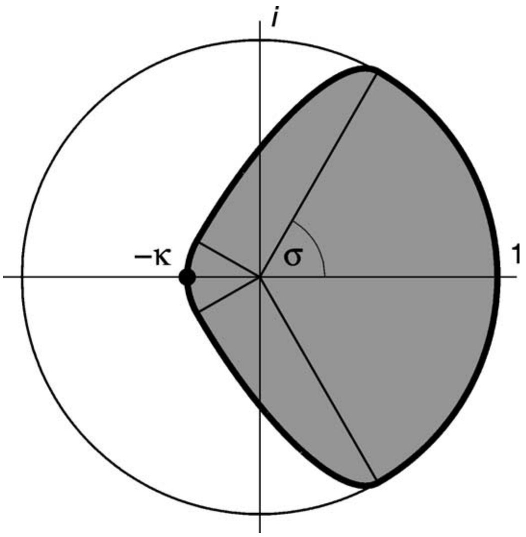

Fig. 3. Axe-shaped domain $\mathcal{A}_{\sigma,K}$ ($\kappa = \sqrt{1 - K^{-2}}$).

with

$$a_{\sigma,K}(\omega) := \begin{cases} 1, & 0 \leqslant |\omega| \leqslant \sigma, \\ \left[\frac{\sin^2(|\omega|-\sigma)}{1-K^{-2}} + \cos^2\big(|\omega|-\sigma\big)\right]^{-1/2}, & \sigma < |\omega| < \sigma + \frac{\pi}{2}, \\ \big(1-K^{-2}\big)^{1/2}, & \sigma + \frac{\pi}{2} \leqslant |\omega| \leqslant \pi. \end{cases} \tag{2.7b}$$

The boundary of $\mathcal{A}_{\sigma,K}$, as specified in (2.7b), consists of a pair of circular arcs with radius 1 respectively $\sqrt{1-K^{-2}}$, softly connected by a pair of elliptic arcs. In the limiting case $K = 1$, $\mathcal{A}_{\sigma,K}$ becomes the sector $\mathcal{A}_{\sigma,1} = \{ae^{i\omega}\colon a \leqslant 1 \text{ and } |\omega| \leqslant \sigma\} = \mathcal{S}_\sigma$.

The following result generalizes Proposition 1.3.

Proposition 2.2. *For $K \geqslant 1$ and arbitrary, fixed $w \in \boldsymbol{D}(A)$ $(Aw \neq 0)$, the following assertions are equivalent*:

$$\frac{\langle Aw, w\rangle}{\|Aw\|\|w\|} \in \mathcal{A}_{\sigma,K}, \tag{2.8a}$$

$$\frac{\|Aw - zw\|}{\|w\|} \geqslant \frac{D_\sigma(z)}{K}, \quad \forall z \in \mathcal{T}_\sigma. \tag{2.8b}$$

Proof (*Sketch*). The detailed proof of Proposition 2.2 is given in Appendix A. It is based on studying, for arbitrary w, the infimum of the w-dependent function (cf. (2.8b))

$$f(d,\theta) := \frac{\|Aw - zw\|^2}{\|w\|^2} - K^{-2} D_\sigma^2(z), \tag{2.9}$$

with respect to the variable $z = de^{i\theta} \in \mathcal{T}_\sigma$. Here, several cases have to be discussed according to the definition of $\mathcal{A}_{\sigma,K}$ and of the function $D_\sigma(z)$. The result of this investigation is visualized in Table 1. For case (ii), three sub-cases concerning the variable θ have to be studied separately, and the relevant infimum is attained for sub-case (iia) (marked by $\bullet$).

In this way we obtain a criterion for (2.8a) which is independent of the variable $z = de^{i\theta}$, and which is obviously equivalent to (2.8a) due to the definition of $\mathcal{A}_{\sigma,K}$. Thus, (2.8a) and (2.8b) are equivalent. $\square$

Our main result is a direct consequence of Proposition 2.2:

Table 1
Structure of proof for Proposition 2.2

case $\omega \in \cdots$	sub-case $\theta \in \cdots$	'$f(d,\theta) \geqslant 0$ $\forall d \geqslant 0$ and $\forall\theta$ in [sub-]case' $\Leftrightarrow$ $\cdots$	
(i) $\omega \in [0,\sigma]$	– $\theta \in (\sigma, 2\pi - \sigma)$	$\omega \leqslant \sigma$	$\bullet$
(ii) $\omega \in \big(\sigma, \sigma + \frac{\pi}{2}\big)$	(iia) $\theta \in (\sigma, \sigma + \frac{\pi}{2})$	$a^2\big[\frac{\sin^2(\omega-\sigma)}{1-K^{-2}} + \cos^2(\omega - \sigma)\big] \leqslant 1$	$\bullet$
	(iib) $\theta \in \big[\sigma + \frac{\pi}{2}, \frac{3\pi}{2} - \sigma\big]$	$a^2 \frac{\sin^2(\omega-\sigma)}{1-K^{-2}} \leqslant 1$	$\circ$
	(iic) $\theta \in \big(\frac{3\pi}{2} - \sigma, 2\pi - \sigma\big)$	weaker than (iia)	$\circ$
(iii) $\omega \in \big[\sigma + \frac{\pi}{2}, \pi\big]$	– $\theta \in (\sigma, 2\pi - \sigma)$	$\frac{a^2}{1-K^{-2}} \leqslant 1$	$\bullet$

Theorem 2.3. *For* $K \geqslant 1$, *the following assertions are equivalent*:

$$\mathcal{R}_N(A) \subseteq \mathcal{A}_{\sigma,K}, \tag{2.10a}$$

$$\left\|(A - zI)^{-1}\right\| \leqslant \frac{K}{D_\sigma(z)}, \quad \forall z \in \mathcal{T}_\sigma \quad (i.e.,\ (2.6)). \tag{2.10b}$$

3. Characterization and computation of $\mathcal{R}_N(A)$

Considering Theorems 2.1 and 2.3, it is tempting to ask whether the set $\mathcal{R}_N(A)$ itself can be characterized by means of inverse inequalities. The basis for such a characterization is given in Section 3.1. In Section 3.2 we discuss the numerical computation of an inclusion for $\mathcal{R}_N(A)$ and present an example.

3.1. Another generalization of Corollary 1.5

Corollary 1.5 characterizes accretive operators by means of the behavior of their resolvents along the negative real axis (or, equivalently, dissipative operators by means of their resolvents along the positive axis). This result generalizes in the sense that an upper bound for the real part of $\mathcal{R}_N(A)$ is equivalent to an inverse inequality along the positive real axis:

Proposition 3.1. *For given* $\kappa \in (0,1)$, *let* $K := (1-\kappa^2)^{-1/2}$. *Then the following assertions are equivalent*:

$$\operatorname{Re} \frac{\langle Aw, w\rangle}{\|Aw\|\|w\|} \leqslant \kappa, \quad \forall w \in \boldsymbol{D}(A)\ (Aw \neq 0), \tag{3.1a}$$

$$\left\|(A - dI)^{-1}\right\| \leqslant \frac{K}{d}, \quad \forall d > 0. \tag{3.1b}$$

Proof. In our standard notation (1.6), relation (3.1a) is equivalent to

$$2d\kappa s - 2dr\cos\omega \geqslant 0, \quad \forall w \in \boldsymbol{D}(A), \tag{3.2}$$

where $d > 0$ can be chosen arbitrarily (it may depend on w). Using the identity (valid for arbitrary $d \in \mathbb{R}$)

$$2d\kappa s \equiv d^2\kappa^2 + s^2 - [d\kappa - s]^2, \tag{3.3}$$

we write

$$\begin{aligned} 2d\kappa s - 2dr\cos\omega &= s^2 - 2dr\cos\omega + d^2\kappa^2 - [d\kappa - s]^2 \\ &= \underbrace{s^2 - 2dr\cos\omega + d^2}_{=\|Aw-dw\|^2 (\text{cf. } (1.9))} - d^2\left(1-\kappa^2\right) - [d\kappa - s]^2. \end{aligned} \tag{3.4}$$

Since $d > 0$ is arbitrary, the following assertions are thus equivalent:

$$(3.1\text{a}) \iff \|Aw - dw\|^2 - d^2\left(1-\kappa^2\right) - [d\kappa - s]^2 \geqslant 0, \quad \forall d > 0,\ \forall w \in \boldsymbol{D}(A), \tag{3.5}$$

$$\iff \|Aw - \underline{d}w\|^2 - \underline{d}^2\left(1-\kappa^2\right) \geqslant 0, \quad \forall w \in \boldsymbol{D}(A), \tag{3.6}$$

where $\underline{d}$ is chosen (depending on w) such that $\underline{d}\kappa - s = 0$, i.e., $\underline{d} := s/\kappa > 0$.

Now we argue as follows: If (3.1a) holds, then

$$\|Aw - dw\|^2 - d^2\bigl(1-\kappa^2\bigr) \geqslant 0, \quad \forall d > 0,\ \forall w \in \boldsymbol{D}(A), \tag{3.7}$$

follows from (3.5). Conversely, (3.7) implies (3.6), which is equivalent to (3.1a). This completes the proof since (3.7) is equivalent to (3.1b). □

Corollary 3.2. *Let A be a bounded operator. For given $\kappa \in (0,1)$, let $K := (1-\kappa^2)^{-1/2}$. Then the following assertions are equivalent*:

$$\operatorname{Re}\frac{\langle Aw, w\rangle}{\|Aw\|\|w\|} \leqslant \kappa, \quad \forall w \in \boldsymbol{D}(A)\ (Aw \neq 0), \tag{3.8a}$$

$$\bigl\|(A - dI)^{-1}\bigr\| \leqslant \frac{K}{d}, \quad \forall d \in \bigl(0, \|A\|/\kappa\bigr]. \tag{3.8b}$$

Proof. An inspection of the proof of Proposition 3.1 shows that the equivalence (3.8a) ⇔ (3.8b) can be proved in a completely analogous way, if '$d > 0$' is replaced by '$d \in (0, \|A\|/\kappa]$' in (3.5) and (3.7). □

The case $\kappa < 0$ can be characterized as follows:

Proposition 3.3. *For given $\kappa \in (-1,0)$, let $K := (1-\kappa^2)^{+1/2}$. Then the following assertions are equivalent*:

$$\operatorname{Re}\frac{\langle Aw, w\rangle}{\|Aw\|\|w\|} \leqslant \kappa, \quad \forall w \in \boldsymbol{D}(A)\ (Aw \neq 0), \tag{3.9a}$$

$$\exists d < 0\colon\ \bigl\|(A - dI)^{-1}\bigr\| \leqslant K|d|. \tag{3.9b}$$

Proof. In our standard notation (1.6), relation (3.9a) is equivalent to

$$2d\kappa s - 2dr\cos\omega \leqslant 0, \quad \forall w \in \boldsymbol{D}(A), \tag{3.10}$$

where $d < 0$ can be chosen arbitrarily (it may depend on w). As in the proof of Proposition 3.1 (cf. (3.3), (3.4)) we can write

$$2d\kappa s - 2dr\cos\omega = \|Aw - dw\|^2 - d^2\bigl(1-\kappa^2\bigr) - [d\kappa - s]^2. \tag{3.11}$$

Since $d < 0$ is arbitrary, the following assertions are thus equivalent:

$$\text{(3.9a)} \iff \forall w \in \boldsymbol{D}(A)\colon\ \exists d < 0\colon\ \|Aw - dw\|^2 - d^2\bigl(1-\kappa^2\bigr) - [d\kappa - s]^2 \leqslant 0, \tag{3.12}$$

$$\iff \forall w \in \boldsymbol{D}(A)\colon\ \|Aw - \underline{d}w\|^2 - \underline{d}^2\bigl(1-\kappa^2\bigr) \leqslant 0, \tag{3.13}$$

where $\underline{d}$ is chosen (depending on w) such that $\underline{d}\kappa - s = 0$, i.e., $\underline{d} := s/\kappa < 0$.

Now we argue as follows: If (3.9a) holds, then

$$\forall w \in \boldsymbol{D}(A)\colon\ \exists d < 0\colon\ \|Aw - dw\|^2 - d^2\bigl(1-\kappa^2\bigr) \leqslant 0, \tag{3.14}$$

follows from (3.13). Conversely, (3.14) implies (3.12), which is equivalent to (3.9a). This completes the proof since (3.14) is equivalent to (3.9b). □

3.2. Numerical computation. Discussion and example

We now consider the finite-dimensional case, say $A \in \mathbb{C}^{n\times n}$, and briefly discuss the problem of numerically exploring $\mathcal{R}(A)$ or $\mathcal{R}_N(A)$. To this end one may either choose a reasonably complete set of vectors $w \in \mathbb{C}^n$ and plot the values $\langle Aw, w\rangle/\langle w, w\rangle$ or $\langle Aw, w\rangle/(\|Aw\|\|w\|)$. A more systematic approach is described in the following.

- Concerning $\mathcal{R}(A)$, it is well known how to scan its boundary (cf. [5]): Choose a grid of values $\theta \in [0, 2\pi]$ and compute, for each θ, the eigenvector w_θ corresponding to the rightmost eigenvalue μ_θ of $\mathrm{Re}(e^{i\theta}A)$. Then,[3] $\langle Aw_\theta, w_\theta\rangle/\langle w_\theta, w_\theta\rangle$ is a boundary point of $\mathcal{R}(A)$. In this way, Figs. 4 and 5 were generated using MATLAB [14].
  The quantity μ_θ is the so-called logarithmic norm of $e^{i\theta}A$, $\mu_\theta = \mathrm{lognorm}(e^{i\theta}A)$; it is the leftmost real number satisfying
  $$\big\langle \mathrm{Re}\big(e^{i\theta}A\big)w, w\big\rangle \leqslant \mu_\theta\langle w, w\rangle, \quad \forall w \in C^n. \tag{3.15}$$
  Therefore, a plot of $\mathcal{R}(A)$ can also be constructed by plotting the lines which are orthogonal to the vector represented by $e^{i\theta}$ at the point $\mu_\theta e^{i\theta}$. The envelope of all these lines gives the boundary of the convex set $\mathcal{R}(A)$.
- For $\mathcal{R}_N(A)$, a nontrivial generalization of latter procedure can be designed on the basis of Propositions 3.1 and 3.3. Again we choose a grid of values $\theta \in [0, 2\pi]$, and for each θ we try to find a good approximation $\tilde\kappa_\theta$ for the best possible, i.e., the leftmost value κ_θ satisfying
  $$\mathrm{Re}\,\frac{\langle e^{i\theta}Aw, w\rangle}{\|e^{i\theta}Aw\|\|w\|} \leqslant \kappa_\theta, \quad \forall w \in \boldsymbol{D}(A)\ (Aw \neq 0). \tag{3.16}$$
  To this end we first compute the eigenvalues of A and $\mu_\theta = \mathrm{lognorm}(e^{i\theta}A)$ and consider several cases:
  (i) If θ is sufficiently close to $\mathrm{Arg}\,\lambda$ for some eigenvalue $\lambda = \lambda(A)$ (plotting accuracy), set $\tilde\kappa_\theta := 1$.
  (ii) Else if μ_θ is sufficiently close to zero (plotting accuracy), set $\tilde\kappa_\theta := 0$.
  (iii) Else if $\mu_\theta > 0$, we use Proposition 3.1, (3.1b), to find an approximation for the optimal value $\kappa_\theta > 0$: If we compute $\widetilde{K}_\theta := \max\{d\|(A - dI)^{-1}\|\}$ for a certain set of test values $d \in (0, D]$, then $\tilde\kappa_\theta := (1 - \widetilde{K}_\theta^{-2})^{1/2}$ is a lower bound for κ_θ. To obtain an upper bound for κ_θ, we extend the set of test values until $D \cdot \tilde\kappa_\theta \geqslant \|A\|$ is satisfied (cf. Corollary 3.2).
  (iv) Else if $\mu_\theta < 0$, we use Proposition 3.3, (3.9b), to find an approximation for the optimal value $\kappa_\theta < 0$: If we compute $\widetilde{K}_\theta := \min\{|d|^{-1}\|((A - dI)^{-1})\|\}$ for a certain set of test values $d \in [-D, 0)$, then $\tilde\kappa_\theta := -(1 - \widetilde{K}_\theta^2)^{1/2}$ is an upper bound for κ_θ.
  Finally, a plot of $\mathcal{R}_N(A)$ is constructed by plotting the lines which are orthogonal to the vector represented by $e^{i\theta}$ at the point $\tilde\kappa_\theta e^{i\theta}$. The envelope of all these lines gives an (approximate) inclusion for the set $\mathcal{R}_N(A)$.
  Here, some nontrivial algorithmic parameters remain unspecified. A more accurate sampling of θ- and d-values will increase the quality of the approximation. Furthermore, the 'upper bound' obtained

[3] In [5], there is a misprint in the description of this algorithm.

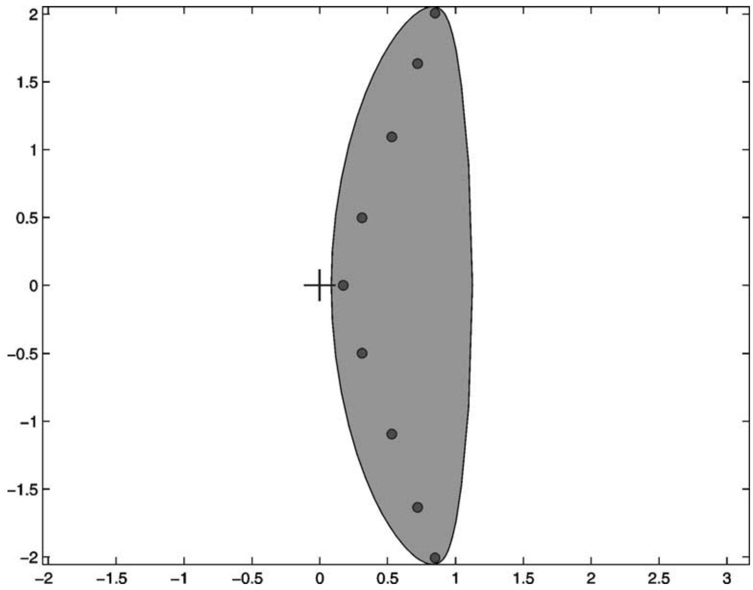

Fig. 4. Numerical range $\mathcal{R}(A)$, for A from (3.17).

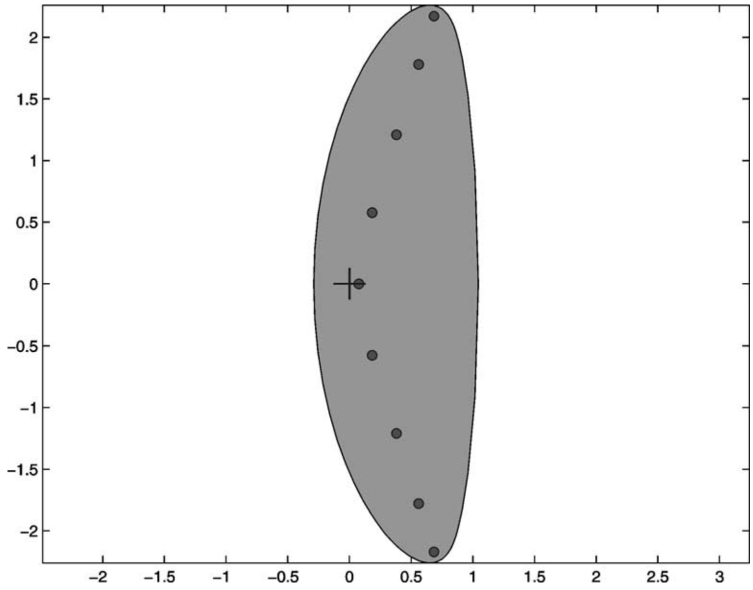

Fig. 5. Numerical range $\mathcal{R}(A)$, for A from (3.18).

in (iii) is only valid up to a discretization error caused by the sampling procedure concerning the parameter d. We do not further dwell on these algorithmic issues here; for the present purpose (cf. Fig. 6) we have used an experimental MATLAB code where a number of parameters can be chosen manually.

The algorithm described above can, e.g., be used for a numerical test of the sectorial property for a given A. Since this procedure is based on resolvent estimates, one may alternatively consider a direct verification of (2.6). However, our approach is more flexible because the problem of numerically determining $\mathcal{R}_N(A)$ and the verification of certain properties of this set are clearly separated.

Apart from these computational issues, the message of this paper is simply this: Resolvent conditions of the type (2.4b) or (2.6) can be equivalently formulated in terms of $\mathcal{R}_N(A)$, and this reformulation directly refers to the behavior of the mapping A itself, without reference to a continuum of resolvents. For theoretical purposes, it is always advantageous to have different but equivalent characterizations at one's disposal.

Example. The (9×9)-matrix

$$A = \begin{pmatrix} 1.00 & -1.00 & & & & \\ 1.00 & 0.50 & -1.00 & & & \\ -0.50 & 1.00 & 0.50 & -1.00 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & -1.00 \\ & & & -0.50 & 1.00 & 0.50 \end{pmatrix}, \tag{3.17}$$

may be considered as the system matrix of an evolution equation $u_t + Au = \cdots$ which emerges from a convection–diffusion problem after spatial discretization. Here, in the interior grid points, a nonsymmetric downstream 3-point scheme has been used for the discretization of the first spatial derivative. The resulting matrix A is not diagonally dominant, and therefore the stability and smoothing behavior of this system for $t > 0$ (which is intimately related to a sectorial property) is not *a priori* obvious. However, a second look show that $\operatorname{lognorm}(-A) \leqslant 0$, and a numerical computation of $\mathcal{R}(A)$ shows that A is sectorial in the conventional sense of Definition 1.1, cf. Fig. 4. (The eigenvalues of A are marked by bullets.)

Now we consider the modified matrix (which is again related to a convection–diffusion problem, but with slightly different coefficients)

$$A = \begin{pmatrix} 0.90 & -1.00 & & & & \\ 1.20 & 0.35 & -1.00 & & & \\ -0.55 & 1.20 & 0.35 & -1.00 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & -1.00 \\ & & & -0.55 & 1.20 & 0.35 \end{pmatrix}. \tag{3.18}$$

Here, the numerical inspection of $\mathcal{R}(A)$ show that A is not sectorial in the conventional sense. However, all eigenvalues of A are contained in the right half plane, cf. Fig. 5. An inclusion for the set $\mathcal{R}_N(A)$, based on the algorithm described above, is shown in Fig. 6 (the bullets correspond to the values $\lambda/|\lambda|$ where λ is an eigenvalue of A). From this and from Theorem 2.3 we presume claim that A from (3.18) is sectorial in the general sense (2.6), with a constant $K > 1$.

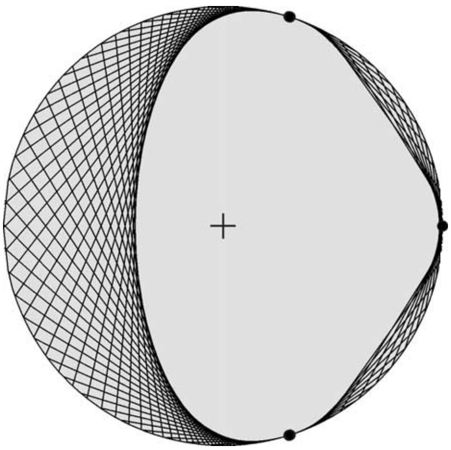

Fig. 6. Inclusion for the normalized numerical range $\mathcal{R}_N(A)$, for A from (3.18).

This example may be called artificial to some extent, but it nicely illustrates our theoretical considerations. We expect that more relevant examples exist, e.g., in the context of evolution equations in several spatial coordinates.

4. On the existence of resolvents

Up to now, our considerations were essentially algebraic. We now briefly turn our attention on a more analytic aspect, namely the question under what conditions on the operator A the inverse $(A - zI)^{-1}$ appearing in our main Theorem 2.3 is defined on the whole space $\boldsymbol{H}$ (cf. the introductory discussion in Section 1.2). In the following we present a sufficient condition of this type; it is closely related to similar conditions for accretive operators.

The material in this section is not self-contained. For the functional analytic background we refer to the relevant literature, cf., e.g., [4] or [9].

Proposition 4.1 below relies on a modification of [4, Theorem 2.3]. We formulate the proof in a way such that, with [4] or [9] at hand, the interested reader can easily check that this modification is correct.

Proposition 4.1. *Let A be a closed operator in a Hilbert space $\boldsymbol{H}$ and assume that, for all $z \in \mathcal{T}_\sigma$, an estimate of the form*

$$\|Aw - zw\| \geqslant C(z)\|w\|, \quad \forall\, w \in \boldsymbol{D}(A), \tag{4.1}$$

is valid for all $z \in \mathcal{T}_\sigma$, with positive constants $C(z)$. Then, $A - zI$ is semi-Fredholm with $\operatorname{nul}(A - zI) = 0$ *and* $\operatorname{def}(A - zI) = \text{const.}$ *for $z \in \mathcal{T}_\sigma$. If* $\operatorname{def}(A - zI) = 0$ *for some $z \in \mathcal{T}_\sigma$, then $\mathcal{T}_\sigma$ is a subset of the resolvent set of A, and for all $z \in \mathcal{T}_\sigma$ the resolvent $(A - zI)^{-1}$ satisfies*

$$\left\|(A - zI)^{-1}\right\| \leqslant \frac{1}{C(z)}. \tag{4.2}$$

Proof. Our argument is essentially the same as in the proof of [4, Theorem 2.3]. First of all, for arbitrary $z \in \mathcal{T}_\sigma$, the fact that $\operatorname{nul}(A - zI) = \dim \ker(A - zI) = 0$ follows directly from (4.1). Therefore,

$(A - zI)^{-1}$ exists and is a bounded closed operator on $\boldsymbol{R}(A - zI)$. Furthermore, $\boldsymbol{R}(A - zI)$ is closed and $A - zI$ is semi-Fredholm with index

$$\operatorname{ind}(A - zI) = -\operatorname{def}(A - zI). \tag{4.3}$$

Now we invoke a stability theorem for the nullity and deficiency of semi-Fredholm operators (cf. [4, Theorem I.3.22] or [9, Theorem IV-5.17]). Together with (4.3), and since $\mathcal{T}_\sigma$ is a connected set, this implies that $\operatorname{def}(A - zI) = \text{const.}$ in $\mathcal{T}_\sigma$. Therefore, if $\operatorname{def}(A - zI) = 0$ for at least one $z \in \mathcal{T}_\sigma$, we have $\boldsymbol{R}(A - zI) = \boldsymbol{H}$ for all $z \in \mathcal{T}_\sigma$. Now, the inequality (4.2) immediately follows from (4.1). □

Our last proposition accompanies our main Theorem 2.3; note that under the specified assumptions A is already 'maximal', i.e., it is necessarily densely defined.

Proposition 4.2. *Assume that A is a closed operator which satisfies* (2.8a) (*cf. Proposition* 2.2). *If there exists at least one $z \in \mathcal{T}_\sigma$ which is contained in the resolvent set of A, then*

- *the same is true for all $z \in \mathcal{T}_\sigma$,*
- *A is densely defined, i.e., $\boldsymbol{D}(A)$ is dense in $\boldsymbol{H}$.*

Proof (*see also* [4, *Theorem* 6.4]). The first assertion follows directly from Proposition 4.1. Furthermore, assume conversely that $\boldsymbol{D}(A)$ is not dense in $\boldsymbol{H}$. Then there exists a nonzero $v \in \boldsymbol{H}$ which is orthogonal to $\boldsymbol{D}(A)$. By assumption on z, $u := (A - zI)^{-1} v$ is well defined and satisfies $0 \neq u \in \boldsymbol{D}(A - zI) = \boldsymbol{D}(A)$, and therefore we obtain

$$0 = \bigl\langle v, (A - zI)^{-1} v, v \bigr\rangle = \bigl\langle (A - zI)u, u \bigr\rangle,$$

hence

$$\langle Au, u \rangle = z \langle u, u \rangle \quad (u \neq 0).$$

This means $z \in \mathcal{S}_\sigma$, contradicting the assumption $z \in \mathcal{T}_\sigma = \mathbb{C} \backslash \mathcal{S}_\sigma$. Therefore $\boldsymbol{D}(A)$ must be dense in $\boldsymbol{H}$. □

Acknowledgement

The author wishes to thank Heinz Langer and both referees for their helpful comments.

Appendix A. Proof of Proposition 2.2

Proof. For $K = 1$, the assertion to be proved is equivalent to Proposition 1.3. From now on we assume $K > 1$ and let $k := 1/K \in (0, 1)$. Furthermore we use the notation from (1.7): $a = r/s$.

Consider an arbitrary $w \in \boldsymbol{D}(A)$ with $Aw \neq 0$ (i.e., $s > 0$). Assertion (2.8b) is equivalent to the requirement that

$$\frac{\|Aw - zw\|^2}{\|w\|^2} \geqslant k^2 D_\sigma^2(z), \tag{A.1}$$

is valid for all $z \in \mathcal{T}_\sigma$. With (1.6), (1.8) this is equivalent to

$$\frac{\|Aw - zw\|^2}{\|w\|^2} - k^2 D_\sigma^2(z) \geqslant 0, \quad \forall z = de^{i\theta} \in \mathcal{T}_\sigma \tag{A.2a}$$

$$\iff s^2 - 2rd\cos(\theta - \omega) + d^2 - k^2 D_\sigma^2(z) \geqslant 0, \quad \forall z = de^{i\theta} \in \mathcal{T}_\sigma \tag{A.2b}$$

$$\iff \inf_{z = de^{i\theta} \in \mathcal{T}_\sigma} \underbrace{\left(s^2 - 2rd\cos(\theta - \omega) + d^2 - k^2 D_\sigma^2(z)\right)}_{=: f(d,\theta)} \geqslant 0. \tag{A.2c}$$

Now we investigate condition (A.2); our goal is to express it equivalently in terms of the w-dependent quantities r, ω and s (cf. (1.6)) but independent of the variable $z = de^{i\theta} \in \mathcal{T}_\sigma$. To this end we shall explicitly compute the infimum in (A.2c).

By symmetry, we may assume without loss of generality that $\omega \in [0, \pi]$ holds ($\omega \in (-\pi, 0)$ can be handled in an analogous way). Concerning $\theta \in \mathcal{T}_\sigma$, it is now convenient (without loss of generality) to choose $\theta \in (\sigma, 2\pi - \sigma)$ (instead of $\theta \in (\sigma, \pi] \cup (-\pi, -\sigma)$).

By definition of $D_\sigma(z)$ (cf. (1.9)), the function $f(d, \theta)$ to be studied[4] in (A.2) and its first and second derivatives with respect to d read as follows.

- $f(d, \theta)$ for $|\theta| \in (\sigma, \sigma + \frac{\pi}{2})$:

$$f(d,\theta) = s^2 - 2rd\cos(\theta - \omega) + d^2\left(1 - k^2 \sin^2(\theta - \sigma)\right),$$

$$\frac{\partial}{\partial d} f(d,\theta) = -2r\cos(\theta - \omega) + 2d\left(1 - k^2 \sin^2(\theta - \sigma)\right),$$

$$\frac{\partial^2}{\partial d^2} f(d,\theta) = 2\left(1 - k^2 \sin^2(\theta - \sigma)\right) > 0. \tag{A.3a}$$

For θ fixed, f this is quadratic with respect to ρ, and the global minimum (with respect to $\rho \in \mathbb{R}$) is attained for $\frac{\partial f}{\partial d} = 0$, i.e., at

$$d = \underline{d}(\theta) := \frac{r\cos(\theta - \omega)}{1 - k^2 \sin^2(\theta - \sigma)}, \tag{A.4}$$

with

$$f\left(\underline{d}(\theta), \theta\right) =: \underline{f}(\theta) = s^2 - \frac{r^2\cos^2(\theta - \omega)}{1 - k^2 \sin^2(\theta - \sigma)}. \tag{A.5}$$

We shall also need the derivative of (A.5): It is given by

$$\underline{f}'(\theta) = \underbrace{\frac{2r^2\cos(\theta - \omega)}{[1 - k^2 \sin^2(\theta - \sigma)]^2}}_{>0}\left[\sin(\theta - \omega) - k^2 \sin(\theta - \sigma)\cos(\omega - \sigma)\right], \tag{A.6}$$

where the $[\cdots]$-term can also be written as

$$\begin{aligned} &\sin(\theta - \omega) - k^2 \sin(\theta - \sigma)\cos(\omega - \sigma) \\ &\quad = \left[1 - k^2\cos^2(\omega - \sigma)\right]\sin(\theta - \omega) - \left[k^2 \sin(\omega - \sigma)\cos(\omega - \sigma)\right]\cos(\theta - \omega). \end{aligned} \tag{A.7}$$

[4] In our notation, the dependence of $f(\cdot)$ on the vector w actually considered is suppressed.

- $f(d,\theta)$ for $|\theta| \in [\sigma + \frac{\pi}{2}, \pi]$:

$$f(d,\theta) = s^2 - 2rd\cos(\theta-\omega) + d^2\bigl(1-k^2\bigr), \tag{A.8a}$$

$$\frac{\partial}{\partial d} f(d,\theta) = -2r\cos(\theta-\omega) + 2d\bigl(1-k^2\bigr), \tag{A.8b}$$

$$\frac{\partial^2}{\partial d^2} f(d,\theta) = 2\bigl(1-k^2\bigr) > 0. \tag{A.8c}$$

For θ fixed, f this is quadratic with respect to ρ, and the global minimum (with respect to $\rho \in \mathbb{R}$) is attained for $\frac{\partial f}{\partial d} = 0$, i.e., at

$$d = \underline{d}(\theta) := \frac{r\cos(\theta-\omega)}{1-k^2}, \tag{A.9}$$

with

$$f\bigl(\underline{d}(\theta),\theta\bigr) =: \underline{f}(\theta) = s^2 - \frac{r^2\cos^2(\theta-\omega)}{1-k^2}. \tag{A.10}$$

It is easy to see that a universal d-independent lower bound for $f(d,\theta)$ is given as follows:

$$f(d,\theta) \geqslant \underline{f}(\theta) \geqslant \begin{cases} s^2, & \cos(\theta-\omega) \leqslant 0, \\ s^2 - \frac{r^2}{1-k^2} < s^2, & \cos(\theta-\omega) > 0. \end{cases} \tag{A.11}$$

We now compute $\inf_{z=de^{i\theta}\in\mathcal{T}_\sigma} f(d,\theta)$. Concerning the given w, we consider three cases:

(i) $\langle Aw, w\rangle \in \mathcal{S}_\sigma$, i.e., $\omega \in [0,\sigma]$.
This case is simple: Assumption (i) immediately implies (2.8b) ($\Leftrightarrow$ (A.2)) for arbitrary $k < 1$ due to Proposition 1.1. Conversely, there is nothing to show because (2.8a) is *a priori* satisfied for case (i) by definition of $\mathcal{A}_{\sigma,K}$. Thus,

$$\inf_{\text{case (i)}} f(d,\theta) \geqslant 0 \iff \omega \leqslant \sigma. \tag{A.12}$$

(ii) $\langle Aw, w\rangle \in \mathcal{T}_\sigma$, with $\omega \in (\sigma, \sigma + \frac{\pi}{2})$.
Here we have

$$\sin(\omega-\sigma) > 0, \qquad \cos(\omega-\sigma) > 0. \tag{A.13}$$

We consider three sub-cases concerning the range of the variable $\theta \in (\sigma, 2\pi - \sigma)$:

(iia) $\theta \in (\sigma, \sigma + \frac{\pi}{2})$. Here, $f(d,\theta)$ takes the form (A.3a), and we have

$$\theta - \omega \in \left(-(\omega-\sigma), \frac{\pi}{2} - (\omega-\sigma)\right] \subseteq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \text{hence } \cos(\theta-\omega) > 0, \tag{A.14}$$

by assumptions (ii), (iia).
– For fixed $\theta \in (\sigma, \sigma + \frac{\pi}{2})$, we have due to (A.14):

$$\inf_{d\geqslant 0} f(d,\theta) = \underline{f}(\theta), \quad \text{as given in (A.5).} \tag{A.15}$$

– We thus know that

$$\inf_{\text{case (iia)}} f(d,\theta) \geqslant 0 \iff \inf_{\theta\in(\sigma,\sigma+\frac{\pi}{2}]} \underline{f}(\theta) \geqslant 0, \tag{A.16}$$

and to determine this infimum we now compute $\arg\min \underline{f}(\theta)$ for $\theta \in (\sigma, \sigma + \frac{\pi}{2})$: Due to (A.14) and (A.6), (A.7), equation $\underline{f}'(\theta) = 0$ is equivalent to

$$\tan(\theta - \omega) = \frac{k^2 \sin(\omega - \sigma)\cos(\omega - \sigma)}{1 - k^2 \cos^2(\omega - \sigma)}. \tag{A.17}$$

In the interval $\theta \in (\sigma, \sigma + \frac{\pi}{2})$, (A.17) has the unique solution[5]

$$\theta = \underline{\theta} := \omega + \operatorname{Arctan}\left(\frac{k^2 \sin(\omega - \sigma)\cos(\omega - \sigma)}{1 - k^2 \cos^2(\omega - \sigma)}\right). \tag{A.18}$$

To verify that $\underline{\theta} = \arg\min_{\theta \in (\sigma, \sigma + \frac{\pi}{2})} \underline{f}(\theta)$ is indeed true, a straightforward inspection taking account of (A.13) shows

$$\underline{f}'(\sigma) = -2r^2 \sin(\omega - \sigma)\cos(\omega - \sigma) < 0, \tag{A.19}$$

and

$$\underline{f}'\left(\sigma + \frac{\pi}{2}\right) = \frac{2r^2 \sin(\omega - \sigma)\cos(\omega - \sigma)}{1 - k^2} > 0. \tag{A.20}$$

– We have verified that

$$\text{(A.16)} \iff \underline{f}(\underline{\theta}) \geqslant 0, \quad \text{with } \underline{\theta} \text{ from (A.18)}. \tag{A.21}$$

Now we evaluate this latter inequality explicitly in terms of $\omega - \sigma$ in the following way. Due to (A.5), inequality $\underline{f}(\underline{\theta}) \geqslant 0$ is equivalent to

$$1 - k^2 \sin^2(\underline{\theta} - \sigma) - a^2 \cos^2(\underline{\theta} - \omega) \geqslant 0. \tag{A.22}$$

Using the first relation in (A.17), which is valid for $\theta = \underline{\theta}$, we now express $\sin^2(\underline{\theta} - \sigma)$ in terms of $\cos^2(\underline{\theta} - \omega)$ and $\cos^2(\omega - \sigma)$. Furthermore, $\cos^2(\underline{\theta} - \omega)$ can be expressed via (A.18) and the identity $\cos^2(\operatorname{Arctan}(\alpha)) \equiv 1/(1 + \alpha^2)$. After some simple but tedious manipulations, this turns (A.22) into the equivalent form

$$\left(1 - a^2 k^2 c^2\right)\left(1 - k^2 c^2\right) - \left(1 - k^2 c^2\right)^2 - k^4 c^2 \left(1 - c^2\right) \geqslant 0, \tag{A.23}$$

where $c^2 := \cos^2(\omega - \sigma) > 0$. Expansion, simplification, and division by $k^2 c^2 \neq 0$ gives

$$\left(1 - k^2\right) - a^2\left(1 - k^2 c^2\right) \geqslant 0, \tag{A.24}$$

and we finally obtain

$$\inf_{\text{case (iia)}} f(d, \theta) \geqslant 0 \iff a^2 \left[\frac{\sin^2(\omega - \sigma)}{1 - k^2} + \cos^2(\omega - \sigma)\right] \leqslant 1. \tag{A.25}$$

[5] By assumption (ii), the right-hand of (A.17) is nonnegative; furthermore it is monotonously increasing as a function of $k \in (0, 1)$, with values in $(0, \cot(\omega - \sigma))$(check $k = 1$). Therefore we have

$$\sigma < \omega \leqslant \underline{\theta} < \omega + \operatorname{Arctan}\big(\cot(\omega - \sigma)\big) = \omega + \left(\tfrac{\pi}{2} - (\omega - \sigma)\right) = \sigma + \tfrac{\pi}{2} \implies \underline{\theta} \in \left(\sigma, \sigma + \tfrac{\pi}{2}\right).$$

(iib) $\theta \in [\sigma + \frac{\pi}{2}, \frac{3\pi}{2} - \sigma]$. Here, $f(d,\theta)$ takes the form (A.8a), and we have

$$\theta - \omega \in \left[\frac{\pi}{2} - (\omega - \sigma), \frac{3\pi}{2} - (\omega + \sigma)\right] \subseteq \left(0, \frac{3\pi}{2} - \sigma\right), \tag{A.26}$$

by assumptions (ii), (iib). Here the case where $\theta - \omega \geqslant \frac{\pi}{2}$ is trivial because $\cos(\theta - \omega) \leqslant 0$, hence $f(d,\theta) \geqslant s^2 \geqslant 0$ holds without any restriction on r, ω (cf. (A.11)).

– We now consider values

$$\theta - \omega \in \left[\frac{\pi}{2} - (\omega - \sigma), \frac{\pi}{2}\right) \subseteq \left(0, \frac{\pi}{2}\right), \quad \text{hence } \cos(\theta - \omega) > 0. \tag{A.27}$$

(Note that the interval $[\frac{\pi}{2} - (\omega - \sigma), \frac{\pi}{2})$ is nonempty by assumption (ii).) For fixed $\theta - \omega \in [\frac{\pi}{2} - (\omega - \sigma), \frac{\pi}{2})$, we have due to (A.27):

$$\inf_{d \geqslant 0} f(d,\theta) = \underline{f}(\theta), \quad \text{as given in (A.10).} \tag{A.28}$$

– We thus know that

$$\inf_{\text{case (iib)}} f(d,\theta) \geqslant 0 \iff \inf_{\theta - \omega \in [\frac{\pi}{2} - (\omega - \sigma), \frac{\pi}{2})} \underline{f}(\theta) \geqslant 0, \tag{A.29}$$

and to determine this infimum we observe that $\underline{f}(\theta)$ is monotonously increasing for (A.27), i.e., in the interval $\theta \in [\sigma + \frac{\pi}{2}, \omega + \frac{\pi}{2})$. Therefore (A.29) is equivalent to

$$0 \leqslant \underline{f}\left(\sigma + \frac{\pi}{2}\right) = s^2 - \frac{r^2 \cos^2((\frac{\pi}{2} - (\omega - \sigma))}{1 - k^2}, \tag{A.30}$$

and we finally obtain

$$\inf_{\text{case (iib)}} f(d,\theta) \geqslant 0 \iff a^2 \frac{\sin^2(\omega - \sigma)}{1 - k^2} \leqslant 1. \tag{A.31}$$

Note that the last inequality in (A.31) is weaker than that one from (A.25), i.e., it follows from the latter.

(iic) $\theta \in (\frac{3\pi}{2} - \sigma, 2\pi - \sigma)$. Here, $f(d,\theta)$ takes the form (A.3a), and we have

$$\theta - \omega \in \left(\frac{3\pi}{2} - (\omega + \sigma), 2\pi - (\omega + \sigma)\right) \subseteq (\pi - 2\sigma, 2\pi - 2\sigma) \subseteq (0, 2\pi), \tag{A.32}$$

by assumptions (ii), (iib). Here the case where $\theta - \omega \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ is trivial because $\cos(\theta - \omega) \leqslant 0$, hence $f(d,\theta) \geqslant s^2 \geqslant 0$ holds without any restriction on r, ω (cf. (A.11)).

– We now consider values

$$\theta - \omega \in \left(\frac{3\pi}{2} - (\omega + \sigma), 2\pi - (\omega + \sigma)\right), \quad \text{where } \cos(\theta - \omega) > 0. \tag{A.33}$$

This range of values is an open interval

$$\theta - \omega \in (\delta_L, \delta_R) \quad \text{contained either in } \left(0, \frac{\pi}{2}\right) \text{ or in } \left(\frac{3\pi}{2}, 2\pi\right). \tag{A.34}$$

Let us consider these alternatives:

– If $(\delta_L, \delta_R) \subseteq (0, \frac{\pi}{2})$, we argue as follows. The infimum of $f(d,\theta)$ for $d > 0$ and $\theta - \omega \in (0, \frac{\pi}{2})$ can be computed in a completely analogous way as for case (iia), and we have

$$\inf_{\text{case (iic)}} f(d,\theta) \geqslant 0 \iff \inf_{\theta-\omega\in(\delta_L,\delta_R)} \underline{f}(\theta) \geqslant 0 \Longleftarrow \inf_{\theta-\omega\in(0,\frac{\pi}{2})} \underline{f}(\theta) \geqslant 0, \tag{A.35}$$

where $\underline{f}(\theta)$, $\underline{f}'(\theta)$ are given by (A.5), (A.6). The solution $\underline{\theta}$ of $\underline{f}'(\theta) = 0$ is given in (A.18), and it is unique in the range $\theta - \omega \in (0, \frac{\pi}{2})$. To verify that $\underline{\theta} = \arg\min_{\theta-\omega\in(0,\frac{\pi}{2})} \underline{f}(\theta)$ is indeed true, a straightforward inspection taking account of (A.13) shows

$$\underline{f}'(\omega) = \underbrace{\frac{2r^2}{(1-k^2\sin^2(\theta-\sigma))^2}}_{>0}\big[0 - k^2\sin(\omega-\sigma)\cos(\omega-\sigma)\big] < 0, \tag{A.36}$$

and

$$\underline{f}'\left(\omega + \frac{\pi}{2}\right) = 0. \tag{A.37}$$

The evaluation of $\underline{f}(\underline{\theta})$ is completely analogous as for case (iia), and therefore

$$\inf_{\theta-\omega\in(0,\frac{\pi}{2})} \underline{f}(\theta) \geqslant 0 \iff a^2\left[\frac{\sin^2(\omega-\sigma)}{1-k^2} + \cos^2(\omega-\sigma)\right] \leqslant 1, \tag{A.38}$$

and due to (A.35) we obtain for case (iic) with $(\delta_L, \delta_R) \subseteq (0, \frac{\pi}{2})$:

$$\inf_{\text{case (iic)}} f(d,\theta) \geqslant 0 \Longleftarrow a^2\left[\frac{\sin^2(\omega-\sigma)}{1-k^2} + \cos^2(\omega-\sigma)\right] \leqslant 1. \tag{A.39}$$

– If $(\delta_L, \delta_R) \subseteq (\frac{3\pi}{2}, 2\pi)$, we have

$$\inf_{\text{case (iic)}} f(d,\theta) \geqslant 0 \iff \inf_{\theta-\omega\in(\delta_L,\delta_R)} \underline{f}(\theta) \geqslant 0 \Longleftarrow \inf_{\theta-\omega\in(\frac{3\pi}{2},2\pi)} \underline{f}(\theta) \geqslant 0, \tag{A.40}$$

where $\underline{f}(\theta)$, $\underline{f}'(\theta)$ are given by (A.5), (A.6). In the range $\theta - \omega \in (\frac{3\pi}{2}, 2\pi)$, $\underline{f}(\theta)$ is monotonously decreasing because (A.7) is negative due to (A.13) and $\cos(\theta-\omega) > 0$, $\sin(\theta-\omega) < 0$. Thus,

$$\underline{f}(\omega + 2\pi) = s^2 - \frac{r^2}{1-k^2\sin^2(\omega-\sigma)} \tag{A.41}$$

gives

$$\inf_{\theta-\omega\in(\frac{3\pi}{2},2\pi)} \underline{f}(\theta) \geqslant 0 \iff \frac{a^2}{1-k^2\sin^2(\omega-\sigma)} \leqslant 1, \tag{A.42}$$

and due to (A.40) we obtain[6] for case (iic) with $(\delta_L, \delta_R) \subseteq (\frac{3\pi}{2}, 2\pi)$:

$$\inf_{\text{case (iic)}} f(d,\theta) \geqslant 0 \Longleftarrow \frac{a^2}{1-k^2\sin^2(\omega-\sigma)} \leqslant 1. \tag{A.43}$$

Note that the last inequality in (A.43) is weaker than the one from (A.25), i.e., it follows from the latter.

[6] The last inequality in (A.43) follows from $1/(1-k^2s^2) \leqslant s^2/(1-k^2) + (1-s^2)$ for arbitrary $k, s \in (0,1)$.

To summarize case (ii), we combine (A.25), (A.31) and (A.39) respectively (A.43), and conclude

$$\inf_{\text{case (ii)}} f(d,\theta) \geqslant 0 \iff a^2\left[\frac{\sin^2(\omega-\sigma)}{1-k^2} + \cos^2(\omega-\sigma)\right] \leqslant 1. \tag{A.44}$$

(iii) $\langle Aw, w\rangle \in \mathcal{T}_\sigma$, with $\omega \in [\sigma + \frac{\pi}{2}, \pi]$.
This case is simple: For $\theta = \omega \in [\sigma + \frac{\pi}{2}, \pi]$ we have $\cos(\theta-\omega) = 1$, and therefore the smaller of the lower bounds in (A.11) is attained in this case. The immediate consequence is

$$\inf_{\text{case (iii)}} f(d,\theta) \geqslant 0 \iff \frac{a^2}{1-k^2} \leqslant 1. \tag{A.45}$$

Finally, the assertion of Proposition 2.2 follows directly from the conclusions we have drawn for case (i), (ii) and (iii) considered above, cf. (A.12), (A.44), and (A.45). (By symmetry, these conclusions are equally valid with $|\omega|$ instead of ω.) □

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