ERROR STRUCTURES
FOR STIFF INITIAL VALUE PROBLEMS

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1 Introduction. Extrapolation based on implicit and semi-implicit methods

Within the convergence theory of discretization methods, the existence of *asymptotic error expansions* in powers of the stepsize h is the classical prerequisite for the theoretical justification of stepsize control mechanisms and acceleration techniques like extrapolation or defect correction. In the present paper we discuss the existence of asymptotic error expansions and their algorithmic utilization in the context of *stiff* nonlinear initial value problems

$$\begin{aligned} y' &= f(t,y), \qquad t \in [0,T] \\ y(0) &= y_0 \end{aligned} \tag{1}$$

Most of the material presented here is discussed in much more detail in a recent series of papers on this subject (cf. [1]–[7]).

Naturally, symmetric schemes are of major interest if one aims to apply Richardson extrapolation. The classical symmetric implicit one-step schemes are the *implicit midpoint rule* (IMR)

$$\begin{aligned} \frac{1}{h}(\eta_\nu - \eta_{\nu-1}) &= f\Big(\tfrac{1}{2}(t_{\nu-1}+t_\nu), \tfrac{1}{2}(\eta_{\nu-1}+\eta_\nu)\Big) \\ \eta_0 &= y_0 \end{aligned} \tag{2}$$

and the *implicit trapezoidal rule* (ITR)

$$\begin{aligned} \frac{1}{h}(\eta_\nu - \eta_{\nu-1}) &= \frac{1}{2}\Big(f(t_{\nu-1},\eta_{\nu-1}) + f(t_\nu,\eta_\nu)\Big) \\ \eta_0 &= y_0 \end{aligned} \tag{3}$$

First attempts to use these methods in extrapolation-based stiff solvers date back to the early Seventies (cf. [9], [11]). However, a rigorous, quantitative analysis of the error structure of the IMR and ITR for nonlinear stiff systems has only been done recently (cf. [3], [4]). It turns out that asymptotic error expansions in the conventional sense do usually not exist; however, as shown in [3], [4], the error structure is sufficiently regular to ensure that acceleration techniques work successfully in most situations. A review of the relevant theory is given in section 2 below. (For further results on error structures for stiff problems cf. [1], [2], [5], [6], [7], [10], [12].)

In the last years, semi-implicit (or linearly-implicit) methods have been proposed as the basis for extrapolation (cf. [8], [10]). The idea behind these methods is to 'freeze' the Jacobian matrix at the beginning of an extrapolation interval and to combine the linearly-implicit integration process with an explicit formula for the non-stiff part of the ODE. Semi-implicit methods are especially suited for the integration of semilinear problems with a constant 'stiff matrix' A and a smooth function $\varphi(y)$:

$$\begin{aligned} y' &= A\,y + \varphi(y) \\ y(0) &= y_0 \end{aligned} \tag{4}$$

The (two-step) *semi-implicit midpoint rule* (SIMR, proposed in [8]) applied to (4) can be written as

$$\begin{aligned}
\eta_0 &= y_0 \\
\frac{1}{h}(\eta_1 - \eta_0) &= A\,\eta_1 + \varphi(\eta_0) \\
\frac{1}{2h}(\eta_{\nu+1} - \eta_{\nu-1}) &= A\,\frac{1}{2}(\eta_{\nu-1} + \eta_{\nu+1}) + \varphi(\eta_\nu) \qquad \nu = 1,2,\ldots,N \\
\bar{\eta}_N &:= \frac{1}{2}(\eta_{N-1} + \eta_{N+1}) \qquad \textit{(smoothing step)}
\end{aligned} \tag{5}$$

This requires only one LU- decomposition for $(I-hA)$; the non-stiff part of (4) is integrated by the explicit midpoint rule. Numerical experience shows that for semilinear problems the extrapolated SIMR is a very efficient integration method. However, an analysis of the error structure of the SIMR (in the spirit of [3], [4]) has not been done so far. (A very simple case is studied in [10]; that paper contains also an analysis of a semi-implicit Euler method). Clearly, the typical imperfections occurring in asymptotic error expansions for fully-implicit methods (cf. section 2 below) must (at least) also be expected for the semi-implicit case.

The SIMR can be applied to general nonlinear stiff systems (1). In the general case, the Jacobian $f_y(t_0, \eta_0)$ (evaluated at the initial point t_0 of the current extrapolation interval) plays the role of A from (5), and $f(t,y) - f_y(t_0,\eta_0)y$ plays the role of $\varphi(y)$. But for problems more general than (4) we must expect to run into troubles: Consider for instance the simple linear problem

$$\begin{aligned}
y' &= A(t)y \\
y(0) &= y_0
\end{aligned} \tag{6}$$

with a time-dependent stiff matrix $A(t)$. Assume that the stiff eigenvalues are of the type $-c(t)/\varepsilon$, $Re(c(t)) > 0$ (ε a small parameter). Then, even for smooth functions $c(t)$,

$$A(t) = A(0) + tA'(0) + \tfrac{t^2}{2}A''(0) + \ldots = A(0) + O\Big(\frac{1}{\varepsilon}\Big), \tag{7}$$

thus the frozen Jacobian $A(0)$ is a very poor approximation for $A(t)$. On the other hand, as demonstrated in [10], semi-implicit methods work successfully for stiff problems of *singulary perturbed* type. The following numerical example serves to illustrate the behavior of extrapolation based on implicit and semi-implicit methods in the singularly perturbed case. Note that all our numerical results were obtained on fixed grids, without any control mechanism. (The question of stepsize control is quite difficult and is not treated here.)

Example: Van der Pol equation.

$$\begin{aligned}
y_1' &= y_2 \\
y_2' &= \frac{1}{\varepsilon}\big[(1-y_1^2)y_2 - y_1\big]
\end{aligned} \tag{8}$$

For $\varepsilon = 10^{-5}$, the initial value

$$y(0) = \begin{pmatrix} 1.5967739602\ldots\text{E+00} \\ -1.0303749391\ldots\text{E+00} \end{pmatrix} \tag{9}$$

defines a solution which remains smooth for $t \in [0, 0.2]$. Our computations were performed on a CDC Cyber 180/860 in double precision arithmetic (96 bit mantissa). Tables 1.1–1.4 show the global errors (measured in the L_2-norm) after extrapolation at $t = 0.2$.

Table 1.1 displays the results for h^2-extrapolation based on the semi-implicit midpoint rule. The SIMR works quite successfully; notice, however, that the accuracy of extrapolation 'stagnates' at the $O(\varepsilon)$-level.

VAN DER POL Eq, epsilon=1E-5: global error at t=0.2 Table 1.1

h	SIMR	1st EX	2nd EX	3rd EX	4th EX	5th EX
1/10	6.557E-02					
1/20	3.907E-02	3.153E-02				
1/40	1.768E-02	1.065E-02	9.261E-03			
1/80	5.850E-03	1.913E-03	1.331E-03	1.205E-03		
1/160	1.575E-03	1.508E-04	3.331E-05	1.276E-05	8.128E-06	
1/320	4.190E-04	3.365E-05	2.584E-05	2.572E-05	2.578E-05	2.579E-05

observed order

0.75					
1.14	1.57				
1.60	2.48	2.80			
1.89	3.67	5.32	6.56		
1.91	2.16	0.37	-1.01	-1.67	

We have modified the SIMR by evaluating the Jacobian not only at the initial point but at each grid point t_ν. Table 1.2 displays the global errors for the resulting semi-implicit updated midpoint rule (SIUMR), which is fully implicit in its essence. The achieved level of accuracy is significantly better than in Table 1.1; but estimating the efficiency one has to bear in mind that for this fully implicit version the computational effort per step is higher than for the SIMR.

VAN DER POL Eq, epsilon=1E-5: global error at t=0.2 Table 1.2

h	SIUMR	1st EX	2nd EX	3rd EX	4th EX	5th EX
1/10	2.123E-01					
1/20	5.327E-02	1.758E-03				
1/40	1.330E-02	1.230E-04	5.095E-05			
1/80	3.329E-03	1.211E-05	9.588E-06	1.045E-05		
1/160	8.328E-04	1.002E-06	3.143E-07	1.671E-07	1.268E-07	
1/320	2.082E-04	6.833E-08	6.917E-09	2.038E-09	1.391E-09	1.268E-09

observed order

1.99					
2.00	3.84				
2.00	3.34	2.41			
2.00	3.60	4.93	5.97		
2.00	3.87	5.51	6.36	6.51	

Table 1.3 shows the results for the IMR without smoothing; similarly as for the SIMR (Table 1.1), extrapolation begins to stagnate at the $O(\varepsilon)$-level.

VAN DER POL Eq, epsilon=1E-5: global error at t=0.2 Table 1.3

h	IMR	1st EX	2nd EX	3rd EX	4th EX	5th EX
1/10	5.439E-02					
1/20	1.747E-02	5.170E-03				
1/40	4.860E-03	6.560E-04	3.550E-04			
1/80	1.260E-03	5.974E-05	1.999E-05	1.467E-05		
1/160	3.208E-04	7.779E-06	4.316E-06	4.067E-06	4.025E-06	
1/320	8.252E-05	3.091E-06	2.779E-06	2.754E-06	2.749E-06	2.748E-06

observed order

	1.64					
	1.85	2.98				
	1.95	3.46	4.15			
	1.97	2.94	2.21	1.85		
	1.96	1.33	0.64	0.56	0.55	

Extrapolation based on the IMR in combination with a smoothing step at the end of an extrapolation interval leads to a considerably higher accuracy. Similar results as for the IMR with smoothing can be obtained by the ITR:

VAN DER POL Eq, epsilon=1E-5: global error at t=0.2 Table 1.4

h	ITR	1st EX	2nd EX	3rd EX	4th EX	5th EX
1/10	2.777E-02					
1/20	6.562E-03	5.076E-04				
1/40	1.619E-03	2.864E-05	3.286E-06			
1/80	4.034E-04	1.742E-06	5.129E-08	6.968E-10		
1/160	1.008E-04	1.078E-07	1.177E-09	3.840E-10	3.853E-10	
1/320	2.519E-05	6.520E-09	2.342E-10	2.193E-10	2.186E-10	2.185E-10

observed order

	2.08					
	2.02	4.15				
	2.00	4.04	6.00			
	2.00	4.01	5.45	0.86		
	2.00	4.05	2.33	0.81	0.82	

For the ITR the achieved accuracy again stagnates at the higher extrapolation steps, but the ‘level of stagnation’ is considerably below that for the SIMR and the IMR. (Of course, also the SIUMR suffers from these stagnations, although to a lesser extent compared to the SIMR.) For a theoretical explanation of these effects cf. section 2.

It must be emphasized that, in a certain sense, singulary perturbed problems are simple because there is only a weak coupling from the stiff to the non-stiff component (cf. section 2). The following example serves to demonstrate the limited applicability of semi-implicit methods in the case of *strong coupling*.

A simple linear example.

Consider a stiff system (6) with $A(t) = T(t)\,\Lambda(t)\,T^{-1}(t)$,

$$\Lambda(t) = \begin{pmatrix} \cos(1+t) & 0 \\ 0 & -\dfrac{1+e^{-(1+t)}}{\varepsilon} \end{pmatrix}, \qquad T(t) = \begin{pmatrix} 1+e^{-(1+t)} & \cos(1+t) \\ \cos(1+t) & 1+e^{-(1+t)} \end{pmatrix} \tag{10}$$

For $\varepsilon = 10^{-10}$ a smooth solution is fixed by the initial value

$$y(0) = \begin{pmatrix} 4.3736149438\ldots\text{E+00} \\ 1.7275456946\ldots\text{E+00} \end{pmatrix} \tag{11}$$

The numerical results given below refer to an autonomous reformulation of this problem; the numbers hardly differ from those obtained for the original, non-autonomous form.

The results for the SIMR are displayed in Table 1.5. (The semi-implicit Euler method proposed in [10] leads to similar results.)

y'=A(t)y, epsilon=1E-10: global error at t=1.0 Table 1.5

h	SIMR	1st EX	2nd EX	3rd EX	4th EX	5th EX
1/4	1.304E+36					
1/8	2.223E+68	2.965E+68				
1/16	2.292+130	3.056+130	3.260+130			
1/32	3.541+249	4.722+249	5.036+249	5.116+249		
1/64	1.920+478	2.560+478	2.731+478	2.774+478	2.785+478	
1/128	*** EXPONENT OVERFLOW ***					

For this example, updating the Jacobian at each grid point t_ν is absolutely necessary. The updated two-step midpoint rule (SIUMR) performs very well:

y'=A(t)y, epsilon=1E-10: global error at t=1.0 Table 1.6

h	SIUMR	1st EX	2nd EX	3rd EX	4th EX	5th EX
1/4	1.366E-02					
1/8	4.091E-03	9.017E-04				
1/16	1.057E-03	4.572E-05	1.134E-05			
1/32	2.663E-04	2.657E-06	2.135E-07	3.686E-08		
1/64	6.669E-05	1.634E-07	2.878E-09	4.656E-10	6.120E-10	
1/128	1.668E-05	1.017E-08	4.244E-11	2.575E-12	7.593E-13	1.619E-13

observed order

1.74					
1.95	4.30				
1.99	4.10	5.73			
2.00	4.02	6.21	6.31		
2.00	4.01	6.08	7.50	9.65	

Table 1.7 shows the results for the IMR (without smoothing):

y'=A(t)y, epsilon=1E-10: global error at t=1.0 Table 1.7

h	IMR	1st EX	2nd EX	3rd EX	4th EX	5th EX
1/4	6.229E-02					
1/8	1.526E-02	4.263E-04				
1/16	3.795E-03	2.687E-05	3.820E-07			
1/32	9.476E-04	1.679E-06	3.751E-09	4.031E-09		
1/64	2.368E-04	1.052E-07	2.366E-10	2.234E-10	2.401E-10	
1/128	5.920E-05	6.794E-09	2.398E-10	2.399E-10	2.400E-10	2.400E-10

observed order

2.03					
2.01	3.99				
2.00	4.00	6.67			
2.00	4.00	3.99	4.17		
2.00	3.95	-0.02	-0.10	0.00	

The extrapolated ITR yields even better results:

y'=A(t)y, epsilon=1E-10: global error at t=1.0 Table 1.8

h	ITR	1st EX	2nd EX	3rd EX	4th EX	5th EX
1/4	1.666E-02					
1/8	4.186E-03	2.909E-05				
1/16	1.048E-03	1.656E-06	1.728E-07			
1/32	2.620E-04	1.014E-07	2.217E-09	4.909E-10		
1/64	6.550E-05	6.309E-09	3.202E-11	2.655E-12	7.408E-13	
1/128	1.638E-05	3.938E-10	4.897E-13	1.074E-14	3.705E-16	3.533E-16

observed order

1.99					
2.00	4.13				
2.00	4.03	6.28			
2.00	4.01	6.11	7.53		
2.00	4.00	6.03	7.95	10.97	

From our numerical experience (only a small selection of our results has been reported here) we may draw the following conclusions:

- Semi-implicit methods work very efficiently for certain problem classes but fail completely when applied to strongly coupled problems. In these latter cases implicit methods must be used. Note that the ITR throughout achieves the highest accuracy.

- An efficient stiff solver should include a suitable monitoring mechanism controlling the necessary updates of the Jacobian. Such a strategy would e.g. be very natural in the case of the SIUMR which embeds the SIMR as a special case.

- Stagnation effects are unavoidable because asymptotic error expansions in the classical sense do not exist for any of the methods considered above (cf. section 2). Whether these stagnations become actually visible in practice depends on the type of problem and its degree of stiffness, on the method used and on the actual tolerance requirements. Clearly, a sound theoretical analysis of the error structure, as indicated in section 2, is an absolutely necessary prerequisite for a robust and efficient error control. Conventional order/stepsize control strategies (based on conventional error expansions) may badly fail.

2 Theoretical Considerations

Now we present a brief overview of our theoretical results from [3] and [4] about the global error structure of the IMR and the ITR.

We consider the class of stiff problems

$$\begin{aligned} y' &= A(t)y + \varphi(t,y) \\ y(0) &= y_0 \end{aligned} \tag{12}$$

where

$$A(t) = T(t)\,\Lambda(t)\,T^{-1}(t), \qquad \Lambda(t) = \begin{pmatrix} c_1(t) & 0 \\ 0 & -\dfrac{c_2(t)}{\varepsilon} \end{pmatrix} \tag{13}$$

with smooth functions $T(t)$, $T^{-1}(t)$, $c_1(t)$ and $c_2(t)$. $\varepsilon > 0$ is a small parameter characterizing the stiffness. It is assumed that $Re(c_2(t)) \geq \kappa > 0$. $\varphi(t,y)$ and its derivatives are assumed to be of moderate size ($O(\varepsilon^0)$); in some special cases (e.g. singulary perturbed problems like (8)) even $O(\varepsilon^{-1})$ can be admitted, as far as compatible with the existence of a smooth solution (denoted by $y(t)$).

The transformation $z(t) := T^{-1}(t)y(t)$ casts (12) into the form

$$\begin{aligned} z'(t) &= \Lambda(t)z(t) + D(t)z(t) + T^{-1}(t)\varphi(t, T(t)z(t)) \\ z(0) &= T^{-1}(0)y_0 \end{aligned} \tag{14}$$

with

$$-T^{-1}(t)T'(t) =: D(t) = \begin{pmatrix} d_{11}(t) & d_{12}(t) \\ d_{21}(t) & d_{22}(t) \end{pmatrix} \tag{15}$$

The off-diagonal elements of $D(t)$ characterize the degree of coupling that remains after transformation. We call the problem *strongly coupled* if $d_{12}(t)$ and $d_{21}(t)$ are $O(1)$. If $d_{12}(t) = O(\varepsilon)$ or $d_{21}(t) = O(\varepsilon)$ then the coupling from the stiff to the non-stiff component is called *weak*, or vice versa. (Semilinear problems (4) are weakly coupled; problems of singular perturbation type show a weak coupling stiff $\to$ non-stiff.)

Our analysis in [3] and [4] is based on (discrete and continuous) singular perturbation techniques. For the IMR, e.g., the following result has been established in [4]:

Assume $\varepsilon \leq Ch^2$ with a moderate constant C. Then the global error of the IMR admits an asymptotic expansion

$$\eta_\nu - y(t_\nu) = h^2 e_2(t_\nu) + h^4 e_4(t_\nu;\chi) + \ldots + R_\nu \tag{16}$$

where the $e_{2i}(t)$ are smooth solutions of the so-called 'variational equations' (cf. [4]*); $e_2(t)$, e.g., is a smooth solution of*

$$e_2'(t) = f_y(t,y(t))\,e_2(t) - \frac{1}{24}y'''(t) + \frac{1}{8}f_y(t,y(t))\,y''(t) \tag{17}$$

The $e_{2i}(t;\chi)$, $i \geq 2$ depend smoothly on the parameter

$$\chi := \frac{\varepsilon}{h^2}. \tag{18}$$

The remainder term R_ν reads

$$R_\nu = T(t_\nu)\begin{pmatrix} h^4\alpha_4(t_\nu;\chi)(-1)^\nu + \dots \\ h^2\beta_2(t_\nu;\chi)(-1)^\nu + h^4\beta_4(t_\nu;\chi)(-1)^\nu + \dots \end{pmatrix} \tag{19}$$

where $\alpha_4(t;\chi)$, $\beta_2(t;\chi)$ and $\beta_4(t;\chi)$ are smooth functions in t depending smoothly on χ. $\beta_2(t;\chi)$ is a solution of

$$\beta_2'(t;\chi) = \Big(d_{22}(t) - \frac{4\,\chi}{c_2(t)}\Big)\beta_2(t;\chi); \tag{20}$$

the equations defining $\alpha_4(t;\chi)$ and $\beta_4(t;\chi)$ are given in [4].

For the ITR a similar result is valid; but the oscillating error terms (cf.(19)) occur only beginning at the $O(h^4)$-level. If $\varepsilon \leq Ch^2$ is not satisfied (mildly stiff case) then the amplitude of the oscillating remainder term R_ν is strongly decaying; for very small h the oscillations vanish (cf. [4] for details).

Note that our results can be applied inductively (induction w.r.t. several integration intervals with different stepsizes h) and are therefore valid for nonequidistant grids. Also the effects of a transient phase can be understood (cf. [1]–[4] for details).

These results show that in the stiff case the global error does not admit an asymptotic expansion in the conventional sense. The error structure is of a more delicate type:

- The expansions are not valid uniformly for $h \to 0$ but different representations of the global error apply for different subdomains of the 'ε-h - plane'.
- Oscillating and/or decaying error terms occur in a dominant way. The actual behavior of the amplitude of these terms depends on the degree of stiffness, i.e., on the ratio of ε and h.
- Even the smooth components of the global error are not independent of h; for $\varepsilon \leq Ch^2$, e.g., the $e_{2i}(t)$ $(i \geq 2)$ depend smoothly on $\chi = \varepsilon/h^2$.

The semi-implicit (updated) two-step midpoint rule (SIMR resp. SIUMR) can presumably be analyzed by similar techniques as in [3] and [4] for the IMR and ITR.

Extrapolation based on the IMR or ITR.

Recall that in classical, non-stiff situations there exists a 'pure' asymptotic error expansion in powers of h^2; hence for fixed $t = t_\nu$ the global error is essentially a polynomial in h^2. This provides the immediate theoretical justification for h^2-extrapolation; it is easy to show that extrapolation achieves the error level $O(h_0^2\,h_1^2\cdots h_n^2)$ (here the h_i denote the stepsizes used).

In stiff situations, the global error cannot be represented by a polynomial in h^2 but depends on h^2 in a more complicated way (for the case $\varepsilon \leq Ch^2$ cf. (16)–(20) above). Our theoretical results, which describe this dependence in an explicit way, can be used to predict the efficiency of extrapolation in the stiff case. In particular, the occurrence of stagnation phenomena can be explained in a satisfactory way. The reader interested in the technical details is referred to [6].

For the case that ε is small compared to h, extrapolation can thus be shown to behave in the following way (we quote from [6]):

If the finest stepsize h_n satisfies $\varepsilon \leq Ch_n^{2n+2}$ with some moderate constant C, then all steps of h^2-extrapolation based on the IMR deliver the full conventional order. The order of extrapolation begins to reduce as soon as the relation $\varepsilon \leq Ch_i^{2i+2}$ is violated. The accuracy of extrapolation begins to stagnate and the achievable level of accuracy is limited by $O(\varepsilon)$.

For a numerical illustration cf. Tables 1.3 and 1.7.

As mentioned above, the error structure of the ITR is 'more perfect' than for the IMR (the 'irregular', oscillating error components have an additional factor h^2). As a consequence, one extrapolation step will show the full order $O(h_1^4)$ under the milder assumption $\varepsilon \leq Ch_1^2$; furthermore, the 'level of stagnation' is $O(\varepsilon h^2)$.

If one applies *smoothing* at the end of the extrapolation interval then the efficiency of extrapolation based on the IMR further improves (because smoothing damps the influence of the irregular error components). For the ITR, smoothing will usually not be necessary.

Mildly stiff case. Local vs. global extrapolation.

If the stepsizes h_i used for extrapolation do not satisfy $\varepsilon \leq Ch_i^2$, the amplitude of the oscillating error components is not smooth but becomes exponentially decaying (cf. Theorem 4.1 of [4]). Therefore a 'pure' asymptotic expansion reappears by and by as the integration proceeds. The efficiency of extrapolation is poor immediately after the start (or after a change of stepsize), but global extrapolation works successfully sufficiently far away from the starting point. Local extrapolation (where the integration is 're-started' after each extrapolation interval) does not share this nice property. For an experimental illustration, cf. [6].

The error structure in the case of weak coupling.

For a number of special cases it can be shown that the oscillating error components (cf. (19)) have an *additional factor* ε (cf. [3]). This is the case

- for the IMR if $d_{21}(t) = O(\varepsilon)$ (weak coupling non-stiff $\rightarrow$ stiff) and if $\varphi(t, y)$ and its derivatives are $O(\varepsilon^0)$
- for the ITR
  - if $d_{12}(t) = O(\varepsilon)$ (weak coupling stiff $\rightarrow$ non-stiff)
  - if $d_{21}(t) = O(\varepsilon)$ (weak coupling non-stiff $\rightarrow$ stiff) and if $\varphi(t, y)$ and its derivatives are $O(\varepsilon^0)$

(cf. (15) for the meaning of $D(t) = (d_{ij}(t))$). In particular, these results apply to semilinear problems (4) ($D(t) \equiv 0$) and problems of singularly perturbed type ($d_{12}(t) = O(\varepsilon)$). As a consequence, the efficiency of extrapolation is particularly good in these cases; the 'level of stagnation' is about $O(\varepsilon^2)$ (IMR without smoothing) or $O(\varepsilon^2 h^2)$ (IMR with smoothing or ITR).

Defect correction.

Another acceleration technique which can be successfully used for the efficient integration of stiff problems is iterated defect correction (IDeC). An analysis of IDeC based on the IMR or ITR can be done on the basis of our theoretical results about error structures. The efficiency of IDeC depends crucially on the smoothness of the global error with respect to t. (16)–(20) show that for $\varepsilon \leq Ch^2$ the global error of the IMR can indeed be described by a smooth function in t (similarly for the ITR). It can thus be shown that, for $\varepsilon \leq Ch^2$, k IDeC steps yield the optimal order $O(h^{2k+2})$; the technicals details of our analysis are given in [6].

If $\varepsilon \leq Ch^2$ is not satisfied the global error is not smooth in t but contains rapidly decaying components. As a consequence, the order of IDeC breaks down at the first grid points but reappears at the later grid points. Therefore a global connection strategy is usually preferable (similarly as for extrapolation). A further discussion as well as numerical examples can be found in [6].

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