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A Quantitative Discrete H^2-Regularity Estimate for the Shortley-Weller Scheme in Convex Domains

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Summary. We investigate the discrete H^2-regularity properties of the Shortley-Weller discretization of Poisson's equation (with homogeneous Dirichlet boundary condition) in bounded convex domains $\Omega \subset \mathbb{R}^2$. It is shown that the regularity constant is 1 independent of the mesh size h if the H^2-seminorm is defined in a way assigning less weight to the (unsymmetric) differences near the boundary.

Subject Classifications: AMS(MOS): 65N99; CR: G1.8.

1. Introduction

The purpose of the present paper is to study the discrete H^2-regularity properties of the so-called "Shortley-Weller"-discretization in bounded convex domains $\Omega \subset \mathbb{R}^2$. The Shortley-Weller scheme is the usual finite difference discretization for elliptic boundary value problems of second order in nonrectangular domains. It is based on "regular" second order difference approximations in the interior whereas near the boundary, where symmetric schemes cannot be applied in general, unsymmetric first order differencing is used. This method has well-known stability properties; the fact that the resulting discrete problem is unsymmetric is of minor relevance. It is also well-known that the convergence order is 2 despite the reduced order of consistency near the boundary. This simple discretization is easy to use, generally applicable and often used in practice.

Stability (i.e., boundedness of the inverse operator) of such a scheme is of course very important but, on the other hand, a fairly weak property: For an elliptic boundary value problem of second order a natural requirement is that not only the solution itself but also the (first and) second derivatives of the solution should be bounded by the right hand side of the differential equation (at least for homogeneous boundary data). This is called "H^2-regularity"; whether a problem is regular in this sense depends on the shape of the domain Ω (and, of course, on the smoothness of the data functions). Convexity of Ω is, for instance, sufficient (but not necessary if the boundary is smooth enough). It is now very natural to require that the discrete problem should have the analogous property: The second difference quotients of the discrete solution should be bounded by the right hand side of the discrete problem – uniformly for $h \to 0$ (where h denotes the mesh size). This requirement goes far beyond stability.

The question of regularity is not only natural but of utmost importance within the analysis of discretization methods. In particular, regularity properties play an important role within the convergence theory for advanced numerical solution techniques like

i) multigrid methods or
ii) defect correction methods.

Within the theory of multigrid methods, regularity estimates like (1.4) below are used to derive the so-called "approximation property" which enables a sharp estimate for the error of a coarse-grid approximation (for a problem on a finer grid) in terms of (fine-grid) residuals. Many convergence results are based on such an estimate (cf. for instance Hackbusch [3]). In a similar way, estimates for the contraction rate of certain defect correction methods can be obtained on the basis of regularity properties. Defect correction is an iterative way to obtain a high-order solution by inverting a low-order scheme and applying correction steps based on high-order residuals ("defects"). Regularity inequalities enable us to bound the difference between the low order ("basic") and the high-order ("target") scheme in terms of a residual norm; this leads to estimates for the norm of the amplification operator of such a method (cf. for instance Auzinger [1]).

For simple model problems (as for instance Poisson's equation in a rectangular domain), H^2-regularity estimates can be obtained in an elementary way by Fourier techniques. For nonrectangular domains the analysis is much more complicated; such an analysis has been done by several authors. Dryja [2] proved discrete H^2-regularity for systems of elliptic difference equations in a convex polyhedron (see also [1] for a special case). In [4], Hackbusch presented a general criterion together with a number of applications. The latter paper contains a result for the Shortley-Weller scheme, but no explicit (quantitative) bounds are given.

In the present paper we give a quantitative regularity analysis for the Shortley-Weller discretization, applied to Poisson's equation with homogeneous Dirichlet boundary data in an arbitrary convex bounded domain $\Omega \subset \mathbb{R}^2$.

Let us at first describe the scheme and introduce some terminology. On a sequence of uniform grids $\mathbb{G}_h \subset \mathbb{R}^2$ (with mesh size h), the Laplace operator is approximated by the usual five-point formula except near the boundary $\partial\Omega$ where the second derivatives are replaced by unsymmetric difference quotients. We shall use the following denotations: Let Ω_h denote the set of grid points $P=(x, y)$ contained in the interior of Ω. $\partial\Omega_h$ is the intersection of $\partial\Omega$ with the grid lines. Ω_h splits up into $\Omega_h = \Omega_h^0 + \Omega_h'$ where Ω_h' is the set of grid points P near the boundary, that is, $P \pm h e_x \notin \Omega$ or $P \pm h e_y \notin \Omega$ (e_x, e_y denote the unit vectors in x- and y-direction, resp.); see Fig. 1.

Consider a grid function u: $\Omega_h \to \mathbb{R}$ assuming zero boundary values and extend u by zero to the infinite grid $\mathbb{G}_h$. Forward and backward differences are denoted by ∂_x, ∂_y and $\bar{\partial}_x$, $\bar{\partial}_y$, resp.:[1]

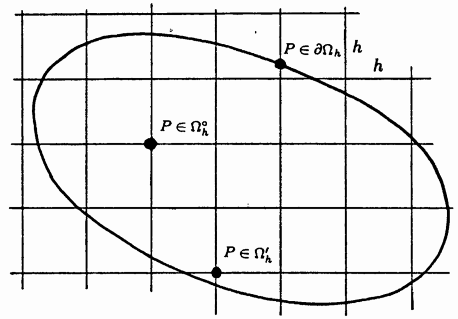

Fig. 1. Discretization of Ω

$$\partial_x u(P) := u(P+h e_x) - u(P), \qquad \bar{\partial}_x u(P) := u(P) - u(P - h e_x),$$
$$\partial_y u(P) := u(P+h e_y) - u(P), \qquad \bar{\partial}_y u(P) := u(P) - u(P - h e_y). \tag{1.1}$$

Let L denote the Shortley-Weller operator in its (undivided) difference form:

$$Lu = L_x u + L_y u, \tag{1.2a}$$

where $L_x u(P) := \bar{\partial}_x \partial_x u(P)$ and $L_y u(P) := \bar{\partial}_y \partial_y u(P)$ for $P \in \Omega_h^0$. For $P \in \Omega_h'$, $\bar{\partial}_x \partial_x$ and/or $\bar{\partial}_y \partial_y$ are replaced by the respective unsymmetric differences, for instance

$$L_x u(P) := \frac{2}{1+\lambda} u(P+h e_x) - \frac{2}{\lambda} u(P) \tag{1.2b}$$

for $P \in \Omega_h'$ and $P - \lambda h e_x \in \partial\Omega_h$ $(0 < \lambda \leqq 1)$.

The Euclidean product of grid functions u, v, taken over some set $\Theta_h \subset \mathbb{G}_h$ of grid points, is denoted by

$$\langle u, v \rangle_{\Theta_h} := \sum_{P \in \Theta_h} u(P) \cdot v(P). \tag{1.3a}$$

The Euclidean (L_2-) norm is

$$\|u\|_{L_2(\Theta_h)} := \sqrt{\langle u, u \rangle_{\Theta_h}}. \tag{1.3b}$$

Our aim is to derive a *discrete H^2-regularity estimate*

$$|u|_{H^2(\Omega_h)} \leqq C \, \|Lu\|_{L_2(\Omega_h)} \tag{1.4}$$

[1] For convenience of notation we use a formulation in terms of (undivided) differences instead of difference quotients. Furthermore, we often drop the mesh index h, writing u, L, ... instead of u_h, L_h, ...

(with a constant C independent of h), where $|\cdot|_{H^2(\Omega_h)}$ is a discrete H^2-seminorm (involving second order differences) which has to be defined in a suitable way. Our particular definition of $|\cdot|_{H^2(\Omega_h)}$ will become clear in the subsequent Sections.

The estimate (1.4) is, in principle, covered by the results of Hackbusch [4]. However, the constant C is not explicitly determined in [4]; moreover, the definition of $|\cdot|_{H^2(\Omega_h)}$ in [4] does not include unsymmetric differences of the type (1.2b) near the boundary (due to technical reasons). Our results, derived in Sects. 2 and 3 below, show that only a slight "weighting" of these unsymmetric differences is necessary within the definition of $|\cdot|_{H^2(\Omega_h)}$ to ensure that the regularity constant is $C=1$ (as in the continuous case – cf. for instance Ladyzhenskaya [5]).

The paper is concluded by some remarks on general domains with a smooth (C^2-) boundary.

2. Representation of $\|Lu\|_{L_2(\Omega_h^0)}-|u|_{H^2(\Omega_h^0)}$ by a Sum along the Boundary

We shall now transform the "interior" norm $\|Lu\|_{L_2(\Omega_h^0)}$, which does not involve unsymmetric differences, into a term corresponding to an "interior" H^2-seminorm plus a "boundary term" (which will be investigated in the sequel). This is done by means of a double summation by parts (recall that $\mathbb{G}_h$ denotes the infinite, uniform grid):

$$\begin{aligned}\|Lu\|^2_{L_2(\Omega_h^0)}&=\langle\bar\partial_x\partial_x u+\bar\partial_y\partial_y u,\ \bar\partial_x\partial_x u+\bar\partial_y\partial_y u\rangle_{\Omega_h^0}\\&=\|\bar\partial_x\partial_x u\|^2_{L_2(\Omega_h^0)}+\|\bar\partial_y\partial_y u\|^2_{L_2(\Omega_h^0)}+2\langle\bar\partial_x\partial_x u,\ \bar\partial_y\partial_y u\rangle_{\Omega_h^0}\end{aligned}$$

with

$$\begin{aligned}2\langle\bar\partial_x\partial_x u,\ \bar\partial_y\partial_y u\rangle_{\Omega_h^0}&=2\langle\bar\partial_x\partial_x u,\ \bar\partial_y\partial_y u\rangle_{\mathbb{G}_h}-2\langle\bar\partial_x\partial_x u,\ \bar\partial_y\partial_y u\rangle_{\mathbb{G}_h-\Omega_h^0}\\&=2\langle\partial_x\partial_y u,\ \partial_x\partial_y u\rangle_{\mathbb{G}_h}-2\langle\bar\partial_x\partial_x u,\ \bar\partial_y\partial_y u\rangle_{\mathbb{G}_h-\Omega_h^0}\\&=2\langle\partial_x\partial_y u,\ \partial_x\partial_y u\rangle_{\Omega_h^+}-B,\end{aligned}$$

where B is a sum taken along the boundary:

$$B:=2\langle\bar\partial_x\partial_x u,\ \bar\partial_y\partial_y u\rangle_{\mathbb{G}_h-\Omega_h^0}-2\langle\partial_x\partial_y u,\ \partial_x\partial_y u\rangle_{\mathbb{G}_h-\Omega_h^+}. \qquad (2.1)$$

Here, Ω_h^+ denotes the set of grid points $P\in\Omega_h$ for which the mixed difference $\partial_x\partial_y u(P)$ is well-defined in terms of interior values, i.e., P, $P+he_x$, $P+he_y$ and $P+h(e_x+e_y)$ are contained in Ω_h. Defining the "interior" H^2-seminorm

$$|u|^2_{H^2(\Omega_h^0)}:=\|\bar\partial_x\partial_x u\|^2_{L_2(\Omega_h^0)}+2\,\|\partial_x\partial_y u\|^2_{L_2(\Omega_h^+)}+\|\bar\partial_y\partial_y u\|^2_{L_2(\Omega_h^0)}, \qquad (2.2)$$

we have

$$|u|^2_{H^2(\Omega_h^0)}=\|Lu\|^2_{L_2(\Omega_h^0)}+B. \qquad (2.3)$$

The boundary term B remains to be studied. In the rest of the present Section we derive a more explicit representation of B in the form

$$B=\sum_{P\in\Omega_h'}[\beta(P)+\gamma(P)], \qquad (2.4)$$

where we define $\beta(P)$ by

$$\beta(P):=2\,\bar\partial_x\partial_x u(P)\,\bar\partial_y\partial_y u(P),\qquad P\in\Omega_h', \qquad (2.5)$$

hence $\sum_{P\in\Omega_h'}\beta(P)=2\langle\bar\partial_x\partial_x u,\ \bar\partial_y\partial_y u\rangle_{\Omega_h'}$. Note that $\Omega_h'\subset\mathbb{G}_h-\Omega_h^0$. With

$$\Omega_h'':=(\mathbb{G}_h-\Omega_h^0)-\Omega_h', \qquad (2.6)$$

$\gamma(P)$ must satisfy the relation

$$\sum_{P\in\Omega_h'}\gamma(P)=2\langle\bar\partial_x\partial_x u,\ \bar\partial_y\partial_y u\rangle_{\Omega_h''}-2\langle\partial_x\partial_y u,\ \partial_x\partial_y u\rangle_{\mathbb{G}_h-\Omega_h^+}, \qquad (2.7)$$

but the definition of $\gamma(P)$ for a particular $P\in\Omega_h'$ is not a priori obvious. We shall now, for each $P\in\Omega_h'$,

i) split $\beta(P)$ into two terms, one of them involving $L_x u(P)\,L_y u(P)$,

ii) define $\gamma(P)$ by assigning a "local portion" of the sum on the right hand side of (2.7) to $P\in\Omega_h'$ in a natural way,

iii) collect and rearrange terms in $\beta(P)+\gamma(P)$.

To this end we distinguish three cases, according to the number of neighbours of $P\in\Omega_h'$ which lie outside Ω. These cases will be referred to as "case (1), (2) or (3)", and P will be called to be of "type (1), (2) or (3)", respectively. The following abbreviations will be used: For a grid point P under consideration, u simply denotes $u(P)$. Similarly, $\bar\partial_x\partial_x u:=\bar\partial_x\partial_x u(P)$, $L_x u:=L_x u(P)$, etc. The neighbours of P are denoted as $P_W:=P-he_x$, $P_E:=P+he_x$, $P_S:=P-he_y$, $P_N:=P+he_y$ (that means, P_W is the neighbour in "western" direction, etc.); furthermore, $u_W:=u(P_W)$, and so on. "Case (4)", i.e., $P\pm he_x\notin\Omega$ and $P\pm he_y\notin\Omega$, is not considered.

Case 1. As an example we consider $P\in\Omega_h'$ such that $P_W\notin\Omega$ (Fig. 2).

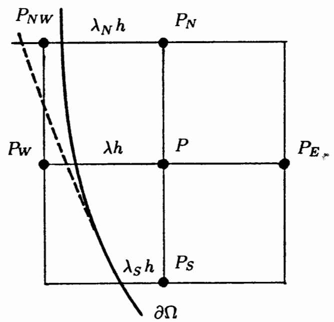

Fig. 2. $P\in\Omega_h'$ (Case 1)

λh, $\lambda_S h$, $\lambda_N h$ are the distances between P, P_S, P_N and $\partial\Omega$ (in western direction). The convexity of Ω implies

$$\lambda_S-2\lambda+\lambda_N\leqq 0; \qquad (2.8)$$

it is possible that $1<\lambda_N \leqq 2$ if $\lambda_S \leqq 1$ (dotted boundary curve in Fig. 2) and vice versa. If $\lambda_N>1$, the "northwestern" neighbour $P_{NW}=P+h(-e_x+e_y)$ of P is contained in Ω_h' and is of the type (2) or (3).

By definition of L_x (cf. (1.2b)) we have

$$\bar{\partial}_x \partial_x u = \frac{1+\lambda}{2} L_x u + \left(\frac{1}{\lambda}-1\right) u. \tag{2.9}$$

Due to (2.9) and $\bar{\partial}_y \partial_y u = L_y u$, $\beta(P)$ splits into

$$\begin{aligned}\beta(P) &= 2\bar{\partial}_x \partial_x u\, \bar{\partial}_y \partial_y u = 2\left[\frac{1+\lambda}{2} L_x u + \left(\frac{1}{\lambda}-1\right) u\right] \bar{\partial}_y \partial_y u \\ &= \frac{1+\lambda}{2}\, 2 L_x u\, L_y u + 2\left(\frac{1}{\lambda}-1\right) u\, \bar{\partial}_y \partial_y u \\ &= \frac{1+\lambda}{2} [(Lu)^2-(L_x u)^2-(L_y u)^2] + 2\frac{1}{\lambda} u\, \bar{\partial}_y \partial_y u - 2u\, \bar{\partial}_y \partial_y u \end{aligned} \tag{2.10a}$$

with

$$\begin{aligned} 2\frac{1}{\lambda} u\, \bar{\partial}_y \partial_y u - 2u\, \bar{\partial}_y \partial_y u &= \frac{1}{\lambda}(2 u u_S - 2u^2) + \frac{1}{\lambda}(2 u u_N - 2u^2) \\ &\quad + (2u^2 - 2 u u_S) + (2u^2 - 2 u u_N). \end{aligned} \tag{2.10b}$$

Those terms in (2.10b) which depend on λ are rewritten as

$$\frac{1}{\lambda}(2 u u_S - 2u^2) = \frac{1}{\lambda}\left(\frac{\lambda_S}{\lambda}-1\right) u^2 + \frac{1}{\lambda}\left[2 u u_S - \left(1+\frac{\lambda_S}{\lambda}\right) u^2\right], \tag{2.10c}$$

and analogously for $1/\lambda\,(2 u u_N - 2u^2)$, yielding

$$\begin{aligned} 2\frac{1}{\lambda} u\, \bar{\partial}_y \partial_y u &= \frac{1}{\lambda}\left(\frac{\lambda_S}{\lambda} - 2 + \frac{\lambda_N}{\lambda}\right) u^2 \\ &\quad + \frac{1}{\lambda}\left[2 u u_S - \left(1+\frac{\lambda_S}{\lambda}\right) u^2\right] + \frac{1}{\lambda}\left[2 u u_N - \left(1+\frac{\lambda_N}{\lambda}\right) u^2\right] \\ &\leqq \frac{1}{\lambda}\left[2 u u_S - \left(1+\frac{\lambda_S}{\lambda}\right) u^2\right] + \frac{1}{\lambda}\left[2 u u_N - \left(1+\frac{\lambda_N}{\lambda}\right) u^2\right] \end{aligned} \tag{2.10d}$$

due to convexity (cf. (2.8)).

Now let $\gamma(P)$ be defined as follows: Consider those terms within (2.7) which involve $u=u(P)$. By definition of Ω_h^+ and Ω_h'' (cf. (2.1), (2.6)) these terms are

$$\begin{aligned} &2\bar{\partial}_x \partial_x u_W\, \bar{\partial}_y \partial_y u_W \quad (=0 \text{ if } \lambda_N \leqq 1), \\ &-2(\bar{\partial}_x \bar{\partial}_y u)^2, \quad -2(\bar{\partial}_x \partial_y u)^2 \end{aligned} \tag{2.11a}$$

(cf. Fig. 2). It appears to be natural to "assign" each non-zero term to P with a factor 1/2 (which means that, for instance, $-2(\bar{\partial}_x \bar{\partial}_y u)^2$ is "equidistributed" between P and P_S)[2]. Thus we may write

$$\gamma(P) := \bar{\partial}_x \partial_x u_W\, \bar{\partial}_y \partial_y u_W - (\bar{\partial}_x \bar{\partial}_y u)^2 - (\bar{\partial}_x \partial_y u)^2. \tag{2.11b}$$

Now we distinguish two cases: If $\lambda_N \leqq 1$ we have $P_{NW} \notin \Omega$, hence $\bar{\partial}_x \partial_x u_w\, \bar{\partial}_y \partial_y u_W = 0$ and

$$\gamma(P) = -(u-u_S)^2 - (u-u_N)^2. \tag{2.12a}$$

Otherwise,

$$\gamma(P) = u\, u_{NW} - (u-u_S)^2 - (u-u_N+u_{NW})^2. \tag{2.12b}$$

Combining (2.10) with (2.12a, b) and rearranging terms we finally obtain

$$\beta(P)+\gamma(P) = \frac{1+\lambda}{2} [(Lu)^2-(L_x u)^2-(L_y u)^2] + \delta_S(P) + \delta_N(P) \tag{2.13}$$

with a "southern" term $\delta_S(P)$ satisfying

$$\delta_S(P) \leqq \frac{1}{\lambda}\left[2 u u_S - \left(1+\frac{\lambda_S}{\lambda}\right) u^2\right] + (u^2 - u_S^2) \tag{2.14a}$$

and a "northern" term $\delta_N(P)$ similar to (2.14a) if $\lambda_N \leqq 1$; otherwise,

$$\begin{aligned} \delta_N(P) &\leqq \frac{1}{\lambda}\left[2 u u_N - \left(1+\frac{\lambda_N}{\lambda}\right) u^2\right] \\ &\quad + (u^2 - u_N^2) + u_{NW}(-u + 2u_N - u_{NW}). \end{aligned} \tag{2.14b}$$

Case 2. As an example we consider $P \in \Omega_h'$ such that $P_W \notin \Omega$ and $P_S \notin \Omega$ (Fig. 3)[3].

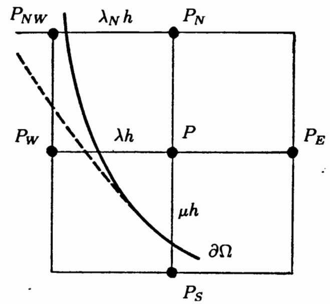

Fig. 3. $P \in \Omega_h'$ (Case 2)

[2] For P_S (or P_N) of type (1) this is obviously a reasonable choice because it is "symmetric". For cases (2) and (3) we shall define $\gamma(P)$ in such a way that all cases are compatible among each other (and, of course, such that (2.7) is satisfied – cf. (2.18a) and (2.25a) below). This will enable a "purely local" estimation of $\beta(P)+\gamma(P)$ in Sect. 3

[3] Situations like P_W, $P_E \in \Omega$ but P_S, $P_N \notin \Omega$ are not considered

λh, $\lambda_N h$ are the distances between P, P_N and $\partial\Omega$ in western, μh the distance between P and $\partial\Omega$ in southern direction. The convexity of Ω implies

$$\frac{\lambda_N}{\lambda} \leqq 1 + \frac{1}{\mu}; \tag{2.15}$$

it is possible that $\lambda_N > 1$ (dotted boundary curve in Fig. 3), and λ_N may be "arbitrarily" large (if $\mu \approx 0$). If $1 < \lambda_N \leqq 2$, P_{NW} is of the type (2) or (3); for $\lambda_N > 2$, P_{NW} is of the type (1). W.r.t. the eastern direction the situation is analogous (as far as compatible with convexity).

By definition of L (cf. (1.2b)) we have

$$\bar\partial_x \partial_x u = \frac{1+\lambda}{2} L_x u + \left(\frac{1}{\lambda} - 1\right) u, \qquad \bar\partial_y \partial_y u = \frac{1+\mu}{2} L_y u + \left(\frac{1}{\mu} - 1\right) u. \tag{2.16}$$

Due to (2.16) we obtain after some simple manipulations

$$\begin{aligned} \beta(P) = 2\bar\partial_x \partial_x u\, \bar\partial_y \partial_y u = {} & \frac{1+\lambda}{2}\,\frac{1+\mu}{2}\,[(Lu)^2 - (L_x u)^2 - (L_y u)^2] \\ & + \frac{1}{\lambda}\left[2 u u_N - \left(2 + \frac{1}{\mu}\right) u^2\right] + 3u^2 - 2 u u_N \\ & + \frac{1}{\mu}\left[2 u u_E - \left(2 + \frac{1}{\lambda}\right) u^2\right] + 3u^2 - 2 u u_E. \end{aligned} \tag{2.17}$$

Now we define $\gamma(P)$ in a similar way as for case (1) (note that $\bar\partial_x \partial_x u_W \bar\partial_y \partial_y u_W = 0$ if $\lambda_N \leqq 1$, similarly for $\bar\partial_x \partial_x u_S \bar\partial_y \partial_y u_S$):

$$\begin{aligned} \gamma(P) := {} & \bar\partial_x \partial_x u_W\, \bar\partial_y \partial_y u_W + \bar\partial_x \partial_x u_S\, \bar\partial_y \partial_y u_S \\ & - (\bar\partial_x \partial_y u)^2 - (\partial_x \bar\partial_y u)^2 - 2(\bar\partial_x \bar\partial_y u)^2 \end{aligned} \tag{2.18a}$$

(note that the term $\bar\partial_x \bar\partial_y u$ does not involve any other point from Ω_h besides P, cf. Fig. 3). $\gamma(P)$ splits up into a "northern" and an "eastern" term:

$$\gamma(P) = \gamma_N(P) + \gamma_E(P) \tag{2.18b}$$

with

$$\begin{aligned} \gamma_N(P) &:= \bar\partial_x \partial_x u_W\, \bar\partial_y \partial_y u_W - (\bar\partial_x \partial_y u)^2 - (\bar\partial_x \bar\partial_y u)^2, \\ \gamma_E(P) &:= \bar\partial_x \partial_x u_S\, \bar\partial_y \partial_y u_S - (\partial_x \bar\partial_y u)^2 - (\bar\partial_x \bar\partial_y u)^2. \end{aligned} \tag{2.18c}$$

Consider for instance $\gamma_N(P)$. We distinguish two cases: For $\lambda_N \leqq 1$ we have

$$\gamma_N(P) = -(u - u_N)^2 - u^2; \tag{2.19a}$$

otherwise,

$$\gamma_N(P) = u u_{NW} - (u - u_N + u_{NW})^2 - u^2. \tag{2.19b}$$

Combining (2.17)–(2.19) and rearranging terms we finally obtain

$$\beta(P) + \gamma(P) = \frac{1+\lambda}{2}\,\frac{1+\mu}{2}\,[(Lu)^2 - (L_x u)^2 - (L_y u)^2] + \delta_N(P) + \delta_E(P) \tag{2.20}$$

with a "northern" term given by

$$\delta_N(P) = \frac{1}{\lambda}\left[2 u u_N - \left(2 + \frac{1}{\mu}\right) u^2\right] + 3u^2 - 2 u u_N + \gamma_N(P), \tag{2.21a}$$

similarly for $\delta_E(P)$. For $\lambda_N \leqq 1$ we obtain (cf. (2.19a))

$$\begin{aligned} \delta_N(P) &= \frac{1}{\lambda}\left[2 u u_N - \left(1 + \frac{\lambda_N}{\lambda}\right) u^2\right] + \frac{1}{\lambda}\left(\frac{\lambda_N}{\lambda} - 1 - \frac{1}{\mu}\right) u^2 \\ &\quad + 3u^2 - 2 u u_N - (u - u_N)^2 - u^2 \\ &\leqq \frac{1}{\lambda}\left[2 u u_N - \left(1 + \frac{\lambda_N}{\lambda}\right) u^2\right] + (u^2 - u_N^2) \end{aligned} \tag{2.21b}$$

due to convexity (cf. (2.15)). For $\lambda_N > 1$ we write (without using convexity for the moment)

$$\begin{aligned} \delta_N(P) = {} & \frac{1}{\lambda}\left[2 u u_N - \left(2 + \frac{1}{\mu}\right) u^2\right] \\ & + (u^2 - u_N^2) + u_{NW}(-u + 2u_N - u_{NW}) \end{aligned} \tag{2.21c}$$

(cf. (2.19b)). An analogous representation holds for $\delta_E(P)$.

Case 3. As an example we consider $P \in \Omega_h'$ such that $P_W \notin \Omega$, $P_S \notin \Omega$, $P_N \notin \Omega$ (Fig. 4).

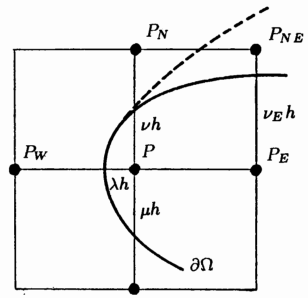

Fig. 4. $P \in \Omega_h'$ (Case 3)

λh, μh, νh are the distances between P and $\partial\Omega$ in western, southern and northern, $\nu_E h$ the distance between P_E and $\partial\Omega$ in northern direction. The convexity of Ω implies

$$\frac{\nu_E}{\nu} \leqq 1 + \frac{1}{\lambda} \tag{2.22}$$

(cf. Fig. 4 and the analogous situation in Fig. 3 above).

By definition of L_x (cf. (1.2b)) and with $L_y u = -\frac{2}{\mu\nu} u$ we have

$$\bar{\partial}_x \partial_x u = \frac{1+\lambda}{2} L_x u + \left(\frac{1}{\lambda} - 1\right) u, \qquad \bar{\partial}_y \partial_y u = \frac{\mu+\nu}{2} L_y u + \left(\frac{1}{\mu} + \frac{1}{\nu} - 2\right) u. \tag{2.23}$$

Due to (2.23) we obtain after some simple manipulations

$$\begin{aligned} \beta(P) = 2\bar{\partial}_x \partial_x u\, \bar{\partial}_y \partial_y u = {} & \frac{1+\lambda}{2}\,\frac{\mu+\nu}{2}\,[(Lu)^2 - (L_x u)^2 - (L_y u)^2] \\ & + \frac{1}{\mu}\left[2 u u_E - \left(2 + \frac{2}{\lambda}\right) u^2\right] + 4u^2 - 2u u_E \\ & + \frac{1}{\nu}\left[2 u u_E - \left(2 + \frac{2}{\lambda}\right) u^2\right] + 4u^2 - 2u u_E. \end{aligned} \tag{2.24}$$

Now we define $\gamma(P)$ in a similar way as for case (2) (note that $\bar{\partial}_x \partial_x u_w \bar{\partial}_y \partial_y u_W = 0$; furthermore, $\bar{\partial}_x \partial_x u_N \bar{\partial}_y \partial_y u_N = 0$ if $\nu_E \leqq 1$, similarly for $\bar{\partial}_x \partial_x u_S \bar{\partial}_y \partial_y u_S$):

$$\begin{aligned} \gamma(P) := {} & \bar{\partial}_x \partial_x u_S \bar{\partial}_y \partial_y u_S + \bar{\partial}_x \partial_x u_N \bar{\partial}_y \partial_y u_N \\ & - 2(\bar{\partial}_x \bar{\partial}_y u)^2 - (\partial_x \bar{\partial}_y u)^2 - 2(\bar{\partial}_x \partial_y u)^2 - (\partial_x \partial_y u)^2 \end{aligned} \tag{2.25a}$$

(note that the terms $\bar{\partial}_x \bar{\partial}_y u$ and $\bar{\partial}_x \partial_y u$ do not involve any other point from Ω_h besides P, cf. Fig. 4). $\gamma(P)$ splits up into a "southeastern" and a "northeastern" term in an obvious way:

$$\gamma(P) = \gamma_{SE}(P) + \gamma_{NE}(P). \tag{2.25b}$$

W.r.t. $\gamma_{NE}(P)$, for instance, we distinguish two cases: For $\nu_E \leqq 1$ we have

$$\gamma_{NE}(P) = -(u - u_E)^2 - 2u^2; \tag{2.26a}$$

otherwise,

$$\gamma_{NE}(P) = u u_{NE} - (u - u_E + u_{NE})^2 - 2u^2. \tag{2.26b}$$

Combining (2.24)–(2.26) and rearranging terms we finally obtain

$$\beta(P) + \gamma(P) = \frac{1+\lambda}{2}\,\frac{\mu+\nu}{2}\,[(Lu)^2 - (L_x u)^2 - (L_y u)^2] + \delta_{SE}(P) + \delta_{NE}(P); \tag{2.27}$$

for $\nu_E \leqq 1$, the northeastern term $\delta_{NE}(P)$ reads

$$\begin{aligned} \delta_{NE}(P) = {} & \frac{1}{\nu}\left[2 u u_E - \left(1 + \frac{\nu_E}{\nu}\right) u^2\right] + \frac{1}{\nu}\left(\frac{\nu_E}{\nu} - 1 - \frac{1}{\lambda}\right) u^2 - \frac{1}{\lambda\nu} u^2 \\ & + 4u^2 - 2u u_E - (u - u_E)^2 - 2u^2 \\ \leqq {} & \frac{1}{\nu}\left[2 u u_E - \left(1 + \frac{\nu_E}{\nu}\right) u^2\right] + (u^2 - u_E^2) \end{aligned} \tag{2.28a}$$

due to convexity (cf. (2.22)) and $-(\lambda\nu)^{-1} u^2 \leqq 0$. For $\nu_E > 1$ we write (without using convexity for the moment)

$$\begin{aligned} \delta_{NE}(P) \leqq {} & \frac{1}{\nu}\left[2 u u_E - \left(2 + \frac{1}{\lambda}\right) u^2\right] \\ & + (u^2 - u_E^2) + u_{NE}(-u + 2u_E - u_{NE}) \end{aligned} \tag{2.28b}$$

(cf. (2.26b)). An analogous representation holds for $\delta_{SE}(P)$.

3. Estimation of the Boundary Terms

We are now in a position to estimate the boundary term $B = \sum_{P \in \Omega_h} [\beta(P) + \gamma(P)]$. In particular, we shall show that the sum over the "δ-terms" (cf. (2.14), (2.21), (2.28)) along the boundary is nonpositive. This can by proved by local considerations; we distinguish three cases.

Case A. Consider a pair of points from Ω_h' which are neighbours in a horizontal or vertical direction. We assume, for example, a situation as shown in Fig. 5 with $0 < \lambda$, $\lambda_N \leqq 1$.

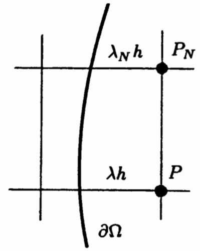

Fig. 5. $P, P_N \in \Omega_h'$ (Case A)

Note that we do not specify the "type" of P or P_N (in the sense of Sect. 2); the argumentation below is valid in any case.

The considerations of Sect. 2 show that there is a northern term $\delta_N(P)$[4] (originating from $\beta(P) + \gamma(P)$) satisfying

$$\delta_N(P) \leqq \frac{1}{\lambda}\left[2 u u_N - \left(1 + \frac{\lambda_N}{\lambda}\right) u^2\right] + (u^2 - u_N^2) \tag{3.1a}$$

(cf. (2.14a), (2.21b) or (2.28a) in their appropriate versions); similarly, there is a southern term $\delta_S(P_N)$ (originating from $\beta(P_N) + \gamma(P_N)$) satisfying

$$\delta_S(P_N) \leqq \frac{1}{\lambda_N}\left[2 u_N u - \left(1 + \frac{\lambda}{\lambda_N}\right) u_N^2\right] + (u_N^2 - u^2). \tag{3.1b}$$

[4] or "$\delta_{NW}(P)$" if P is of the type (3) (cf. (2.28a))

The sum of these terms is bounded by

$$\delta_N(P)+\delta_S(P_N)\leqq\left(\frac{1}{\lambda}+\frac{1}{\lambda_N}\right)2\,u\,u_N-\frac{1}{\lambda}\left(1+\frac{\lambda_N}{\lambda}\right)u^2-\frac{1}{\lambda_N}\left(1+\frac{\lambda}{\lambda_N}\right)u_N^2. \quad (3.2\text{a})$$

Using the inequality $2\,u\,u_N\leqq\varepsilon u^2+\varepsilon^{-1}u_N^2$ with $\varepsilon=\lambda_N/\lambda$ we easily obtain

$$\delta_N(P)+\delta_S(P_N)\leqq 0. \quad (3.2\text{b})$$

Case B. Consider a pair of points from Ω_h' which are neighbours in a diagonal direction and are both of the type (2) or (3). We assume, for example, a situation as shown in Fig. 6 with $0<\mu,\ \lambda_{NW}\leqq 1$.

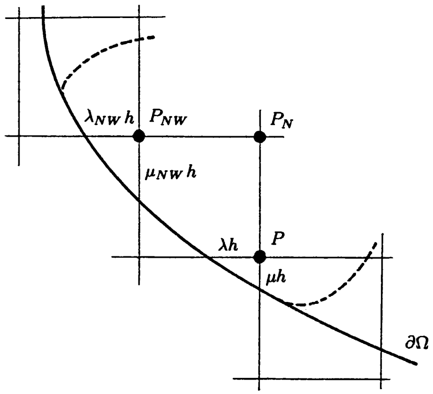

Fig. 6. $P, P_{NW}\in\Omega_h'$ (Case B)

The convexity of Ω implies

$$\frac{\mu}{\lambda}\leqq\frac{1-\mu_{NW}}{1-\lambda},\qquad \frac{\lambda_{NW}}{\mu_{NW}}\leqq\frac{1-\lambda}{1-\mu_{NW}}. \quad (3.3)$$

The considerations of Sect. 2 show that there is a northern term $\delta_N(P)$[5] (originating from $\beta(P)+\gamma(P)$) satisfying

$$\delta_N(P)\leqq\frac{1}{\lambda}\left[2\,u\,u_N-\left(2+\frac{1}{\mu}\right)u^2\right]$$
$$+(u^2-u_N^2)+u_{NW}(-u+2\,u_N-u_{NW}) \quad (3.4\text{a})$$

(cf. (2.21c) or (2.28b) in their appropriate versions); similarly, there is an eastern term $\delta_E(P_{NW})$ (originating from $\beta(P_{NW})+\gamma(P_{NW})$) with

$$\delta_E(P_{NW})\leqq\frac{1}{\mu_{NW}}\left[2\,u_{NW}\,u_N-\left(2+\frac{1}{\lambda_{NW}}\right)u_{NW}^2\right]$$
$$+(u_{NW}^2-u_N^2)+u(-u_{NW}+2\,u_N-u). \quad (3.4\text{b})$$

[5] or "$\delta_{NW}(P)$" if P is of the type (3) (cf. (2.28b))

Let $\sigma:=1+\lambda^{-1}$, $\tau:=1+\mu_{NW}^{-1}$. Using the inequality $2(\sigma u+\tau u_{NW})\,u_N\leqq\varepsilon(\sigma u+\tau u_{NW})^2+\varepsilon^{-1}u_N^2$ with $\varepsilon=1/2$, we bound the sum of (3.4a) and (3.4b) from above by an expression which does not depend on u_N:

$$\delta_N(P)+\delta_E(P_{NW})\leqq\left[\frac{1}{2}\sigma^2-(\sigma-1)\left(2+\frac{1}{\mu}\right)\right]u^2$$
$$+\left[\frac{1}{2}\tau^2-(\tau-1)\left(2+\frac{1}{\lambda_{NW}}\right)\right]u_{NW}^2+2\left(\frac{1}{2}\sigma\tau-1\right)u\,u_{NW}. \quad (3.5)$$

The inequality $2\,u\,u_{NW}\leqq\varepsilon u^2+\varepsilon^{-1}u_{NW}^2$ yields with $\varepsilon=(\sigma-2)/(\tau-2)>0$[6]

$$\delta_N(P)+\delta_E(P_{NW})\leqq\left[\frac{1}{2}\sigma^2-(\sigma-1)\left(2+\frac{1}{\mu}\right)+\frac{\sigma-2}{\tau-2}\left(\frac{1}{2}\sigma\tau-1\right)\right]u^2$$
$$+\left[\frac{1}{2}\tau^2-(\tau-1)\left(2+\frac{1}{\lambda_{NW}}\right)+\frac{\tau-2}{\sigma-2}\left(\frac{1}{2}\sigma\tau-1\right)\right]u_{NW}^2. \quad (3.6)$$

Using the inequality

$$-\frac{1}{\mu}\leqq-(\sigma-2)\frac{\tau-1}{\tau-2}, \quad (3.7)$$

which is equivalent to the first relation in (3.3), we easily conclude that the factor [...] which appears with u^2 in (3.6) is nonpositive. [...] u_{NW}^2 can be estimated analogously; hence

$$\delta_N(P)+\delta_E(P_{NW})\leqq 0. \quad (3.8)$$

Case C. Consider a pair of points from Ω_h' which are neighbours in a diagonal direction, one of them being of the type (2) or (3) and the other one of the type (1). We assume, for example, a situation as shown in Fig. 7 with $0<\mu\leqq 1$,

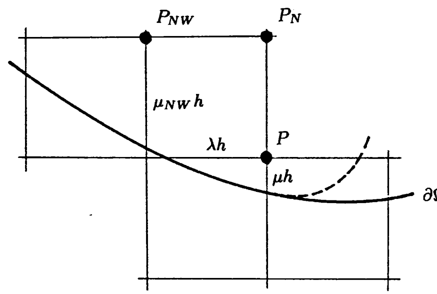

Fig. 7. $P, P_{NW}\in\Omega_h'$ (Case C)

[6] For the special case $\lambda=\mu_{NW}=1$ $(\sigma=\tau=2)$ we choose $\varepsilon=1$ and immediately obtain $\delta_N(P)+\delta_E(P_{NW})\leqq 0$

$\lambda_{NW} > 1$. (Note that P and P_{NW} cannot be both of the type (1) – this would contradict convexity.)

The considerations of Sect. 2 show that there is a northern term $\delta_N(P)$ (originating from $\beta(P)+\gamma(P)$) satisfying (3.4a) above. Since P_{NW} is of the type (1), the corresponding eastern term $\delta_E(P_{NW})$ originating from $\beta(P_{NW})+\gamma(P_{NW})$ satisfies

$$\delta_E(P_{NW}) \leqq \frac{1}{\mu_{NW}}\left[2u_{NW}u_N - \left(1+\frac{1+\mu}{\mu_{NW}}\right)u_{NW}^2\right] + (u_{NW}^2 - u_N^2) + u(-u_{NW}+2u_N-u) \tag{3.9}$$

(cf. (2.14b) in its appropriate version). Let $\sigma := 1+\lambda^{-1}$, $\tau := 1+\mu_{NW}^{-1}$. In a similar way as for case (B) (cf. (3.5)) we bound the sum of (3.4a) and (3.9) from above by

$$\delta_N(P)+\delta_E(P_{NW}) \leqq \left[\frac{1}{2}\sigma^2-(\sigma-1)\left(2+\frac{1}{\mu}\right)\right]u^2 + \left[\frac{1}{2}\tau^2-(\tau-1)\left(1+\frac{1+\mu}{\mu_{NW}}\right)\right]u_{NW}^2 + 2\left(\frac{1}{2}\sigma\tau-1\right)u\,u_{NW}. \tag{3.10}$$

The inequality $2u\,u_{NW} \leqq \varepsilon u^2 + \varepsilon^{-1}u_{NW}^2$ yields with $\varepsilon = \mu_{NW}/\mu = [\mu(\tau-1)]^{-1}$:

$$\delta_N(P)+\delta_E(P_{NW}) \leqq \left[\frac{1}{2}(\sigma-2)^2 - \frac{1}{\mu(\tau-1)}\,\frac{1}{2}(\sigma-2)(\tau-2)\right]u^2 + \left[-\frac{1}{2}\tau(\tau-2)+\mu(\tau-1)\frac{1}{2}(\sigma-2)\tau\right]u_{NW}^2. \tag{3.11}$$

Using the inequality[7]

$$\mu(\tau-1) \leqq \frac{\tau-2}{\sigma-2}, \tag{3.12}$$

which holds due to convexity (cf. (3.3) and (3.7) above), we easily conclude that both factors [...] within (3.11) are nonpositive; hence

$$\delta_N(P)+\delta_E(P_{NW}) \leqq 0. \tag{3.13}$$

4. Conclusion

By our considerations of Sects. 2 and 3 we have covered all possible situations which may occur near the boundary of a convex domain Ω (at least if the grid is not too coarse). Summing up (2.13), (2.20) and (2.27) (in their appropriate versions) over all $P \in \Omega_h'$ and taking account of (3.2b), (3.8) and (3.13), we obtain the following estimate for B (cf. (2.1)):

$$B \leqq \sum_{P\in\Omega_h} (Lu)^2 - \sum_{P\in\Omega_h} \omega(P)[(L_x u)^2 + (L_y u)^2] \tag{4.1a}$$

with the weight function $\omega(P)$ ($P \in \Omega_h'$) defined by

$$\omega(P) := \begin{cases} 1/2 & \text{for } P \text{ of type (1)},\\ 1/4 & \text{for } P \text{ of type (2)},\\ 0 & \text{elsewhere.}\end{cases} \tag{4.1b}$$

(4.1) is valid for arbitrary bounded convex $\Omega \subset \mathbb{R}^2$; in this general formulation we must choose $\omega(P)=0$ for P of type (3) because the factor $(\mu+\nu)/2$ in (2.27) may become arbitrarily small. Thus, the unsymmetric differences $L_x u(P)$, $L_y u(P)$, P of type (3), are not included[8] in our definition of $|\cdot|_{H^2(\Omega_h)}$ (cf. (2.2) for the definition of $|\cdot|_{H^2(\Omega_h^0)}$):

$$|u|_{H^2(\Omega_h)}^2 := |u|_{H^2(\Omega_h^0)}^2 + \|\omega L_x u\|_{L_2(\Omega_h')}^2 + \|\omega L_y u\|_{L_2(\Omega_h')}^2. \tag{4.2}$$

Summarizing, we obtain from (2.3) and (4.1) the desired H^2-regularity estimate

$$|u|_{H^2(\Omega_h)}^2 \leqq \|Lu\|_{L_2(\Omega_h^0)}^2 + \|Lu\|_{L_2(\Omega_h')}^2 = \|Lu\|_{L_2(\Omega_h)}^2 \tag{4.3}$$

for $|\cdot|_{H^2(\Omega_h)}$ as defined in (4.2).

For general (nonconvex) domains with a C^2-boundary, a direct quantitative regularity estimate (strengthening the results of Hackbusch [4]) appears feasible but would be much more involved. For a point $P \in \Omega_h'$ of the type (1), for example (cf. Fig. 2), one has to use – instead of (2.8) – the inequality

$$|\lambda_S - 2\lambda + \lambda_N| \leqq Ch \tag{4.4}$$

(which is valid for $\partial\Omega$ of class C^2) in order to estimate the term $(\lambda_S - 2\lambda + \lambda_N)u^2/\lambda^2$ which appears in (2.10d); further, u/λ has to be expressed by a first difference of u (without the factor $1/\lambda$) on the basis of the definition of the Shortley-Weller operator. To obtain the desired estimate one will then proceed in a similar way as in the continuous case (cf. Ladyzhenskaya [5]).

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[7] For the special case $\lambda = \mu_{NW} = 1$ ($\sigma = \tau = 2$) the desired estimate follows directly from (3.11)

[8] This is of course a mild restriction