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DCMG01: A Multigrid Code with Defect Correction to Solve $\Delta U - c(x,y)\,U = f(x,y)$ (on Ω) $U = g(x,y)$ (on $\partial\Omega$) on Nonrectangular Bounded Domains Ω with High Accuracy*) **)

*) DCMG01 is an extension of the Multigrid program MG01 which is part of the GMD Multigrid library.

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Preface

The code DCMG01 presented here is an extension of the GMD-code MG01. MG01 is a multigrid-program for the fast solution of Helmholtz' equation with variable coefficient in (quite) general 2D-domains with Dirichlet boundary conditions.

MG01 uses the usual 5-point difference star (consistency of order 2). On the basis of Hans Stetter's defect correction (DC) approach, Winfried Auzinger has developed DCMG01 by combining MG01 and the DC-method. DCMG01 achieves an accuracy of order 4.

Hans Stetter and Winfried Auzinger were so kind as to make DCMG01 available to the GMD. We are very grateful for this extension of the GMD-collection of MG codes.

We believe that this a very good example of the fruitful effects of the informal exchange of scientific ideas aiming at a coordinated development of numerical software.

Ulrich Trottenberg,
Gesellschaft für Mathematik und Datenverarbeitung mbH, Sankt Augustin

CONTENTS

1. INTRODUCTION

DCMG01 is an extension of the Multigrid program MG01 (cf. [4], Chapter 10; for more details cf. [3]) to solve the Helmholtz equation with Dirichlet boundary conditions,

$$\begin{aligned} \Delta U - c(x,y)\, U &= f(x,y) \quad (\text{on } \Omega) \\ U &= g(x,y) \quad (\text{on } \partial\Omega) \end{aligned} \tag{1.1}$$

by difference methods in general bounded domains $\Omega \subset R^2$ with high accuracy. The extension is based on a *defect correction principle* (cf. [1], [2]). DCMG01 has been developed and tested in FORTRAN 77 (single precision) on a CDC CYBER 170 computer (48 bit mantissa).

This report describes mainly the properties of the extension and how to use it, but gives only little information about the basic solver. Before using DCMG01, one should be familiar with MG01. For a detailed description of MG01 we refer to [3]. All what is said about MG01 in [3] remains valid for DCMG01 unless otherwise stated.

Sections 2, 4, 5 and 6 show when and how to use DCMG01; Section 3 describes the method. Section 7 gives some examples. Finally, Section 8 contains some details about the program structure and the internal organization.

2. PROBLEMS WHICH CAN BE SOLVED BY DCMG01

Due to the way the grids are organized within MG01, it can only be used if the domain Ω is *convex w.r.t. a certain direction* (which has to be assigned to the y-coordinate direction, cf. [3]). Furthermore, the defect correction method which is used by DCMG01 can only be applied without restrictions if the boundary $\partial\Omega$ is sufficiently smooth (compared to the mesh size of the finest grid). See Sections 3 and 5 for details.

DCMG01 produces a *high order approximation* of the solution of (1.1). Its order of consistency is 4 whereas the basic solver is consistent of order 2. This means that the approximation error decreases like h^4 for decreasing h (where h is the mesh size), *if* the true solution is sufficiently smooth; otherwise, the term "high order" makes no sense. Theoretically, this smoothness is described by the existence and continuity of certain derivatives. For practical purposes, we note the following *restrictions:*

– The problems should be of definite type (i.e. $c(x,y) \geq 0$); the Helmholtz function c(x,y) should not be too large in size (compared to the finest mesh size h), say: c(x,y) not significantly larger than h^{-2}.

– The functions c(x,y), f(x,y), g(x,y) and the boundary curve $\partial\Omega$ should be well-behaved: No strong oscillations, no jumps in size, no discontinuities of the lower derivatives.

If these conditions are fulfilled, DCMG01 yields an approximate solution to (1.1) of high accuracy with comparatively little additional effort. The error will be significantly smaller than the error of the "basic approximation" delivered by MG01.

If there is some unsmoothness in the problem or if there are singularities due to corners of the domain, it is to be expected that high accuracy will not be achieved and the approximation quality will hardly be better (or even somewhat worse) than with MG01. A similar situation arises in the case of a Helmholtz function which is very large in size: Here, the basic solver already achieves high accuracy due to a very small stability constant. Examples are given in Section 7.

In addition, it has to be noted that the shape of the domain has some influence on the contraction properties of the iteration performed within DCMG01. For the "multigrid component" this effect is well-known (cf. [3]); for the "defect correction component" a similar dependence is observed. See Section 7 for numerical evidence.

3. DISCRETIZATION AND ALGORITHM

3.1 Combination of Multigrid with Defect Correction

Let $L_h\, u_h = f_h$ represent the linear system arising from the discretization of (1.1) which is used within MG01 (cf. [3], Section 3). The multigrid iteration reads

$$u_h^{(i+1)} := u_h^{(i)} - C_h(L_h\, u_h^{(i)} - f_h), \tag{3.0}$$

where C_h represents one "multigrid cycle". The basic idea of *defect correction* is to use a residual (or "defect") of higher order in (3.0); i.e.

$$u_h^{(i+1)} := u_h^{(i)} - C_h(L_h'\, u_h^{(i)} - f_h'), \tag{3.1}$$

where $L_h'\, u_h = f_h'$ represents a high order discretization of (1.1). A fixed point u_h' of (3.1) satisfies $L_h'\, u_h' = f_h'$. L_h' and f_h' are described in Section 3.2.

A closer analysis shows that the contraction properties of the iteration (3.1) depend crucially on the smoothness of the "algebraic error" (i.e. the error of $u_h^{(i)}$ w.r.t. the fixed point). Therefore, in order to ensure a similar contraction rate as for (3.0), an extra smoothing step should be performed before calculating the defect (cf. [1], [2]):

$$\begin{aligned} \bar{u}_h^{(i)} &:= u_h^{(i)} - S_h(L_h u_h^{(i)} - f_h), \\ u_h^{(i+1)} &:= \bar{u}_h^{(i)} - C_h(L'_h \bar{u}_h^{(i)} - f'_h). \end{aligned} \tag{3.3}$$

Here, the smoothing step is the same relaxation procedure which is part of the basic multigrid solver.

Note that a fixed point of (3.3) is *not* the precise solution of $L'_h u_h = f'_h$. However, high accuracy is retained; moreover, (3.3) is more robust than (3.1), cf. [1].

(3.0), (3.1), (3.3) correspond to IDC = 0,1,3 within a call of DCMG01 (see Section 4). In most cases, (3.3) can be replaced by the iteration

$$\begin{aligned} \bar{u}_h^{(i)} &:= u_h^{(i)} - S_h(L_h u_h^{(i)} - f'_h); \\ u_h^{(i+1)} &:= \bar{u}_h^{(i)} - C_h(L'_h \bar{u}_h^{(i)} - f'_h), \end{aligned} \tag{3.2}$$

where f'_h is used instead of f_h in the smoothing step. (3.2) corresponds to IDC = 2 and requires less storage than (3.3). In fact, IDC = 2 is the standard choice. See Section 4 for details.

The implementation of the defect correction has been done in the following way. Within MG01, SUBROUTINE CSFIX performs one multigrid cycle (3.0), where $u_h^{(i)}$ and f_h are given input data. To use CSFIX as a blackbox routine, (3.1) is re-written as

$$u_h^{(i+1)} := u_h^{(i)} - C_h(L_h u_h^{(i)} - d_h^{(i)}),$$

where

$$d_h^{(i)} := \Delta L_h u_h^{(i)} + f'_h, \quad \Delta L_h := L_h - L'_h.$$

Thus, f_h is replaced by $d_h^{(i)}$ which has to be calculated in each step.

DCMG01 works in the *"Full Multigrid"* mode which gives a solution with high accuracy and approximation order 4. (For the choice of parameters to ensure this, we refer to Section 4.) In addition, on the finest level the iteration can be continued up to the (numerical) fixed point. This may be of interest for research purposes.

In the Full Multigrid mode, no defect correction is used on the coarsest grid.

3.2 The defect-defining discretization

In [3], Section 3, the "basic discretization" is given by

$$\Delta_h u_h - c(x,y)\ u_h = f(x,y), \quad (x,y) \in \Omega_h,$$
$$u_h = g(x,y), \quad (x,y) \in \partial\Omega_h.$$

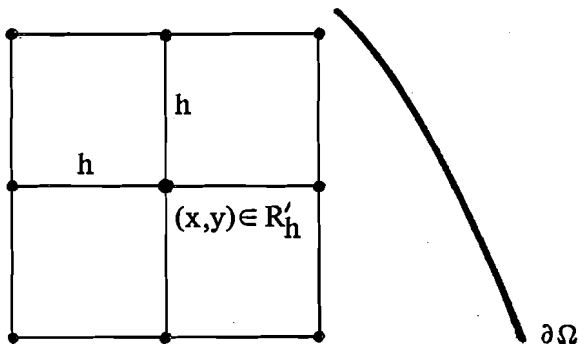
Here Ω_h and $\partial\Omega_h$ are the sets of interior and boundary grid points, respectively. Δ_h is a discrete, stable Laplace operator.

The high-order discretization $L'_h\ u_h = f'_h$ which is used for calculating the defect can be written as

$$\Delta'_h u_h - W_h\, c(x,y)\ u_h = W_h\, f(x,y), \quad (x,y) \in \Omega_h,$$
$$u_h = \quad g(x,y), \quad (x,y) \in \partial\Omega_h.$$

Δ'_h is a discrete Laplace operator of higher order; W_h is a local weighting operator.
Δ'_h and W_h are chosen as follows:

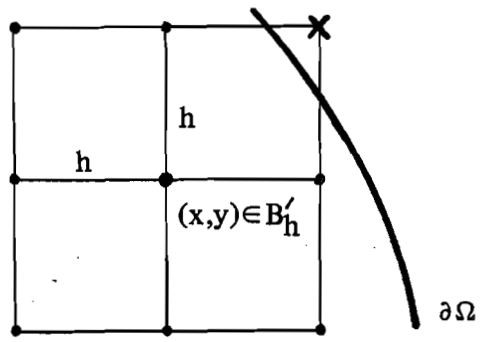
– Let $R_h \subset \Omega_h$ be the set of regular interior grid points (cf. [3]): For $(x,y) \in R_h$, the 4 grid neighbors in coordinate directions are also interior grid points. Let $R'_h \subset R_h$ denote the set of points for which, in addition, all "diagonal" grid neighbors are interior grid points:



For $(x,y) \in R'_h$, DCMG01 uses the formulas

$$\Delta'_h := \frac{1}{6h^2} \begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix}, \quad W_h := \frac{1}{12} \begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix}. \qquad (3.4)$$

– For $(x,y) \in B_h' := R_h \backslash R_h'$ it is not possible to apply Δ_h' in the sense of (3.4): Here,



Here, Δ_h' and W_h are simply defined by

$$\Delta_h' := \Delta_h, \quad W_h := \frac{1}{12} \begin{bmatrix} & 1 & \\ 1 & 8 & 1 \\ & 1 & \end{bmatrix} \tag{3.5}$$

where Δ_h is the symmetric 5-point formula. Thus, for $(x,y) \in B_h'$, L_h' agrees with L_h. It would also be sufficient to choose $W_h := I_h$ (identity operator).

– Let $B_h := \Omega_h \backslash R_h$ denote the set of irregular interior grid points (cf. [3]). If the curvature of the boundary $\partial\Omega$ is not too strong w.r.t. the mesh size h, and if the mesh has been placed in a sensible way, it can be expected that one of the following *standard situations* will (almost) always arise:

(1) One neighbor on the boundary:

If, e.g., the distance of $(x,y) \in B_h$ to the boundary in western direction is λh, Δ_h' and W_h are defined by

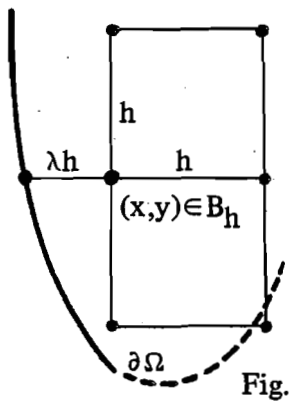
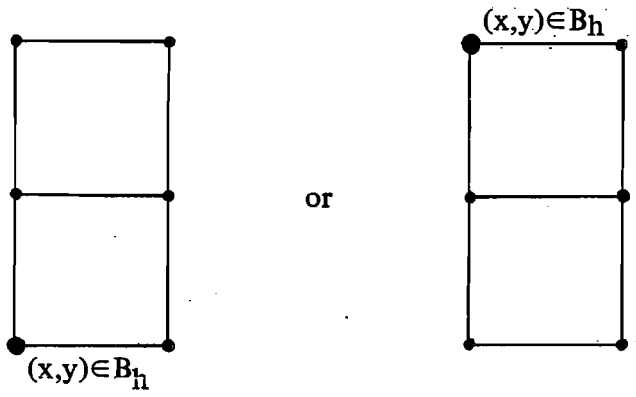


Fig. 3.1

$$\Delta_h' := \Delta_h + \frac{1}{3}(1-\lambda)\,\frac{1}{h^2} \begin{bmatrix} -1 & 1 \\ 2 & -2 \\ -1 & 1 \end{bmatrix},$$

$$W_h := I_h + \frac{1}{3}(1-\lambda) \begin{bmatrix} -1 & 1 \end{bmatrix}. \tag{3.6}$$

If the situation is such that the difference stencil used in (3.6) cannot be formed (dashed curve in Fig. 3.1), it is still possible that one of the "shifted differences"



exists. This leads to an obvious alternative formulation of Δ_h' in (3.6).

(2) Two neigbors on the boundary:

Let there be a neighbor of $(x,y) \in B_h$ on the boundary in both coordinate directions, e.g.: like in Fig. 3.2:

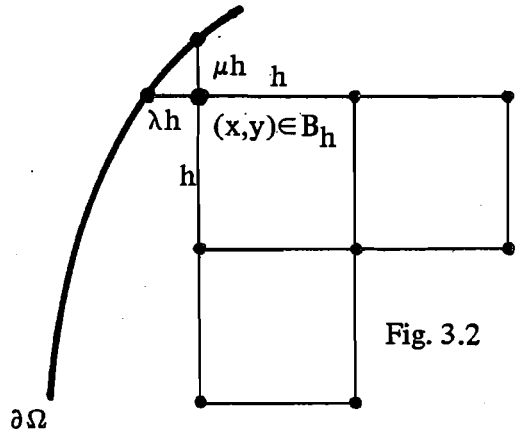


Fig. 3.2

Here, the "shifted version" of (3.6) is used in both coordinate directions in an obvious way.

If there are further points $(x,y) \in B_h$ not conforming to one of the standard situations, DCMG01 simply uses

$$\Delta_h' := \Delta_h \quad , \qquad W_h := I_h \, , \tag{3.7}$$

where Δ_h is the Shortley-Weller formula. This has the consequence that the approximation order is 3 instead of 4 and will cause DCMG01 to issue a special WARNING: Such points will be called *"very irregular points"*; see Section 5.

A theoretical justification of the choice for L_h' and f_h' described above is given in [1]. The main result is that, if (3.7) is excluded, the approximation order of the method is, in fact, 4.

4. THE INPUT OF DCMG01

4.1 General Description

The parameter list of DCMG01 contains only input parameters. It consists of the parameters of MG01 (cf. [3]) and two more parameters IDC and IDIMC:

SUBROUTINE DCMG01 (XA, YA, H0, M, A, B, FL, FH, DBW, DBE, ICONST, C, G, F, ITYP, ITER, IDC, IDIMQ, IDIMR, IDIMC)

The parameters IDC and IDIMC have the following meaning:

– IDC – specifies the type of iteration to be performed. IDC may be set to 0, 1, 2 or 3:
  - 0 – no defect correction: DCMG01 works like MG01; see (3.0).
  - 1 – defect correction before every multigrid cycle; see (3.1).
  - 2 – like IDC = 1, with 1 smoothing step (performed by SUBROUTINE RXC1); see (3.2). *IDC = 2 is the standard value.*
  - 3 – like IDC = 2, with smoothing in the sense of (3.3) instead of (3.2).

For the optimal choice of IDC (in connection with the parameter ITER) we refer to Section 4.2.

– IDIMC – dimension of the COMMON-array IRRDC in the calling program. See Section 4.3 for details.

The following sample program calling DCMG01 is similar to the example given in [3], Section 4. IDC≠0 requires allocation of additional storage place (COMMON/DCKEY/):

```
PROGRAM DRIVER
REAL                Q, QIRR, XA, YA
INTEGER             IRRDC
INTEGER             IDIMR, IDIMQ, IDIMC, IDC
EXTERNAL            F, G, C, DBE, DBW, FH, FL
COMMON/QSPACE/      Q(20000)
COMMON/RSPACE/      QIRR(10000)
COMMON/DCKEY/       IRRDC(1000)
XA = . . .
YA = . . .
  :
IDC = . . .
IDIMQ =   20000
IDIMR =   10000
IDIMC =    1000
CALL DCMG01 (XA,YA,. . ., IDC, IDIMQ, IDIMR, IDIMC)
STOP
END
```

The computed solution can be retrieved from the COMMON-array Q as described in [3], Section 4.

4.2 The Parameter IDC

If the given problem is "smooth" in the sense of Section 2, and if the Helmholtz function c(x,y) is of moderate size (relative to h^{-2}),

$$\text{IDC} = 2 \quad \text{together with} \quad \text{ITYP} = 3 \text{ (or 2)}, \quad \text{ITER} = 0, \tag{4.1}$$

is sufficient to solve the problem *to the level of the (high order) truncation error.*

Use of the V-Cycle instead of the W-Cycle (this means ITYP = 2 or even 1 instead of ITYP = 3 in (4.1)) will give an approximation which is not so accurate as with ITYP = 3, but still significantly better than with IDC = 0, and may be optimal from the efficiency point of view. For smooth problems, ITYP = 2 will often be sufficient in practice.

If the boundary of the domain is nicely smooth, it will be sufficient to replace IDC = 2 by IDC = 1 in (4.1). On the other hand, for domains with corners or a sharply curved boundary, IDC = 1 may lead to an error which is large relative to the truncation error.

If, under the conditions stated for the use of (4.1), the problem is to be solved with an error which is *very close (some %) to the truncation error,* an additional iteration step on the finest level should be used:

$$\text{IDC} = 2 \quad \text{together with} \quad \text{ITYP} = 3, \quad \text{ITER} = 1. \tag{4.2}$$

(Again, ITYP = 2 will often be more efficient). IDC = 1 instead of IDC = 2 in (4.2) may yield a result of similar quality, but not in every situation, as the contraction may deteriorate with IDC = 1, depending on the shape of the domain.

If the Helmholtz function c(x,y) is *not* of moderate size (i.e. not significantly smaller than h^{-2}), IDC = 2 must be replaced by IDC = 3 in (4.1), (4.2). IDC = 3 is more costly, but its use is indispensable in such a situation. Also, IDC = 1 should not be used because it will cause very poor convergence.

If there is doubt about the smoothness of the problem, if there are reentrant corners or if c(x,y) is pathologically large, use IDC = 0.

In any case, iteration up to the fixed point can be achieved by calling additional iterations on the finest level via the parameter ITER (cf. [3], Section 4). See Section 7 for typical contraction rates with ITYP = 3.

4.3 The Parameter IDIMC, Storage Allocation

In addition to the COMMON-blocks /QSPACE/ and /RSPACE/, (cf. [3], Section 4), a call of DCMG01 with IDC ≠ 0 requires the declaration of

COMMON / DCKEY / IRRDC ,

where IRRDC is an integer array. The actual value of the parameter IDIMC should be the dimension of IRRDC in the calling program. The dimension of IRRDC must be at least the *total number of all irregular interior grid points* (added over all grids).

For IDC = 0, the COMMON-block /DCKEY/ need not be declared or the dimension of IRRDC may be set to 1. In this case, the parameter IDIMC will not be used.

How to choose the dimensions of Q and QIRR:

– QIRR (COMMON/RSPACE/):
  Use the same dimension as required for MG01: at least *13 times* the total number of irregular interior grid points (i.e., 13 times the dimension of IRRDC).

– Q (COMMON/QSPACE/):
  The dimension required depends on the choice of the parameters IDC, ITYP and ITER: If
  IDC = 0 or
  IDC < 3 and ITYP < 4 and ITER = 0,
  (e.g. when using the "smooth solver" (4.1)), use the same dimension as required for MG01: at least n times the total number of interior grid points (added over all grids), where

  n = 2 if ICONST = 1,
  n = 3 if ICONST = 0.

  Otherwise, (e.g. when (4.2) is used), the dimension of Q must be at least (n+1) times this number.

The subroutine DCSIZE contained in the DCMG01 package can be used to calculate the minimal array dimensions for any combination of parameter values. The parameter list of DCSIZE consists of the parameters of SIZE (cf. [3]) and 3 more parameters:

SUBROUTINE DCSIZE (M, XA, YA, H0, ICONST, A, B, FL, FH, DBW, DBE, ITYP, ITER, IDC).

DCSIZE prints the requested information.

5. THE OUTPUT OF DCMG01

For the retrieval of information after a successful run of DCMG01 see [3], Section 5. Beyond the output for MG01, only some additional ERROR or WARNING messages may occur. The meaning of these messages is explained in the following:

– DIMENSION ERROR OCCURED ON GRID XX:
DIMENSION XXXX IS TOO SMALL FOR ARRAY IRRDC

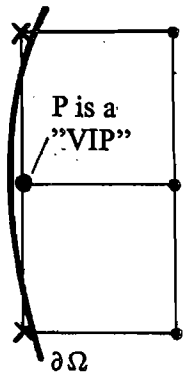
This message is self-explanatory. Program execution is stopped; use SUBROUTINE DCSIZE to calculate the required dimension and make sure that IDIMC is the actual dimension of IRRDC in the calling program.

– WARNING: XXX "VERY IRREGULAR POINTS" ON GRID XXX

This message means that on one of the coarser grids "very irregular points" (in the sense of Section 3.2) have been detected. This will often occur on the coarsest levels, depending mainly on the curvature of the boundary (compared to the mesh size). This situation should normally *not* arise on the *finest grid;* in this case, there is a special WARNING:

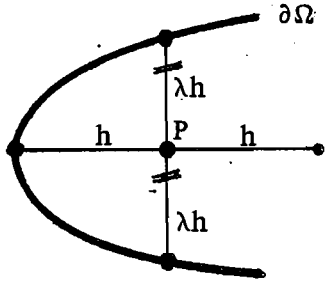
– WARNING: XXX "VERY IRREGULAR POINTS" ON FINEST GRID XXX
  – BOUNDARY UNSMOOTH OR GRID TOO COARSE
  – DEFECT CORRECTION MAY BECOME INEFFICIENT

This may *sometimes* happen even if the boundary is very smooth, e.g.:



In most cases, however, the occurrence of "very irregular points" on the finest grid is the consequence of sharp corners or strong curvature of the boundary. Here, full efficiency of the defect correction method cannot be expected (cf. Sections 2 and 3).

Remark. Sometimes, an irregular point is such that the basic discretization is symmetric, e.g.:



Here, there is *no* need to use a special formula for calculating the defect. All possible symmetric situations will be automatically verified by DCMG01 and will *not* be marked "very irregular". Within the check for symmetry, a relative tolerance of 10^{-10} is used.

6. COMPUTING TIMES

Calling DCMG01 with IDC ≠ 0 requires some additional effort:

In the *preparation phase,* DCMG01 (with IDC ≠ 0)

- analyzes the geometrical situation near all irregular grid points and stores information about it for later use,
- calculates the weighted right hand side for the defect-defining discretization (exception: IDC = 3; in this case the weighting is included in the solution phase).

In the *solution phase,* DCMG01 (with IDC ≠ 0) executes the following computations before every cycle (on any level, except the coarsest one):

- 1 relaxation step with SUBROUTINE RXC1 (only for IDC = 2 or 3)
- weighting of right hand side (only for IDC = 3)
- calculation of the modified right hand side for defect correction, depending on the actual approximation (IDC = 1,2,3).

Assuming use of the W-Cycle-FMG (ITYP = 3, ITER = 0), this will cost additional computing time compared to IDC = 0. Typical values, including the preparation phase, are

+	15 – 20 %	if	IDC = 1,
+	20 – 25 %	if	IDC = 2,
+	30 – 35 %	if	IDC = 3.

One further iteration step on the finest level will cost about 50% of the FMG computing time. (This is approximately true for any choice of IDC.)

As pointed out in [3], for the solution of "smooth problems", the W-cycle causes superfluous effort for IDC = 0, whereas its use might be necessary to obtain full accurary with IDC ≠ 0. However, IDC ≠ 0 + W-cycle will need not more than twice the computing time consumed by IDC = 0 + V-cycle (ITYP = 2).

Therefore, the use of defect correction is very efficient for smooth problems: The ratio

accuracy / work unit

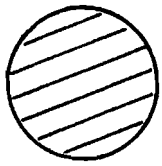
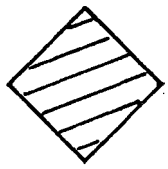
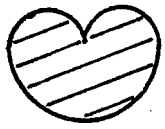
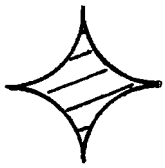
is considerably improved. See Section 7 for numerical evidence.

7. EXAMPLES

The following tables contain some results obtained by DCMG01 for problems where the exact solution U(x,y) is known. The values of h (finest mesh size) relate to domains of diameter 1. All errors are measured in the Euclidean norm.

Table 1: Poisson's equation, $U(x,y) = \sin(\pi(x+2y))$, on different domains.
W-cycle FMG (ITYP = 3).

$\|ERR\|^{FMG}$	–	error w.r.t. U(x,y) after FMG.
$\|ERR\|^{FMG+1}$	–	error w.r.t. U(x,y) after FMG and one further iteration step (ITER = 1)
$\|ERR\|^{FIX}$	–	error of the fixed point of the iteration w.r.t. U(x,y)
p^{FIX}	–	order of $\|ERR\|^{FIX}$, observed when refining from 2h to h
IDC	–	parameter controlling the type of defect correction (cf. Section 4)

Ω;h	$\|ERR\|^{FMG}$	$\|ERR\|^{FMG+1}$	$\|ERR\|^{FIX}$	p^{FIX}	IDC
1  $h = \frac{1}{64}$	.304E–3	———	.307E–3	1.95	0
	.139E–5	.110E–5	.110E–5	4.20	1
	.143E–5	.115E–5	.115E–5	4.15	2
2  $h = \frac{3}{5} \times \frac{1}{64}$	.934E–2	———	.954E–2	1.98	0
	.327E–3	.108E–3	.104E–3	4.04	1
	.148E–3	.109E–3	.109E–3	4.00	2.
3  $h = \frac{1}{64}$	.248E–2	———	.250E–2	2.01	0
	.933E–4	.967E–5	.113E–4	4.41	1
	.682E–4	.100E–4	.921E–5	4.56	2
4  $h = \frac{1}{64}$	.272E–2	———	.270E–2	1.99	0
	.401E–4	.435E–4	.449E–4	3.89	1
	.484E–4	.446E–4	.448E–4	3.89	2
5  $h = \frac{1}{64}$	.103E–2	———	.105E–2	1.92	0
	.343E–4	.115E–4	.105E–4	4.23	1
	.163E–4	.108E–4	.108E–4	4.24	2

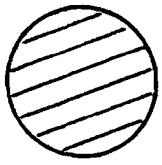
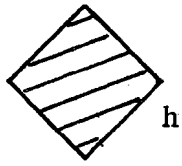
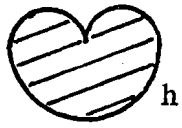
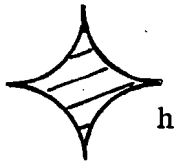
The accuracy improves by approximately 2 orders of magnitude if IDC ≠ 0. This gain of accuracy is achieved by a relatively modest additonal effort; cf. Section 6.

For IDC ≠ 0, FMG (with ITER = 0) sometimes yields an error very close to the truncation error $\|ERR\|^{FIX}$. But there are also situations where this is not true, e.g. in example 3 above: Here, the FMG error is a certain multiple of the truncation error. In such a situation IDC = 2 is superior to IDC = 1.

For all domains treated above, IDC ≠ 0 with ITER = 1 yields a solution which is absolutely satisfying.

Note that the difference between the fixed points for IDC = 1 and IDC = 2 is minimal; cf. Section 3.

Table 2: Examples like in Table 1; Table 2 shows $\|ERR\|^{FMG}$ for the V-cycle (ITYP = 1,2)

Ω;h	ITYP = 1	ITYP = 2	IDC
1  $h = \frac{1}{64}$	.293E–3 .612E–5 .258E–5	.297E–3 .175E–5 .169E–5	0 1 2
2  $h = \frac{3}{5} \times \frac{1}{64}$	.863E–2 .942E–3 .297E–3	.921E–2 .361E–3 .169E–3	0 1 2
3  $h = \frac{1}{64}$	.221E–2 .179E–3 .654E–4	.229E–2 .755E–4 .535E–4	0 1 2
4  $h = \frac{1}{64}$	.359E–2 .241E–3 .684E–4	.292E–2 .159E–3 .142E–3	0 1 2
5  $h = \frac{1}{64}$	.959E–3 .634E–4 .228E–4	.101E–2 .345E–4 .164E–4	0 1 2

As pointed out in Section 4, the V-cycle causes some loss of accuracy, but sometimes at the gain of improved efficiency. E.g. in example 1, ITYP = 2 is fully competitive, whereas in example 3 ITYP = 1 is *not* inferior to ITYP = 3, consuming only about half the computing time. This effect is due to the poor convergence behaviour of the defect correction component which dominates the contraction behaviour of the combined process. Note, however, that the full potential accuracy cannot be expected with the V-cycle.

Table 3: Behaviour of DCMG01 for a problem with a typical singularity occuring at a reentrant corner (with inner angle $\alpha\pi$):

$$U(x,y) = r^{\frac{1}{\alpha}} \sin\frac{\varphi}{\alpha}, \text{ Poisson's equation,}$$

where (r,φ) denote polar coordinates w.r.t. the singular point.

Ω;h \ $\|ERR\|^{FMG}$	IDC = 0	IDC = 1	IDC = 2
6  $\alpha = 3/2$, $h = 1/64$	.339E–3	.306E–3	.307E–3
7  $\alpha = 2$, $h = 1/64$	.161E–2	.149E–2	.146E–2

Although there is still some improvement for IDC ≠ 0, the level of accuracy is not better than for IDC = 0. From the efficiency point of view, IDC ≠ 0 is inferior.

Table 4: Solving the Helmholtz equation with $c(x,y) \equiv 10^i$, $i = 0,1,...,7$. The exact solution is $U(x,y) = \sin(\pi(x+2y))$. Results are given for ITYP = 3 and ITER = 1; the domain treated is the unit circle, $h = \frac{1}{64}$.

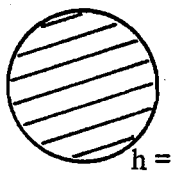
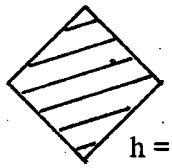
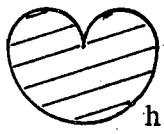
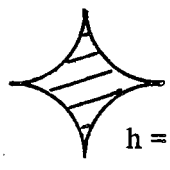
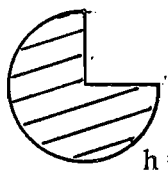
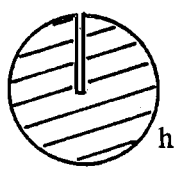
c(x,y)	IDC = 0	IDC = 1	IDC = 2	IDC = 3
10^0	.301E–3	.109E–5	.114E–5	.108E–5
10^1	.263E–3	.102E–5	.110E–5	.102E–5
10^2	.116E–3	.749E–6	.131E–5	.783E–6
10^3	.190E–4	.429E–6	.715E–5	.483E–6
10^4	.298E–5	.531E–6	.473E–4	.228E–6
10^5	.686E–6	.274E–5	.427E–3	.145E–6
10^6	.102E–6	.555E–5	.122E–2	.400E–7
10^7	.109E–7	.648E–5	.147E–2	.487E–8

For c(x,y) increasing in size, serveral effects are observed:

- IDC = 0 gives better and better results.
- IDC = 2 delivers catastrophic results if c(x,y) is close to or larger than h^{-2}.
- IDC = 3 is always best; however, the improvement against IDC = 0 decreases.
- IDC = 1 is a good as IDC = 3 as long as c(x,y) is small. For larger c(x,y) this is not true because of a very poor contraction rate (approximately $\frac{2}{3}$) in this case.

Table 4 may serve as a guideline for choosing parameters solving the Helmholtz equation with large c(x,y).

Table 5: Contraction rates for the W-Cycle (ITYP = 3) and IDC = 0,1,2 in different domains. The figures relate to Poisson's equation; the iteration was started after FMG-interpolation. The contraction rates are measured in the spectral norm.

Ω ;h	1st STEP	2nd STEP	3rd STEP	4th STEP	5th STEP	IDC
1  $h = \frac{1}{64}$	.005	.013	.026	.016	.031	0
	.027	.093	.147	.210	.237	1
	.017	.018	.032	.036	.038	2
2  $h=\frac{3}{5}\times\frac{1}{64}$	.011	.012	.032	.020	.031	0
	.060	.093	.168	.221	.246	1
	.020	.020	.033	.038	.041	2
3  $h = \frac{1}{64}$	.003	.006	.012	.023	.030	0
	.075	.087	.188	.217	.236	1
	.070	.045	.046	.047	.048	2
4  $h = \frac{1}{64}$	.019	.088	.090	.091	.091	0
	.051	.084	.151	.231	.245	1
	.035	.063	.067	.067	.067	2
5  $h = \frac{1}{64}$	.008	.007	.019	.016	.020	0
	.041	.076	.156	.209	.239	1
	.017	.023	.026	.031	.030	2
6  $h = \frac{1}{64}$	.062	.048	.049	.048	.048	0
	.046	.060	.301	.248	.257	1
	.039	.030	.031	.030	.031	2
7  $h = \frac{1}{64}$	.105	.093	.094	.094	.094	0
	.089	.052	.153	.388	.290	1
	.076	.069	.070	.069	.070	2

For IDC = 1, the contraction rate, which is nicely small at the beginning, becomes very unsatisfactory after few iteration steps (cf. Section 3).

In the case of IDC = 2, the contraction behavior is almost always better than for IDC = 1, does not deteriorate and is hardly worse than the pure W-Cycle contraction (IDC = 0).

8 SUBROUTINES USED BY DCMG01

The following subroutines which are contained in the MG01 package have been changed and renamed:

MG01	→	DCMG01,	
FMG	→	DCFMG	(called by DCMG01),
SIZE	→	DCSIZE.	

The parameter list of DCMG01 is described in Section 4. Furthermore, for technical reasons a minor extension of RXC1 was necessary. RXC1 has not been renamed since the extended version is fully compatible with the original version.

Six subroutines have been added. They will only be used when DCMG01 is called with IDC ≠ 0:

IRRTYP	called by	DCFMG,
D3SYMM	called by	IRRTYP,
WEIGHF	called by	DCFMG,
DEFDC	called by	DCFMG,
DIFF3	called by	DEFDC,
GRDBOX	called by	DEFDC.

The following remarks contain a short description of these subroutines.

(1) SUBROUTINE IRRTYP (K, KHELP, LIN, NVIP, IDIMC, IERROR)

Analyzes the irregular interior grid points on grid K and selects the type of formula to be used for calculating the defect. On exit, the array IRRDC (COMMON/DCKEY/) contains the corresponding information in an internal format for later use within the subroutines WEIGHF and DEFDC. Furthermore, NVIP is set to the number of "very irregular points" on grid K (in the sense of Section 3.2).
IRRTYP is called once per level within the preparation phase. Successful execution results in IERROR = 0. If the dimension IDIMC (cf. Section 4) is too small, IRRTYP is terminated with IERROR = 1 and an error message is printed (cf. Section 5). IERROR = 1 will cause DCMG01 to terminate execution.
Subroutine used: D3SYMM.

(2) LOGICAL FUNCTION D3SYMM (IB, I, J)

Used within IRRTYP. Returns .TRUE. if regular 3rd difference (type specified by IB) near grid point specified by I, J can be computed using inner grid values (cf. Section 3.2). If this is not possible, D3SYMM = .FALSE.

(3) SUBROUTINE WEIGHF (K, KF, KF1, ICOPY)

Performs weigthting of function values on grid KF. The weighting operator W_h is described in Section 3.2. The result (right hand side of defect-defining discretization) is put onto grid KF1. If ICOPY ≠ 0, the result is also copied back to grid KF.
WEIGHF uses the information stored in COMMON /DCKEY/.
If DCMG01 is called with IDC = 1 or 2, WEIGHF is once per level used within the preparation phase. If IDC = 3, WEIGHF is called several times within the solution phase (once per cycle).

(4) SUBROUTINE DEFDC (IDC, K, M)

Computes the modified right hand side (see Section 3) for defect correction, depending on the actual approximation stored on grid K.
DEFDC uses the information stored in COMMON /DCKEY/.
For IDC ≠ 0, DEFDC is called before each MG-Cycle (except on the coarsest grid).
Subroutines used: DIFF3, GRDBOX.

(5) REAL FUNCTION DIFF3 (IB, I, J)

Used within DEFDC. Returns regular 3rd difference (type specified by IB) near grid point specified by I, J using inner grid values. (The existence of this 3rd difference has been checked by IRRTYP by a call of D3SYMM; the corresponding information is passed to DEFDC via COMMON/DCKEY/.

(6) SUBROUTINE GRDBOX (I, J1, J2)

Used within DEFDC. Returns J1, J2 so that for grid column I, the grid points (I, J) with $J1 \leqslant J \leqslant J2$ are exactly those points where the full 4th order formula (3.4) can be applied (cf. Section 3).
Because the domain Ω is convex w.r.t. y-direction, the only possible values are
J1 = JMINX(I) + 1 or J1 = JMINX(I) + 2,
J2 = JMAXN(I) – 1 or J2 = JMAXN(I) – 2.
(See [3] for the meaning of JMINX(———), JMAXN(———) within the grid organization.)
This means that at the southern as well as at the northern end of each grid column, there is at most one grid point which is a regular point but does not allow (3.4) to be applied. (Here, DEFDC uses (3.5); see Section 3.2.)

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