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Defect Correction for Nonlinear Elliptic Difference Equations*

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Summary. The present paper is concerned with the study of a high-order defect correction technique for discretizations of nonlinear elliptic boundary value problems. The convergence of the method is analyzed in general and, in more detail, for a specific example. The algorithmic combination of defect correction and multigrid techniques is also discussed.

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1. Introduction

High-order discrete solutions to differential equations can be obtained by defect correction (DC-)techniques in a variety of situations. Among other applications of the DC idea, discretizations of elliptic boundary value problems have recently been considered by several authors (cf. [1, 2, 4, 5]). More specifically, its combination with fast multigrid solvers is of actual interest.

The present paper, with its emphasis on the nonlinear case, is intended to complete the presentation in Auzinger [1] which contains a linear analysis. In [1] it has been shown, for a specific application, that the convergence rates are

– independent of the discretization level, and
– sufficiently good to ensure that the usual multigrid efficiency can be obtained (when a multigrid method is used within the solution step).

In Sect. 2 we present the nonlinear versions of DC and DC with smoothing. We formulate a general convergence criterion and discuss the influence of smoothing on contraction rates and accuracy. Section 3 deals with the algorithmic combination of DC and (linear or nonlinear) multigrid. Finally, we present an example in Sect. 4.

Some preliminaries: Assume that an (elliptic boundary value) problem

$$Fu = c \tag{1.1}$$

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is approximated on a sequence of discretization levels (with "mesh size" h) by

$$\tilde{F}_h u_h = \tilde{c}_h, \tag{1.2a}$$

and

$$F_h u_h = c_h, \tag{1.2b}$$

respectively.

Let (1.2a), (1.2b) be consistent of order $\tilde{p}$ and p, resp., with $p>\tilde{p}$; i.e., the truncation errors behave like $O(h^{\tilde{p}})$, $O(h^p)$. (A typical case is $p=2\tilde{p}$.) Assume that (1.2a) is stable. (1.2a), (1.2b) are referred to as the "basic" and "target"-discretization of (1.1).

2. Nonlinear Defect Correction and Smoothing

For a pair (1.2) of equations, the well-known version B of the (nonlinear) defect correction iteration reads

$$\tilde{F}_h u_h^{(i+1)} := \tilde{F}_h u_h^{(i)} - [F_h u_h^{(i)} - c_h], \qquad i=0,1,\ldots \tag{2.1}$$

(cf. [2, 6]). Each step within (2.1) requires the solution of an equation like (1.2a) with a modified right hand side. A solution u_h' of (1.2b) is also a fixed point of (2.1).

(Within the context of multigrid applications, it is natural to proceed from (2.1) since nonlinear FAS-multigrid may be interpreted as a defect correction scheme of type B in a slightly generalized sense.)

The convergence properties of (2.1) depend on the quality of the approximation

$$\tilde{F}_h \approx F_h. \tag{2.2}$$

For applications involving high-order difference schemes, it is well known from linear models (cf. [1, 2]) that the *actual* convergence rates are governed by smoothness properties of the error of the intermediate solution $u_h^{(i)}$ which enters the defect computation. It is therefore natural to consider the following extension of (2.1):

$$\bar{u}_h^{(i)} := u_h^{(i)} - S_h[F_h u_h^{(i)} - c_h],$$
$$\tilde{F}_h u_h^{(i+1)} := \tilde{F}_h \bar{u}_h^{(i)} - [F_h \bar{u}_h^{(i)} - c_h], \tag{2.3}$$

where S_h is a suitable smoothing operator, assumed linear. (In general, S_h will originate from a linearization process; thus $S_h = S_h^{(i)}$. Our assumption that S_h is fixed is a convenient simplification.) The fixed point u_h' remains unchanged.

A further modification, which is of particular importance for our application, has been proposed in [2]: Replace the first step within (2.3) by a smoothing step w.r.t. the *basic* discretization:

$$\bar{u}_h^{(i)} := u_h^{(i)} - \tilde{S}_h[\tilde{F}_h u_h^{(i)} - \tilde{c}_h],$$
$$\tilde{F}_h u_h^{(i+1)} := \tilde{F}_h \bar{u}_h^{(i)} - [F_h \bar{u}_h^{(i)} - c_h]. \tag{2.4}$$

Obviously, (2.4) cannot have the same fixed point as (2.3). For the linear case, a representation of the "shifted" fixed point is given in [1] (see also [2]). The result from [1] shows that (2.4) (in its linear version) may be interpreted as a DC iteration with a modified, "stabilized" target discretization. No essential loss of accuracy (i.e., no order reduction) takes place if the problem is "sufficiently smooth".

In the following we briefly discuss the convergence properties of the nonlinear iterative scheme (2.4). Some notation: Let u_h^* denote the restriction of the exact solution of (1.1). The truncation errors of (1.2a, b) are

$$\tilde{\lambda}_h := \tilde{F}_h u_h^* - \tilde{c}_h, \tag{2.5a}$$
$$\lambda_h := F_h u_h^* - c_h. \tag{2.5b}$$

Define

$$\Delta F_h := \tilde{F}_h - F_h, \tag{2.6}$$
$$\tilde{R}_h := I_h - \tilde{S}_h \tilde{F}_h. \tag{2.7}$$

The "one-step-formulation" of (2.4) reads

$$\tilde{F}_h u_h^{(i+1)} := \Delta F_h(\tilde{R}_h u_h^{(i)} + \tilde{S}_h \tilde{c}_h) + c_h. \tag{2.4'}$$

(2.4′) implicitly defines an operator G_h: $u_h^{(i+1)} := G_h u_h^{(i)}$. We shall now formulate sufficient conditions for the contractivity of G_h. To this end, let

$$\|\cdot\|_\alpha, \quad \|\cdot\|_\beta, \tag{2.8}$$

be norms and let

$$\mathscr{B}_h := \{u_h : \|u_h - u_h^*\|_\alpha \leqq \rho\}, \qquad \rho = O(1), \tag{2.9}$$

be some ball around u_h^* such that the following estimates hold independently of h for all $u_h, v_h \in \mathscr{B}_h$:

$$\|G_h u_h - G_h v_h\|_\alpha \leqq C_1 \|\tilde{F}_h G_h u_h - \tilde{F}_h G_h v_h\|_\beta, \tag{2.10}$$
$$\|\Delta F_h(\tilde{S}_h \tilde{c}_h + \tilde{R}_h u_h) - \Delta F_h(\tilde{S}_h \tilde{c}_h + \tilde{R}_h v_h)\|_\beta \leqq C_2 \|u_h - v_h\|_\alpha. \tag{2.11}$$

Assume further

$$\|G_h u_h^* - u_h^*\|_\alpha \leqq C_1' \|\tilde{F}_h G_h u_h^* - \tilde{F}_h u_h^*\|_\beta, \tag{2.10'}$$
$$\|\Delta F_h(u_h^* - \tilde{S}_h \tilde{\lambda}_h) - \Delta F_h u_h^*\|_\beta \leqq C_2' \|\tilde{S}_h \tilde{\lambda}_h\|_\alpha. \tag{2.11'}$$

The constant C_1 is a "measure of regularity" for $\tilde{F}_h$.

The following consistency estimates are required:

$$\|\tilde{S}_h \tilde{\lambda}_h\|_\alpha \leqq C h^q, \tag{2.12a}$$

where we expect $\tilde{p} < q \leqq p$, and

$$\|\lambda_h\|_\beta \leqq C h^p. \tag{2.12b}$$

C is some generic h-independent constant. Whereas (2.12b) is seen to be natural, (2.12a) may not always be obvious. A concrete discussion of (2.12) is referred to Sect. 4.

Proposition 2.1. *Assume* (2.10)–(2.12) *and*

$$C_1\, C_2 \leqq m < 1 \quad \textit{independent of } h. \tag{2.13}$$

Then, the operator G_h *(defined by* (2.4′)*) maps* $\mathscr{B}_h$ *into itself and is contractive on* $\mathscr{B}_h$ *with*

$$\| G_h\, u_h - G_h\, v_h \|_\alpha \leqq m \| u_h - v_h \|_\alpha. \tag{2.14}$$

In $\mathscr{B}_h$, *there is a unique fixed point* $\hat{u}_h$ *satisfying*

$$\| \hat{u}_h - u_h^* \|_\alpha \leqq \frac{1}{1-m}\, C_1' \,[\| \lambda_h \|_\beta + C_2' \| \tilde{S}_h\, \tilde{\lambda}_h \|_\alpha]. \tag{2.15}$$

Thus, $\| \hat{u}_h - u_h^* \|_\alpha = O(h^q)$ *(with q from* (2.12a)*).*

Proof. a) By definition of G_h and by (2.10), (2.11),

$$\begin{aligned} \| G_h\, u_h - G_h\, v_h \|_\alpha &\leqq C_1 \| \tilde{F}_h\, G_h\, u_h - \tilde{F}_h\, G_h\, v_h \|_\beta \\ &= C_1 \| \Delta F_h(\tilde{S}_h\, \tilde{c}_h + \tilde{R}_h\, u_h) + c_h - \Delta F_h(\tilde{S}_h\, \tilde{c}_h + \tilde{R}_h\, v_h) - c_h \|_\beta \\ &\leqq C_1\, C_2 \| u_h - v_h \|_\alpha, \qquad \text{which implies (2.14).} \end{aligned}$$

b) We show next that $G_h\, u_h \in \mathscr{B}_h$ if $u_h \in \mathscr{B}_h$:

$$\begin{aligned} \| G_h\, u_h - u_h^* \|_\alpha &\leqq \| G_h\, u_h - G_h\, u_h^* \|_\alpha + \| G_h\, u_h^* - u_h^* \|_\alpha \\ &\leqq m \| u_h - u_h^* \|_\alpha + \| G_h\, u_h^* - u_h^* \|_\alpha \qquad \text{by (2.14).} \end{aligned}$$

Furthermore, by (2.10′), (2.11′), (2.12) and by definition of G_h:

$$\begin{aligned} \| G_h\, u_h^* - u_h^* \|_\alpha &\leqq C_1' \| \tilde{F}_h\, G_h\, u_h^* - \tilde{F}_h\, u_h^* \|_\beta \\ &= C_1' \| \Delta F_h(\tilde{R}_h\, u_h^* + \tilde{S}_h\, \tilde{c}_h) + c_h - F_h\, u_h^* - \tilde{F}_h\, u_h^* + F_h\, u_h^* \|_\beta \\ &= C_1' \| \Delta F_h(u_h^* - \tilde{S}_h\, \tilde{\lambda}_h) - \Delta F_h\, u_h - \lambda_h \|_\beta \\ &\leqq C_1'\, C_2' \| \tilde{S}_h\, \tilde{\lambda}_h \|_\alpha + C_1' \| \lambda_h \|_\beta = O(h^p). \end{aligned}$$

Thus,

$$\begin{aligned} \| G_h\, u_h - u_h^* \|_\alpha &\leqq m \| u_h - u_h^* \|_\alpha + O(h^p) \\ &< \| u_h - u_h^* \|_\alpha \qquad (h \text{ sufficiently small}). \end{aligned}$$

c) Estimate (2.15) is an immediate consequence of b). (Replace u_h by $\hat{u}_h = G_h\, \hat{u}_h$.) □

Proposition 2.1 carries over in an obvious way to the special case $\tilde{S}_h = 0$ (defect correction without smoothing, $\tilde{R}_h = I_h$).

If ΔF_h is a linear operator, $\hat{u}_h$ can be given a more explicit interpretation:

Corollary 2.2. *Let the assumptions of Proposition* 2.1 *hold, and assume that* ΔF_h *is linear. Then, the fixed point* $\hat{u}_h$ *satisfies*

$$\hat{F}_h\, \hat{u}_h = \hat{c}_h, \tag{2.16a}$$

where

$$\begin{aligned} \hat{F}_h &:= F_h + \Delta F_h\, \tilde{S}_h\, \tilde{F}_h, \\ \hat{c}_h &:= c_h + \Delta F_h\, \tilde{S}_h\, \tilde{c}_h. \end{aligned} \tag{2.16b}$$

Proof. By linearity of ΔF_h, the fixed point equation

$$\tilde{F}_h\, \hat{u}_h = \Delta F_h[(I_h - \tilde{S}_h\, \tilde{F}_h)\, \hat{u}_h + \tilde{S}_h\, \tilde{c}_h] + c_h$$

can be recombined into

$$\tilde{F}_h\, \hat{u}_h - \Delta F_h\, \hat{u}_h + \Delta F_h\, \tilde{S}_h\, \tilde{F}_h\, \hat{u}_h = c_h + \Delta F_h\, \tilde{S}_h\, \tilde{c}_h.$$

This results in (2.16) since $F_h = \tilde{F}_h - \Delta F_h$. □

Let us conclude this Section with some brief remarks on *linearization* techniques. Assume that $\tilde{H}_h$ is an invertible linear operator which is an approximation of $\tilde{F}_h$ (e.g. Newton or Quasi-Newton linearization: $\tilde{H}_h = \tilde{F}_h'(\ldots)$). $\tilde{F}_h$ has to be replaced by $\tilde{H}_h$ on *both* sides of equation (2.1):

$$\tilde{H}_h \cdot (u_h^{(i+1)} - u_h^{(i)}) := -[F_h\, u_h^{(i)} - c_h]. \tag{2.17}$$

This can be interpreted as a "direct" linearization of the target equation. Note that replacing $\tilde{F}_h$ by $\tilde{H}_h$ only on the left hand side of (2.1) will not work: A simple reasoning shows that the resulting change of the fixed point will completely destroy the order of convergence. In other words: Defect correction and linearization have to be performed *simultaneously*.

In view of this observation, one will also use $\tilde{H}_h$ instead of $\tilde{F}_h$ within the smoothing step of (2.4) (although this is not so essential there; cf. [1]).

The extension of Proposition 2.1 to the linearized iteration (2.17) requires estimates for $\| \tilde{H}_h - F_h \|_{\beta,\alpha}$.[1] Since

$$\| \tilde{H}_h - F_h \|_{\beta,\alpha} \leqq \| \tilde{H}_h - \tilde{F}_h \|_{\beta,\alpha} + \| \Delta F_h \|_{\beta,\alpha}, \tag{2.18}$$

no essential difficulties will arise because

$$\| \tilde{H}_h - \tilde{F}_h \|_{\beta,\alpha} \leqq C_3 < 1 \tag{2.19}$$

can normally be expected (with C_3 sufficiently small) if $\tilde{H}_h$ is a reasonable approximation for $\tilde{F}_h$.

If $\tilde{H}_h = \tilde{H}_h^{(i)} = \tilde{F}_h'(u_h^{(i)})$, the influence of the linearization will be negligible once that the Newton method is quadratically convergent. (On the other hand, it is obvious that using the "full" Newton method will usually not be efficient.)

3. Defect Correction and Multigrid: Algorithmic Patterns

The iterative process described below arises from the combination of the DC-iteration (2.1) with a nonlinear 2-level FAS-cycle (cf. [3, 4]) for the solution step. $\tilde{S}_h$ is a linear smoothing operator (already introduced in Sect. 2). $I_{2h,h}$

[1] Here we use $\| \cdot \|_{\beta,\alpha}$ as a concise denotation for Lipschitz bounds of the respective nonlinear operators

and $I_{h,2h}$ are linear restriction and prolongation operators, respectively.

$$\begin{aligned}\tilde{d}_h &:= \tilde{F}_h u_h^{(i)} - [F_h u_h^{(i)} - c_h]\\ \bar{u}_h^{(i)} &:= u_h^{(i)} - \tilde{S}_h[\tilde{F}_h u_h^{(i)} - \tilde{d}_h]\\ \tilde{F}_{2h} v_{2h} &:= \tilde{F}_{2h} I_{2h,h} \bar{u}_h^{(i)} - I_{2h,h}[\tilde{F}_h \bar{u}_h^{(i)} - \tilde{d}_h]\\ u_h^{(i+1)} &:= \bar{u}_h^{(i)} + I_{h,2h}[v_{2h} - I_{2h,h} \bar{u}_h^{(i)}]\end{aligned} \tag{3.1}$$

In the general multigrid formulation, $\tilde{F}_{2h}$ is again recursively substituted by smoothing sweeps and coarse grid corrections.

Notice that, by definition of $\tilde{d}_h$,

$$\tilde{F}_h u_h^{(i)} - \tilde{d}_h = F_h u_h^{(i)} - c_h \tag{3.2}$$

holds. Thus, the composite iterative procedure (3.1) is similar to a standard two-grid scheme for the solution of $F_h u_h = c_h$.

If we use the accelerated version of DC in the sense of (2.4), the composite iterations reads

$$\begin{aligned}\bar{u}_h^{(i)} &:= u_h^{(i)} - \tilde{S}_h[\tilde{F}_h u_h^{(i)} - \tilde{c}_h]\\ \tilde{d}_h &:= \tilde{F}_h \bar{u}_h^{(i)} - [F_h \bar{u}_h^{(i)} - c_h]\\ \bar{\bar{u}}_h^{(i)} &:= \bar{u}_h^{(i)} - \tilde{S}_h[\tilde{F}_h \bar{u}_h^{(i)} - \tilde{d}_h]\\ \tilde{F}_{2h} v_{2h} &:= \tilde{F}_{2h} I_{2h,h} \bar{\bar{u}}_h^{(i)} - I_{2h,h}[\tilde{F}_h \bar{\bar{u}}_h^{(i)} - \tilde{d}_h]\\ u_h^{(i+1)} &:= \bar{\bar{u}}_h^{(i)} + I_{2h,h}[v_{2h} - I_{2h,h} \bar{\bar{u}}_h^{(i)}]\end{aligned} \tag{3.3}$$

The third step can be rewritten as

$$\bar{\bar{u}}_h^{(i)} := \bar{u}_h^{(i)} - \tilde{S}_h[F_h \bar{u}_h^{(i)} - c_h]. \tag{3.4}$$

It is obviously reasonable to omit (3.4) since it immediately follows the initial smoothing sweep (accelerating DC convergence). This results in Hackbusch's version of multigrid + defect correction (see [4, 5] for a linear analysis). A disadvantage of this simplification is that defect correction and the multigrid process are no longer clearly separated.

If we use the linearized version (2.17) of defect correction, linear ("CS-scheme") multigrid can be employed to solve the linearized equations. (For a comparison of nonlinear and linearized multigrid we refer to Stüben and Trottenberg [7]).

Another technique involving a pair of different discretizations within a multigrid scheme has been proposed by Brandt [3]: The so-called "double-discretization" scheme introduces high-order defects on *each* discretization level. In its nonlinear version, both (i.e., the "basic" and the "target"-)defects should be used in parallel throughout whole of the process. (See [3] for a detailed discussion.)

4. Application: A Weakly Nonlinear Boundary Value Problem

In this final Section we discuss the convergence properties of the defect correction iteration (cf. Proposition 2.1) for the weakly nonlinear sample problem

$$\begin{aligned}Lu(x,y) + f(x,y,u(x,y)) &= g(x,y), \quad (x,y)\in\Omega\subseteq\mathbb{R}^2,\\ u(x,y) &= b(x,y), \quad (x,y)\in\partial\Omega,\end{aligned} \tag{4.1}$$

where L is the negative Laplace operator $-\Delta$ and f is a diagonal, Lipschitz continuous operator satisfying the monotonicity condition

$$\frac{\partial}{\partial u}(x,y,u) \geqq 0. \tag{4.2}$$

We shall restrict our considerations to the case that Ω is a rectangular domain. (See Auzinger [1] for a linear analysis in general domains.) In contrast to [1], we are not going into all of the details but try to outline the typical argumentation.

Let Ω_h be an equidistant mesh (mesh size h) with boundary points $\partial\Omega_h$, and let $\bar{\Omega}_h := \Omega_h + \partial\Omega_h$. We consider the basic discretization $\tilde{F}_h u_h = \tilde{c}_h$ defined by

$$\begin{aligned}\tilde{L}_h u_h(x,y) + f(x,y,u_h(x,y)) &= g(x,y), \quad (x,y)\in\Omega_h,\\ u_h(x,y) &= b(x,y), \quad (x,y)\in\partial\Omega_h,\end{aligned} \tag{4.3a}$$

where L_h denotes the usual five-point-discretization of $-\Delta$. Assume further that $F_h u_h = c_h$ is given by the well-known, 4th order "Mehrstellenverfahren":

$$\begin{aligned}L_h u_h(x,y) + W_h f(x,y,u_h(x,y)) &= g(x,y), \quad (x,y)\in\Omega_h,\\ u_h(x,y) &= b(x,y), \quad (x,y)\in\partial\Omega_h,\end{aligned} \tag{4.3b}$$

with the nine-point stencil L_h and the local weighting operator W_h (cf. [1]). Thus, the difference operator $\Delta F_h = \tilde{F}_h - F_h$ reads

$$\Delta F_h u_h(x,y) = \begin{cases}\Delta L_h u_h(x,y) + (I_h - W_h) f(x,y,u_h(x,y)), & (x,y)\in\Omega_h,\\ 0 & (x,y)\in\partial\Omega_h,\end{cases} \tag{4.4}$$

with the stencil representations

$$\Delta L_h \sim \frac{1}{6h^2}\begin{bmatrix}1 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 1\end{bmatrix}, \quad I_h - W_h \sim \frac{1}{12}\begin{bmatrix} & -1 & \\ -1 & 4 & -1\\ & -1 & \end{bmatrix}. \tag{4.4'}$$

The simplest example of a smoothing operator is of Jacobi type:

$$\tilde{S}_h u_h(x,y) = \omega\frac{h^2}{4} u_h(x,y), \quad (x,y)\in\Omega_h \tag{4.5}$$

with a suitable damping factor $\omega < 1$.

The conditions (2.10)–(2.13) will now be discussed for a natural choice of norms $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$. Some notation: Define the inner product

$$\langle u_h, v_h\rangle := h^2 \sum_{(x,y)\in\bar{\Omega}_h} u_h(x,y)\, v_h(x,y) \tag{4.6}$$

and the Euclidean (or discrete H^0-)norm

$$\|u_h\|_{H^0}:=\langle u_h, u_h\rangle^{\frac{1}{2}}. \tag{4.7}$$

A discrete H^1-seminorm is defined as usual via first difference quotients. The equality

$$\langle \tilde{L}_h u_h, u_h\rangle = |u_h|^2_{H^1} \tag{4.8}$$

can easily be shown by partial summation. (The inequality $\langle \tilde{L}_h u_h, u_h\rangle \geqq c|u_h|^2_{H^1}$ holds in much more general situations.)

H^1-estimates. Assume

$$\|\cdot\|_\alpha = |\cdot|_{H^1}, \qquad \|\cdot\|_\beta := |\cdot|_{H^{-1}}, \tag{4.9}$$

where $|\cdot|_{H^{-1}}$ denotes the dual of $|\cdot|_{H^1}$. (Note that $|\cdot|_{H^1}$ is a norm since Ω is bounded.)[2]

Lemma 4.1. *For the present example,* (2.10) *and* (2.10′) *hold with* $C_1 = C_1' = 1$, *since for arbitrary* u_h, v_h:

$$|u_h - v_h|_{H^1} \leqq |\tilde{L}_h(u_h - v_h) + f(u_h) - f(v_h)|_{H^{-1}}. \tag{4.10}$$

Proof. By the mean value theorem,

$$f(x, y, u_h(x, y)) - f(x, y, v_h(x, y)) = \gamma\cdot(u_h - v_h)(x, y), \tag{4.11}$$

where $\gamma = \gamma(x, y, u_h(x, y), v_h(x, y)) \geqq 0$ by assumption (4.2). Thus

$$\langle \tilde{L}_h(u_h - v_h) + f(u_h) - f(v_h), u_h - v_h\rangle \geqq \langle \tilde{L}_h(u_h - v_h), u_h - v_h\rangle$$

holds for arbitrary u_h, v_h. The result follows from (4.8) by standard argumentation (cf. [1]). □

(2.11) and (2.11′) are an immediate consequence of the following Lemma. Recall that $\tilde{R}_h := I_h - \tilde{S}_h \tilde{F}_h$.

Lemma 4.2. *Assume*

$$|f(x, y, u) - f(x, y, v)| \leqq \bar{\gamma}|u - v|. \tag{4.12}$$

For $\tilde{S}_h$ *as defined in* (4.5) (*with an appropriate damping factor* ω), *the estimate*

$$|\Delta L_h(\tilde{R}_h u_h - \tilde{R}_h v_h) + (I_h - W_h)[f(z_h + \tilde{R}_h u_h) - f(z_h + \tilde{R}_h v_h)]|_{H^{-1}} \leqq C_2|u_h - v_h|_{H^1} \tag{4.13}$$

holds uniformly in h for arbitrary u_h, v_h, z_h *and*

$$C_2 = \tfrac{1}{12} + O(\bar{\gamma}h). \tag{4.13′}$$

Proof (sketch). Split the left hand side of (4.13) into

$$\begin{aligned} &|\Delta L_h(\tilde{R}_h u_h - \tilde{R}_h v_h) + (I_h - W_h)[f(z_h + \tilde{R}_h u_h) - f(z_h + \tilde{R}_h v_h)]|_{H^{-1}} \\ &\quad \leqq |\Delta L_h(I_h - \tilde{S}_h \tilde{L}_h)(u_h - v_h)|_{H^{-1}} + |\tilde{S}_h[f(u_h) - f(v_h)]|_{H^{-1}} \\ &\qquad + |(I_h - W_h)[f(z_h + \tilde{R}_h u_h) - f(z_h + \tilde{R}_h v_h)]|_{H^{-1}}. \end{aligned}$$

[2] More precisely: $|\cdot|_{H^1}$ is a norm on the space of grid functions with zero (Dirichlet-) boundary values; this is sufficient for our purpose

a) The first, linear term is bounded by

$$|\Delta L_h(I_h - \tilde{S}_h \tilde{L}_h)(u_h - v_h)|_{H^{-1}} \leqq C|u_h - v_h|_{H^1}$$

with $C = \frac{1}{12}$, as has been shown in [1]. An estimate of this type is natural for an operator like ΔL_h; however, the relatively small threshold $\frac{1}{12}$ is only achieved by virtue of the so-called "smoothing property" of $(I_h - \tilde{S}_h \tilde{L}_h)$, which holds for the choice $\omega = \frac{1}{2}$ if $\bar{\gamma}$ is small compared with h^{-2}. ($\Delta L_h u_h$ is "small" if u_h is smooth.)

The remaining terms can be estimated as follows:

b) By (4.8), it is easy to see that $|u_h|_{H^{-1}} = |\tilde{L}_h^{-1} u_h|_{H^1}$. Thus,

$$\begin{aligned} |\tilde{S}_h[f(u_h) - f(v_h)]|_{H^{-1}} &= |\tilde{L}_h^{-1} \tilde{S}_h[f(u_h) - f(v_h)]|_{H^1} \\ &\leqq Ch^2|f(u_h) - f(v_h)|_{H^1} \leqq Ch^2 h^{-1}\|f(u_h) - f(v_h)\|_{H^0} \\ &\leqq CC'h\bar{\gamma}|u_h - v_h|_{H^1}. \end{aligned}$$

Here we used (4.12), the stability of $\tilde{L}_h$ w.r.t. $|\cdot|_{H^1}$, and the inequalities

$$|u_h|_{H^1} \leqq Ch^{-1}\|u_h\|_{H^0},$$
$$\|u_h\|_{H^0} \leqq C'|u_h|_{H^1} \text{ ("discrete Poincaré inequality").}$$

c) Since $I_h - W_h = \dfrac{h^2}{12}\tilde{L}_h$ (cf. (4.4′)),

$$\begin{aligned} &|(I_h - W_h)[f(z_h + \tilde{R}_h u_h) - f(z_h + \tilde{R}_h v_h)]|_{H^{-1}} \\ &\quad \leqq \tfrac{1}{12}h^2 h^{-1}\|f(z_h + \tilde{R}_h u_h) - f(z_h + \tilde{R}_h v_h)\|_{H^0} \\ &\quad \leqq \tfrac{1}{12}h\bar{\gamma}\|\tilde{R}_h u_h - \tilde{R}_h v_h\|_{H^0} \\ &\quad \leqq \tfrac{1}{12}h\bar{\gamma}\|(I_h - \tilde{S}_h \tilde{L}_h)(u_h - v_h)\|_{H^0} + \tfrac{1}{12}h\bar{\gamma}h^2\|f(u_h) - f(v_h)\|_{H^0} \\ &\quad \leqq \tfrac{1}{12}hCC'\bar{\gamma}|u_h - v_h|_{H^1} \end{aligned}$$

follows in a similar way as above. □

Remark. It can be shown that $O(\bar{\gamma}h)$ can be replaced by $O(\bar{\gamma}h^2)$ if f is sufficiently smooth.

Lemmata 4.1 and 4.2 imply the contractivity of the defect correction operator since (2.14) is satisfied:

$$C_1 C_2 \leqq \tfrac{1}{12} + O(\bar{\gamma}h) \leqq m < 1. \tag{4.14}$$

Note that $\frac{1}{12}$ would have to be replaced by $\frac{1}{3}$ if $\tilde{S}_h = 0$ (no smoothing) (cf. [1, 2]). This shows that the smoothing step improves the performance of our method considerably.

The consistency conditions (2.12a, b) depend on the degree of smoothness of the exact solution of (4.1). For our choice norms,

$$|\tilde{S}_h \tilde{\lambda}_h|_{H^1} \leqq Ch^q, \qquad |\lambda_h|_{H^{-1}} \leqq Ch^p \tag{4.15a, b}$$

are required. (4.15b) holds with $p = 4$ under the usual differentiability assumptions for the "Mehrstellenverfahren". In fact, the condition is *weaker* for $|\cdot|_{H^{-1}}$

than for $\|\cdot\|_{H^0}$. (4.15a) can be estimated by

$$|\tilde{S}_h \tilde{\lambda}_h|_{H^1} \leqq C h^2 |\tilde{\lambda}_h|_{H^1}.$$

Thus, the optimal result $q=4$ requires some additional smoothness of the truncation error $\tilde{\lambda}_h$ of the basic discretization. It can be seen by Taylor expansion that this can be expected under the same assumptions as above. (However, some problems would arise in the case of an irregular boundary.)

Summing up, we may say that the fixed point $\hat{u}_h$ of the defect correction with smoothing satisfies

$$|\hat{u}_h - u_h^*|_{H^1} \leqq C h^4 \tag{4.16}$$

under conditions which are natural for a 4th order method.

We finally note that H^0-convergence estimates for the DC iteration are more difficult to obtain and have so far only be proved for the simplest linear examples. In contrast, H^1-estimates, as outlined above, will usually be extendible to more general situations without severe complications.

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