New shape functions for triangular p-FEM using integrated Jacobi polynomials

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Abstract

In this paper, the second order boundary value problem $-\nabla\cdot(\mathcal{A}(x,y)\nabla u) = f$ is discretized by the Finite Element Method using piecewise polynomial functions of degree p on a triangular mesh. On the reference element, we define integrated Jacobi polynomials as interior ansatz functions. If $\mathcal{A}$ is a constant function on each triangle and each triangle has straight edges, we prove that the element stiffness matrix has not more than $25/2p^2$ nonzero matrix entries. An application for preconditioning is given. Numerical examples show the efficiency of the proposed basis.

1 Introduction

In this paper, we investigate the following boundary value problem: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $\mathcal{A}(x,y)$ be a matrix which is symmetric and uniformly positive definite in $\overline{\Omega}$. Find $u \in H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_1\}$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\overline{\Gamma_1 \cup \Gamma_2} = \partial\Omega$ such that

$$a_\triangle(u,v) := \int_\Omega (\nabla u)^T \mathcal{A}(x,y)\nabla v = \int_\Omega fv + \int_{\Gamma_2} f_1 v := \langle f, v\rangle_\Omega + \langle f_1, v\rangle_{\Gamma_2} \qquad (1.1)$$

holds for all $v \in H^1_{\Gamma_1}(\Omega)$. Problem (1.1) will be discretized by means of the hp-version of the finite element method using triangular elements $\triangle_s$, $s = 1,\ldots,nel$. Let $\hat{\triangle}$ be the reference triangle and $F_s : \hat{\triangle} \to \triangle_s$ be the (possibly nonlinear) isoparametric mapping to the element $\triangle_s$. We define the finite element space $\mathbb{M} := \{u \in H^1_{\Gamma_1}(\Omega), u\mid_{\triangle_s} = \tilde{u}(F_s^{-1}(x,y)), \tilde{u} \in \mathbb{P}_p\}$, where $\mathbb{P}_p$ is the space of all polynomials of maximal total degree p.

By $\Psi = (\psi_1, \ldots, \psi_N)$, we denote a basis for $\mathbb{M}$ in which the functions $\psi_1, \ldots, \psi_{n_v}$ are the usual hat functions. The functions $\psi_{n_v+(j-1)(p-1)+1}, \ldots, \psi_{n_v+j(p-1)}$ correspond to the edge e_j of the mesh, and vanish on all other edges, i.e. satisfy the condition $\psi_{n_v+(j-1)(p-1)+k-1} \mid_{e_l} = \delta_{j,l} p_k$, where p_k is a polynomial of degree p, $k = 2, \ldots, p$. The support of an edge function is formed by those two elements, which have this edge e_j in common. The remaining basis functions are interior bubble functions consisting of a support containing one element only. These functions vanish on each edge of the mesh. With this definition, the basis functions ψ_i can be divided into three groups,

- the vertex functions,
- the edge bubble functions,
- the interior bubbles,

locally on each element $\triangle_s$, and globally on Ω.
The Galerkin projection of (1.1) onto $\mathbb{M}$ leads to the linear system of algebraic finite element equations

$$\mathcal{K}_\Psi \underline{u} = \underline{f}, \quad \text{where} \quad \mathcal{K}_\Psi = [a_\Delta(\psi_j, \psi_i)]_{i,j=1}^N, \quad \underline{f}_p = [\langle f, \psi_i \rangle + \langle f_1, \psi_i \rangle_{\Gamma_2}]_{i=1}^N. \tag{1.2}$$

The global stiffness matrix $\mathcal{K}_\Psi$ can be expressed by the local stiffness matrices on the elements, i.e.

$$\mathcal{K}_\Psi = \sum_{s=1}^{nel} R_s^T K_s R_s, \tag{1.3}$$

where K_s is the stiffness matrix on the element $\triangle_s$ and R_s denotes the connectivity matrix for the numbering of the shape functions on $\triangle_s$ and Ω.
Using the vector $\underline{u}$, an approximation $u_p = \Psi \underline{u}$ of the exact solution u of (1.1) can be built by the usual finite element isomorphism. In the case of smooth solutions u in parts of the domain Ω, spectral methods, [21], and finite elements of high order (p-version), see e.g. [26], [27], and the references therein, have become more popular for twenty years. For the h-version of the FEM, the polynomial degree p of the shape functions on the elements is kept constant and the mesh-size h is decreased. This is in contrast to the the p-version of the FEM in which the polynomial degree p is increased and the mesh-size h is kept constant. Both ideas, mesh refinement and increasing the polynomial degree, can be combined. This is called the hp-version of the FEM.
The advantage of the p-version in comparison to the h-version is that the solution converges faster to the exact solution with respect to the number of unknowns N. However, the choice of a basis Ψ in which the element stiffness matrix K_s has $\mathcal{O}(N)$ nonzero matrix entries is a difficult question. In the one-dimensional case, i.e. for the differential equation $-u'' + u = f$, one can take the primitives over orthogonal polynomials in order to get a sparse system matrix, see e.g. [20]. In the 2D and 3D case, the choice of a basis which is optimal due to condition number and sparsity of $\mathcal{K}_\Psi$ is not so clear. In [7], several bases have been investigated due to the condition number. In the case of rectangular elements $\triangle_s$ and Laplacian, one can take tensor products of integrated Legendre polynomials, see e.g. [6], [20]. Then, the element stiffness matrix K_s has $\mathcal{O}(N)$ nonzero matrix entries and $\mathcal{K}_\Psi$ can be computed in $\mathcal{O}(N)$ operations via (1.3). However, in the case of a general quadrilateral element $\triangle_s$ with nonparallel opposite edges, most of the orthogonality relations of the rectangular case disappear and K_s (and hence $\mathcal{K}_\Psi$) has $\mathcal{O}(N^2)$ matrix entries. Using a quadrature rule, the cost in order to obtain $\mathcal{K}_\Psi$ is $\mathcal{O}(p^6)$. In [24], tensor products of Lagrangian polynomials on the grid of the Gauss-Lobatto points are proposed. Then, the cost for computing K_s by a quadrature rule is $\mathcal{O}(N^2)$, or $\mathcal{O}(p^4)$. This approach can be extended to the triangular case by the Duffy transformation. Here, the choice of a basis in which K_s has $\mathcal{O}(N)$ matrix entries for some elements $\triangle_s$ is more difficult.

In this paper, we will present basis functions such that the element stiffness matrix K_s has $\mathcal{O}(p^2)$ nonzero matrix entries in the case of piecewise constant coefficients $\mathcal{A}(x,y)$ on the elements $\triangle_s$ and affine linear mappings F_s. Moreover, each nonzero matrix entry can be computed in $\mathcal{O}(1)$ operations. So, the matrix vector multiplication and the generation of the stiffness matrix can be done in $\mathcal{O}(N)$ arithmetical operations.
Let us briefly motivate the construction of the basis functions which is similar to the construction for spectral elements in [12]. The difficulty in the triangular case is the construction of orthogonal polynomials on the triangle with respect to the $L_2(\hat{triangle}$ scalar product. Let $\hat{\triangle}$ be the triangle with the vertices $(-1,-1)$, $(1,-1)$ and $(0,1)$ and set

$$g_{ij}(x,y) = h_{1,i}\left(\frac{2x}{1-y}\right)(1-y)^i h_{2,j}(y), \quad i,j \geq 0, i+j \leq p,$$

where $h_{1,i}$ and $h_{2,i}$ are some polynomials of degree i specified later. In order to satisfy the orthogonality relations, the numbers

$$\begin{aligned} \gamma_{i,j,k,l} &= \int_{\hat{\triangle}} h_{1,i}\left(\frac{2x}{1-y}\right)(1-y)^i h_{2,j}(y) h_{1,k}\left(\frac{2x}{1-y}\right)(1-y)^k h_{2,l}(y)\,\mathrm{d}(x,y) \\ &= \int_{-1}^{1}\int_{\frac{y-1}{2}}^{\frac{1-y}{2}} h_{1,i}\left(\frac{2x}{1-y}\right) h_{1,k}\left(\frac{2x}{1-y}\right)(1-y)^{i+k} h_{2,l}(y) h_{2,j}(y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

have to be zero if $i \neq k$ or $j \neq l$. With the Duffy transformation $z = \frac{2x}{1-y}$, we obtain

$$\gamma_{i,j,k,l} = \frac{1}{2}\int_{-1}^{1} h_{1,i}(z) h_{1,k}(z)\,\mathrm{d}z \int_{-1}^{1} (1-y)^{i+k+1} h_{2,l}(y) h_{2,j}(y)\,\mathrm{d}y.$$

In order to obtain $\gamma_{i,j,k,l} = \delta_{ik}\delta_{jl}\tilde{\rho}_i\hat{\rho}_j$, where δ_{ik} is the Kronecker delta, we have to use polynomials of degree i which are orthogonal with respect to the weight 1, i.e. Legendre polynomials, for $h_{1,i}$. For $h_{2,i}$ polynomials which are orthogonal with respect to the weight function $(1-x)^{2i+1}$, i.e. special Jacobi polynomials, should be used. This should motivate the choice of the functions

$$\phi_{ij}(x,y) = \hat{p}_i^0\left(\frac{2x}{1-y}\right)(1-y)^i \hat{p}_j^{2i-1}(y),$$

where $\hat{p}_i^\alpha$ are the primitives over the orthogonal polynomials of degree $i-1$ with respect to $(1-x)^\alpha$, for the interior bubbles in section 3.
The direct application of this result is the fast generation of the matrix $\mathcal{K}_\Psi$ and the fast matrix vector multiplication $\mathcal{K}_\Psi \underline{u}$ in the case of piecewise constant coefficients and polygonal bounded domains in (1.1). Moreover, there is another application for preconditioning systems of linear algebraic equations arising from the p-version of the finite element method. It is well known from the literature that pcg-methods with domain decomposition preconditioners of Dirichlet-Dirichlet-type are one of the most efficient iterative solvers for systems of the type (1.2), see e.g. [15], [6], [4], [20], [18], [19], [22], [11]. Corresponding to the partition of basis functions $\Psi = [\Psi_V, \Psi_E, \Psi_I] = [\Psi_C, \Psi_I]$, i.e. $C = V \cup E$, let

$$\mathcal{K}_\Psi = \begin{bmatrix} \mathcal{K}_C & \mathcal{K}_{CI} \\ \mathcal{K}_{IC} & \mathcal{K}_I \end{bmatrix} = \begin{bmatrix} I & \mathcal{K}_{CI}\mathcal{K}_I^{-1} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathcal{S} & \mathbf{0} \\ \mathbf{0} & \mathcal{K}_I \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathcal{K}_I^{-1}\mathcal{K}_{IC} & I \end{bmatrix} \tag{1.4}$$

be the block structure of the stiffness matrix with the Schur-complement $\mathcal{S} = \mathcal{K}_C - \mathcal{K}_{CI}\mathcal{K}_I^{-1}\mathcal{K}_{IC}$. The domain decomposition preconditioner will be of the form

$$\mathcal{C} = \begin{bmatrix} I & -E^T \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} C_S & \mathbf{0} \\ \mathbf{0} & C_I \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ -E & I \end{bmatrix}, \tag{1.5}$$

where

- C_I is a preconditioner for $\mathcal{K}_I$,
- C_S is a preconditioner for the Schur-complement $\mathcal{S} = \mathcal{K}_C - \mathcal{K}_{CI}\mathcal{K}_I^{-1}\mathcal{K}_{IC}$ and
- E is the matrix representation of an extension operator acting from the edges of the elements into the interior.

Preconditioners for the Schur-complement have been proposed in [18], [19], [20], [2], [16] and [22]. For C_I, several preconditioners have been developed for the quadrilateral (hexahedron), see e.g. [20], [9], [10], [23]. The papers [6], [8], [3] and [25] deal with the extension operator for the p-version of the FEM using triangular or tetrahedral elements. In [11], see also [17], an algebraic analysis of a preconditioner of the type (1.5) is given. In this paper, we will propose a relatively simple preconditioner for $\mathcal{K}_I$ and (based on this) a matrix representation for the extension operator.
This remaining part of this paper is organized as follows. In section 2, we formulate and prove the most important properties of Jacobi polynomials and their primitives. In section 3, the shape functions on the reference triangle $\hat{\triangle}$ are defined and the main result of this paper, Theorem 3.3, is formulated. In section 4, we give two applications of this result, fast matrix evaluation and preconditioning. Section 5 is very technical, Theorem 3.3 is proved. Finally, we show some numerical experiments in section 6. In the appendix, we give the reader an impression of the computation of the nonzero matrix entries of the element stiffness matrix.
Throughout this paper, the reference triangle $\hat{\triangle}$ is the triangle with the vertices $(-1,-1)$, $(1,-1)$ and $(0,1)$. The parameter *nel* denotes the number of elements, where the parameter p denotes the polynomial degree. By F_s, we denote the isoparametric mapping from $\hat{\triangle}$ to the triangle $\triangle_s$.

2 Properties of Jacobi polynomials with weight $(1-x)^\alpha$

For the definition of our basis functions on the reference element, Jacobi polynomials are required. In this section, we will summarize the most important properties of Jacobi polynomials. We refer the reader to the books of Abramowitz and Stegun, [1], Andrews, Askey and Roy, [5], and Tricomi, [28], for more details. Moreover, we will state and prove some properties which only hold for polynomials with weight $(1-x)^\alpha$.
Let

$$p_n^\alpha(x) = \frac{1}{2^n n!(1-x)^\alpha}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((1-x)^\alpha(x^2-1)^n\right) \quad n\in\mathbb{N}_0, \alpha>-1 \tag{2.1}$$

be the $n-$th Jacobi polynomial with respect to the weight function $(1-x)^\alpha(1+x)^0$. $p_n^\alpha(x)$ is a polynomial of degree n, i.e. $p_n^\alpha \in \mathbb{P}_n((-1,1))$, where $\mathbb{P}_n$ is the space of all polynomials of degree n on the interval. In the special case $\alpha=0$, the functions are called Legendre polynomials. Moreover, let

$$\hat{p}_n^\alpha(x) = \int_{-1}^x p_{n-1}^\alpha(y)\,\mathrm{d}y \quad n\geq 1, \quad \hat{p}_0^\alpha(x)=1 \tag{2.2}$$

be the $n-$th integrated Jacobi polynomial with respect to the weight function $(1-x)^\alpha(1+x)^0$.

Lemma 2.1. *Let p_n^α and $\hat{p}_n^\alpha$ be defined via (2.1) and (2.2). Moreover, let $j,l\in\mathbb{N}_0$ and $\alpha>-1$. Then, we have*

$$p_n^\alpha(-1)=(-1)^n, \quad p_n^\alpha(1)=\begin{pmatrix} n+\alpha \\ n\end{pmatrix}, \quad \frac{\mathrm{d}}{\mathrm{d}x}p_n^\alpha(1)=\frac{n}{2}\begin{pmatrix} n+\alpha+1 \\ n\end{pmatrix}, \tag{2.3}$$

where $\begin{pmatrix} i \\ j \end{pmatrix}$ *is the binomial coefficient* i *over* j*. Moreover, the relations*

$$\begin{aligned}
p_n^\alpha(x) &= \sum_{k=0}^{n} 2^{-n} \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} n+\alpha \\ n-k \end{pmatrix} x^n + \sum_{j=0}^{n-1} \zeta_j x^j, \quad \zeta_j \in \mathbb{R}, && (2.4)\\
p_n^{\alpha-1}(x) &= \frac{1}{\alpha+2n} \left[(\alpha+n)p_n^\alpha(x) - np_{n-1}^\alpha(x)\right], && (2.5)\\
p_{n+1}^\alpha(x) &= \frac{2n+\alpha+1}{(2n+2)(n+\alpha+1)(2n+\alpha)} \left((2n+\alpha+2)(2n+\alpha)x+\alpha^2\right) p_n^\alpha(x) \\
&\quad - \frac{n(n+\alpha)(2n+\alpha+2)}{(n+1)(n+\alpha+1)(2n+\alpha)} p_{n-1}^\alpha(x), \quad n \geq 1, && (2.6)
\end{aligned}$$

and the integral relations

$$\begin{aligned}
\int_{-1}^{1} (1-x)^\alpha p_j^\alpha(x) p_l^\alpha(x)\,\mathrm{d}x &= \rho_j^\alpha \delta_{jl}, \quad \textit{where } \rho_j^\alpha = \frac{2^{\alpha+1}}{2j+\alpha+1}, && (2.7)\\
\int_{-1}^{1} (1-x)^\alpha p_j^\beta(x) q_l(x)\,\mathrm{d}x &= 0 \quad \forall q_l \in \mathbb{P}_l, \alpha-\beta \in \mathbb{N}_0, j > l+\alpha-\beta && (2.8)
\end{aligned}$$

are valid.

Proof.
- Relations (2.3) and (2.4) are direct consequences of (2.1).
- Relation (2.6) is the recurrence for the Jacobi polynomials, [1].
- Relation (2.7) is the orthogonality relation of the Jacobi polynomials.
- Relation (2.8) follows from (2.7):

$$\begin{aligned}
\int_{-1}^{1} (1-x)^\alpha p_j^\beta(x) q_l(x)\,\mathrm{d}x &= \int_{-1}^{1} (1-x)^\beta p_j^\beta(x) q_l(x) (1-x)^{\alpha-\beta}\,\mathrm{d}x \\
&= \int_{-1}^{1} (1-x)^\beta p_j^\beta(x) \tilde{q}_{l+\alpha-\beta}(x)\,\mathrm{d}x,
\end{aligned}$$

  where $\tilde{q}_{l+\alpha-\beta}(x) \in \mathbb{P}_{l+\alpha-\beta}$. Since $\int_{-1}^{1} (1-x)^\beta p_j^\beta(x) q_m(x)\,\mathrm{d}x = 0$ for $m < j$, i.e. $p_j^\beta(x)$ is orthogonal to all polynomials of degree $m < j$ with respect to the scalar product with weight $(1-x)^\beta$, this integral is zero for $j > l+\alpha-\beta$ which proves (2.8).

- Formula (2.5) can be found in e.g., [5]. In order to prove it, the polynomial $p_n^{\alpha-1}$ is represented in the basis of the Jacobi polynomials p_n^α, i.e.

$$p_n^{\alpha-1}(x) = \sum_{i=0}^{n} \gamma_i p_i^\alpha(x), \quad \gamma_i \in \mathbb{R}.$$

  Using (2.8) and (2.7) for $l < n-1$ and $\beta = \alpha - 1$, we have

$$\begin{aligned}
0 = \int_{-1}^{1} (1-x)^\alpha p_n^{\alpha-1}(x) p_l^\alpha(x)\,\mathrm{d}x &= \sum_{i=0}^{n} \gamma_i \int_{-1}^{1} (1-x)^\alpha p_i^\alpha(x) p_l^\alpha(x)\,\mathrm{d}x \\
&= \sum_{i=0}^{n} \gamma_i \delta_{il} \rho_i^\alpha = \rho_l^\alpha \gamma_l.
\end{aligned}$$

So,

$$p_n^{\alpha-1}(x) = \gamma_n p_n^\alpha(x) + \gamma_{n-1} p_{n-1}^\alpha(x).$$

In order to compute the coefficients γ_n and γ_{n-1}, we insert the values at the points $x = \pm 1$. With (2.3), we obtain

$$\gamma_n - \gamma_{n-1} = 1, \quad \binom{n+\alpha}{n}\gamma_n + \binom{n+\alpha-1}{n-1}\gamma_{n-1} = \binom{n+\alpha-1}{n}.$$

Hence, a simple computation gives

$$\gamma_n = \frac{\alpha+n}{\alpha+2n} \quad \text{and} \quad \gamma_{n-1} = -\frac{n}{\alpha+2n},$$

which proves the assertion.

□

The next lemma considers properties of the integrated Jacobi polynomials (2.2).

Lemma 2.2. *Let $l, j \in \mathbb{N}_0$. Let p_n^α and $\hat{p}_n^\alpha$ be defined via (2.1) and (2.2). Then, the recurrence relations*

$$\begin{aligned}
\hat{p}_n^\alpha(-1) &= 0, \quad n \geq 1, &(2.9)\\
\hat{p}_n^\alpha(x) &= \frac{2n+2\alpha}{(2n+\alpha-1)(2n+\alpha)} p_n^\alpha(x) + \frac{2\alpha}{(2n+\alpha-2)(2n+\alpha)} p_{n-1}^\alpha(x)\\
&\quad - \frac{2n-2}{(2n+\alpha-1)(2n+\alpha-2)} p_{n-2}^\alpha(x), \quad n \geq 2, &(2.10)\\
\hat{p}_{n+1}^\alpha(x) &= \frac{2n+\alpha-1}{(2n+2)(n+\alpha)(2n+\alpha-2)} \left((2n+\alpha-2)(2n+\alpha)x + \alpha(\alpha-2)\right) \hat{p}_n^\alpha(x)\\
&\quad - \frac{(n-1)(n+\alpha-2)(2n+\alpha)}{(n+1)(n+\alpha)(2n+\alpha-2)} \hat{p}_{n-1}^\alpha(x), \quad n \geq 1, &(2.11)\\
\hat{p}_n^\alpha(x) &= \frac{1}{2n+\alpha-1}\left(p_n^{\alpha-1}(x) + p_{n-1}^{\alpha-1}(x)\right), \quad n \geq 1 &(2.12)
\end{aligned}$$

and the integral relations

$$\begin{aligned}
\int_{-1}^{1} (1-x)^\alpha \hat{p}_j^\alpha(x) \hat{p}_l^\alpha(x)\,\mathrm{d}x &= 0 \quad \text{if } |j-l| > 2, &(2.13)\\
\int_{-1}^{1} (1-x)^\alpha \hat{p}_j^{\beta+1}(x) q_l(x)\,\mathrm{d}x &= 0 \quad \forall q_l \in \mathbb{P}_l, \alpha-\beta \in \mathbb{N}_0, j > l+1+\alpha-\beta &(2.14)
\end{aligned}$$

are valid.

Proof. Relation (2.9) is a direct consequence of (2.2). In order to show (2.10), we have

$$\begin{aligned}
\int (1-x)^\alpha p_l^\alpha(x)\,\mathrm{d}x &= \frac{1}{2^l l!} \int \frac{\mathrm{d}^l}{\mathrm{d}x^l}\left[(x^2-1)^l (1-x)^\alpha\right]\,\mathrm{d}x\\
&= \frac{1}{2^l l!} \frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}}\left[(x^2-1)^l (1-x)^\alpha\right] + C
\end{aligned}$$

by (2.1). Thus,

$$\int (1-x)^\alpha p_l^\alpha(x)\,\mathrm{d}x = (1-x)^{\alpha+1}(1+x) q_{l-1}(x) + C, \qquad (2.15)$$

where $q_{l-1} \in \mathbb{P}_{l-1}$ and C is some constant. Now, we compute $m_{jl}^{(0)} = \int_{-1}^{1}(1-x)^\alpha \hat{p}_j^\alpha(x) p_l^\alpha(x)\,\mathrm{d}x$ by partial integration. Due to (2.8), $m_{jl}^{(0)} = 0$ for $j < l$. Using (2.15), (2.2) and $\alpha > -1$, one obtains

$$\begin{aligned} m_{jl}^{(0)} &= (1-x)^{\alpha+1}(1+x)q_{l-1}(x)\hat{p}_j^\alpha(x)\Big|_{-1}^{1} - \int_{-1}^{1}(1-x)^{\alpha+1}(1+x)q_{l-1}(x)p_{j-1}^\alpha(x)\,\mathrm{d}x \\ &= \int_{-1}^{1}(1-x)^\alpha(1-x^2)q_{l-1}(x)p_{j-1}^\alpha(x)\,\mathrm{d}x. \end{aligned}$$

The function $(1-x^2)q_{l-1}(x)$ is a polynomial of degree $l+1$. Using (2.8), $m_{jl}^{(0)} = 0$ for $j > l+2$. Hence as in the proof of relation (2.5), one can conclude

$$\hat{p}_n^\alpha(x) = \beta_n p_n^\alpha(x) + \beta_{n-1}p_{n-1}^\alpha(x) + \beta_{n-2}p_{n-2}(x) \tag{2.16}$$

with coefficients β_n, β_{n-1} and β_{n-2}. Using (2.4), one obtains $\beta_n = \frac{2n+2\alpha}{(2n+\alpha-1)(2n+\alpha)}$. The numbers β_{n-1} and β_{n-2} can be computed by inserting $x = \pm 1$ into (2.16) and using (2.3) and (2.9).
In order to prove (2.12), we start from (2.10) and obtain

$$\begin{aligned} \hat{p}_n^\alpha(x) &= \frac{n+\alpha}{2n+\alpha}\frac{p_n^\alpha(x)}{2n+\alpha-1} + \frac{2\alpha p_{n-1}^\alpha(x)}{(2n+\alpha-1)(2n+\alpha)} - \frac{n-1}{2n+\alpha-2}\frac{p_{n-2}^\alpha(x)}{2n+\alpha-1} \\ &= \frac{p_n^{\alpha-1}(x)}{2n+\alpha-1} + \frac{p_{n-1}^{\alpha-1}(x)}{2n+\alpha-1} \end{aligned}$$

by using two times relation (2.5). Relation (2.11) follows from (2.10) and (2.6), whereas relation (2.13) follows from (2.10) and (2.7). Relation (2.14) follows from (2.12). □

The most important results are the formulas (2.10) and (2.6). With relation (2.6), we are able to compute recursively function values of the Jacobi polynomials, relation (2.10) gives a simple formula between the Jacobi and integrated Jacobi polynomials.
Finally, we need two properties of integrated Legendre polynomials, i.e. $\hat{p}_n^0$. Using (2.7), one obtains

$$\hat{p}_n^0(1) = 0 \quad \text{for } n \geq 2. \tag{2.17}$$

A direct consequence of (2.10) with $\alpha = 0$ is

$$(1-y)\hat{p}_{j-1}^0(y) = -\frac{j-1}{2j-3}\hat{p}_{j-2}^0(y) + \hat{p}_{j-1}^0(y) - \frac{j}{2j-3}\hat{p}_j^0(y), \quad j \geq 3. \tag{2.18}$$

3 Element stiffness matrix

In this section, we define the shape functions on the reference element. Then, we will formulate the main theorem of this section in which we state that the element stiffness matrix has about $25/2p^2$ nonzero matrix entries. The parameter p denotes the polynomial degree.

3.1 Definition of the shape functions

Let $\hat{\triangle}$ be the reference triangle with the vertices $(-1,-1)$, $(1,-1)$ and $(0,1)$ and the edges $e_2 = \{(x,y) \in \mathbb{R}^2, -1 \leq y \leq 1, 2x = 1-y\}$, $e_3 = \{(x,y) \in \mathbb{R}^2, -1 \leq y \leq 1, -2x = 1-y\}$ and $e_1 = \{(x,y)) \in \mathbb{R}^2, -1 \leq x \leq 1, y = -1\}$.
Using the polynomials (2.2), we define the shape functions on the element $\hat{\triangle}$:

- Vertex functions:

$$\phi_{V,1/2}(x,y) = \frac{1 \pm 2x - y}{4} \quad \text{and} \quad \phi_{V,3}(x,y) = \frac{1+y}{2}. \tag{3.1}$$

Let $\Phi_V = [\phi_{V,1}, \phi_{V,2}, \phi_{V,3}]$ be the basis of the vertex functions.

- Edge bubble functions: For the edge e_1, we define

$$\phi_{e_1,i}(x,y) = \hat{p}_i^0\left(\frac{2x}{1-y}\right)(1-y)^i, \quad 2 \le i \le p. \tag{3.2}$$

For the remaining two edges e_2 and e_3, the definition is different:

$$\phi_{e_2/e_3,i}(x,y) = \frac{1 \pm 2x - y}{2}\hat{p}_i^0(y), \quad 1 \le i \le p-1. \tag{3.3}$$

By $\Phi_{e_k} = [\phi_{e_k,i}]_{i=2}^p$, $k = 1,2,3$, we denote the bases of the edge bubble functions on the edges e_1, e_2 and e_3 and $\Phi_E = [\Phi_{e_1}, \Phi_{e_2}, \Phi_{e_3}]$ is the basis of all edge bubble functions.

- Interior bubbles: Here, the functions

$$\phi_{ij}(x,y) = \hat{p}_i^0\left(\frac{2x}{1-y}\right)(1-y)^i\hat{p}_j^{2i-1}(y), \quad i+j \le p, i \ge 2, j \ge 1, \tag{3.4}$$

are used. Moreover, $\Phi_I = [\phi_{ij}]_{i \ge 2, j \ge 1}^{i+j \le p}$ denotes the basis of the interior bubbles.

- Finally, let $\Phi = [\Phi_V, \Phi_E, \Phi_I]$ be the set of all shape functions on $\hat{\triangle}$.

The most important properties are summarized in the following proposition.

Proposition 3.1.

- *The functions* (3.4) *are polynomials of degree* $i+j$. *Moreover,* $\phi_{ij}(x,y) = 0$ *holds for* $(x,y) \in e_k$ *with* $k = 1,2,3$. *So,* $\phi_{ij}(\partial\hat{\triangle}) = 0$, $i \ge 2$, $j \ge 1$.

- *The edge bubble functions* (3.2), (3.3) *are polynomials of degree* p *and satisfy the property*

$$\phi_{e_k,i}(e_j) = 0, \quad \text{if } j \ne k, \max\{3-k,1\} \le i \le \max\{p-1, p+1-k\}.$$

- *The vertex functions* (3.1) *are the usual hat functions.*

- *The functions* (3.1), (3.2), (3.3), (3.4) *span a basis in the space of all polynomials of degree* p.

Proof. We start with the first assertion: Since $\hat{p}_i^0$ is a polynomial of degree i, $\hat{p}_i^0\left(\frac{2x}{1-y}\right)(1-y)^i$ is a polynomial of degree i in x and y. Let $\begin{bmatrix} x \\ y \end{bmatrix} \in e_1$. Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad t \in [-1,1].$$

By (3.4), we have $\phi_{ij}(x,y) = \hat{p}_i^0(\frac{2t}{2})2^i\hat{p}_j^{2i-1}(-1)$. Using (2.9) with $\alpha = 2i-1$, one easily concludes $\phi_{ij}(x,y) = 0$ for $(x,y) \in e_1$.
Let $\begin{bmatrix} x \\ y \end{bmatrix} \in e_2$. Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + \frac{t}{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad t \in [-1,1].$$

Using (3.4) and (2.17), we have

$$\phi_{ij}(x,y) = \hat{p}_i^0\left(\frac{2(1/2-t/2)}{1-t}\right)(1-t)^i\hat{p}_j^{2i-1}(t) = \hat{p}_i^0(1)(1-t)^i\hat{p}_j^{2i-1}(t) = 0,$$

which proves the assertion that $\phi_{ij}(x,y) = 0$ for $(x,y) \in e_2$.
By the same arguments, the assertion $\phi_{ij}(x,y) = 0$ for $(x,y) \in e_3$ and the second assertion can be proved. The third assertion is trivial. The last assertion follows from the first three assertions and the linear independence of the functions (3.1), (3.2), (3.3), and (3.4). □

Remark 3.2. *Since, $\hat{p}_0^\alpha(y) \equiv 1$, the edge bubble functions Φ_{e_1} (3.2) can be written as $\phi_{e_1,i}(x,y) = \phi_{i,0}(x,y)$, $i = 2,\ldots,p$.*

3.2 Properties of the element stiffness matrix

Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and let

$$\hat{K} = \int_{\hat{\triangle}} (\nabla\Phi(x,y))^T A \nabla\Phi(x,y)\,\mathrm{d}(x,y) \tag{3.5}$$

is the stiffness matrix on $\hat{\triangle}$ with respect to the basis Φ. According to the partitioning of the basis Φ into vertex functions, edge bubble functions and interior bubble functions, the matrix $\hat{K}$ can be split into 3×3 blocks, i.e.

$$\hat{K} = \begin{bmatrix} \hat{K}_{V,(A)} & \hat{K}_{V,E,(A)} & \hat{K}_{V,I,(A)} \\ \hat{K}_{E,V,(A)} & \hat{K}_{E,(A)} & \hat{K}_{E,I,(A)} \\ \hat{K}_{I,V,(A)} & \hat{K}_{I,E,(A)} & \hat{K}_{I,(A)} \end{bmatrix}, \tag{3.6}$$

where the matrix

$$\hat{K}_{I,(A)} = [a_{ij,kl}]_{(i,j);(k,l)} = \left[\int_{\hat{\triangle}} (\nabla\phi_{ij}(x,y))^T A \nabla\phi_{kl}(x,y)\,\mathrm{d}(x,y)\right]_{(i,j);(k,l)} \tag{3.7}$$

is the block of the interior bubbles.
Now, we are able to formulate the main theorem of this section.

Theorem 3.3. *Let $\hat{K}$ be defined via (3.5)-(3.7). Then, the matrix $\hat{K}$ has $\mathcal{O}(p^2)$ nonzero matrix entries. More precisely, $a_{ij,kl} = 0$ if $|i-k| > 2$ or $|i-k+j-l| > 2$. In the special case of the matrix $\hat{K}_{I,(I)}$, i.e. A is the identity matrix, the entries of $a_{ij,kl}$ are zero if $|i-k| \notin \{0,2\}$ or $|i-k-l+j| > 2$.*

A detailed proof of this theorem will be given in section 5.
Now, we will extend this result to more general triangles $\triangle_s$. Let $\triangle_s$ be a triangle and let $F_s : \hat{\triangle} \mapsto \triangle_s$ be the affine linear mapping from the reference element to $\triangle_s$. On $\triangle_s$, we define the shape functions

$$\Phi_s = \Phi \circ F_s^{-1}. \tag{3.8}$$

Let

$$K_s = \int_{\triangle_s} (\nabla\Phi_s(x,y))^T A \nabla\Phi_s(x,y)\,\mathrm{d}(x,y) \tag{3.9}$$

be the stiffness matrix with respect to the basis Φ_s.

Theorem 3.4. *Let K_s be defined via* (3.9)*. Then, the matrix K_s has $\mathcal{O}(p^2)$ nonzero matrix entries.*

Proof. Since F_s is an affine linear mapping, F_s^{-1} is affine linear too, and ∇F_s^{-1} is constant. By the transformation F_s^{-1} from $\triangle_s$ to $\hat{\triangle}$ and (3.8), we have

$$\begin{aligned} K_s &= \int_{\triangle_s} (\nabla \Phi_s(x,y))^T A \nabla \Phi_s(x,y) \,\mathrm{d}(x,y) \\ &= \int_{\hat{\triangle}} \left[\nabla \Phi(\hat{x},\hat{y})\right]^T (\nabla F_s^{-1})^T A \nabla F_s^{-1} |\det(\nabla F_s)| \nabla \Phi(\hat{x},\hat{y}) \,\mathrm{d}(\hat{x},\hat{y}) \\ &= \int_{\hat{\triangle}} \left[\nabla \Phi(\hat{x},\hat{y})\right]^T A_s \nabla \Phi(\hat{x},\hat{y}) \,\mathrm{d}(\hat{x},\hat{y}), \end{aligned} \tag{3.10}$$

where $A_s = |\det(\nabla F_s)| (\nabla F_s^{-1})^T A \nabla F_s^{-1}$ is a constant positive definite matrix. Using Theorem 3.3, the assertion follows. □

3.3 Modification of the edge bubble functions on the edges e_2 and e_3.

With the definition of the functions (3.2), (3.3), the edge bubble basis functions restricted to an edge are integrated Legendre polynomials, i.e. $\hat{p}_j^0(x), j = 2, \ldots, p$ on the edge e_1 and functions $(1-x)\hat{p}_j^0(x)$, $j = 1, \ldots, p-1$, on the remaining two edges e_2 and e_3. Globally, we will use integrated Legendre polynomials on each edge in order to ensure that our basis functions are globally continuous. So, the basis functions can be extended much simpler to a neighboring element. More precisely, let

$$\tilde{\phi}_{e_2/e_3,j}(x,y) = \frac{1 \pm 2x - y}{2 - 2y} \hat{p}_j^0(y), \quad j = 2, \ldots, p. \tag{3.11}$$

Since $\hat{p}_j^0(1) = 0$, the functions in (3.11) are polynomials of degree p. Moreover, $\tilde{\phi}_{e_k,j}(x,y) = \hat{p}_j^0(y)$ for $(x,y) \in e_k$, $k = 2, 3$. So, these functions are integrated Legendre polynomials on the edges e_2 and e_3. Between the bases (3.11) and (3.3), one has a basis transformation, i.e.

$$\tilde{\Phi}_{e_k} = \left[\tilde{\phi}_{e_k,2}(x,y), \ldots, \tilde{\phi}_{e_k,p}(x,y)\right] = \left[\phi_{e_k,1}(x,y), \ldots, \phi_{e_k,p-1}(x,y)\right] W, \quad k = 2, 3. \tag{3.12}$$

Lemma 3.5. *Let W be defined via* (3.12)*. Then,*

$$W = \begin{bmatrix} -2 & 0 & 0 & \ldots & & & \\ 1 & -1 & 0 & \ldots & & & \\ -\frac{1}{5} & 1 & -\frac{4}{5} & & 0 & \ldots & 0 \\ & -\frac{2}{7} & 1 & & -\frac{5}{7} & 0 & 0 \\ \vdots & & & & \ddots & & \\ & \ldots & 0 & -\frac{p-3}{2p-3} & & 1 & -\frac{p}{2p-3} \end{bmatrix}.$$

Proof. The assertion follows from relation (2.18). □

Now, let

$$\tilde{\Phi} = \left[\Phi_V, \tilde{\Phi}_E, \Phi_I\right] \quad \text{with} \quad \tilde{\Phi}_E = \left[\Phi_{e_1}, \tilde{\Phi}_{e_2}, \tilde{\Phi}_{e_3}\right] \tag{3.13}$$

and

$$\tilde{K} = \int_{\hat{\triangle}} (\nabla \tilde{\Phi}(x,y))^T A \nabla \tilde{\Phi}(x,y) \,\mathrm{d}(x,y), \quad \tilde{K}_s = \int_{\triangle_s} (\nabla \tilde{\Phi}_s(x,y))^T A \nabla \tilde{\Phi}_s(x,y) \,\mathrm{d}(x,y). \tag{3.14}$$

Theorem 3.6. *Let $\tilde{K}$ and $\tilde{K}_s$ be defined via* (3.14)*. Then,*

$$\tilde{K} = \begin{bmatrix} I & & \\ & W_E^T & \\ & & I \end{bmatrix} K \begin{bmatrix} I & & \\ & W_E & \\ & & I \end{bmatrix} \quad \textit{with} \quad W_E = \ \mathrm{blockdiag}\,[I, W, W]\,. \tag{3.15}$$

Moreover, the multiplications $\tilde{K}\underline{u}$ and $\tilde{K}_s\underline{u}$ can be done in $\mathcal{O}(p^2)$ operations.

Proof. The first assertion follows from (3.5), (3.14) and (3.12). The second assertion is a consequence of Theorem 3.3 and Lemma 3.5 for $\tilde{K}$ and Theorem 3.4 and Lemma 3.5 for $\tilde{K}_s$. □

4 Applications

In this section, we give two applications of the results of section 3 for the discretization of the boundary value problem (1.1) using for the *hp*-version of the finite element method.

4.1 Fast *hp*-FEM for straight triangles with sparse matrices

In this subsection, let us assume that

- the domain Ω is bounded by polygons and
- the matrix $\mathcal{A}(x,y)$ is piecewise constant on subdomains $\Omega_i \subset \Omega$ which are bounded by polygons, too.

In order to find an approximate solution of (1.1), we use a finite element mesh of triangles $\triangle_s$, which satisfy the following two properties:

- the matrix $\mathcal{A}(x,y)$ is constant on $\triangle_s$,
- the edges of the element $\triangle_s$ are straight lines.

Let $F_s : \hat{\triangle} \mapsto \triangle_s$ be the affine linear mapping. On each element, we take the basis $\tilde{\Phi}_s$, i.e.

$$\tilde{\Phi}_s = \tilde{\Phi} \circ F_s^{-1}, \tag{4.1}$$

cf. (3.8) and (3.13) with polynomial degree p_s. Let Ψ be the corresponding global basis and let $p = \max_s p_s$ be the maximal polynomial degree over all triangles $\triangle_s$.

Theorem 4.1. *Let us assume that each element $\triangle_s$ of the triangulation is bounded by straight lines and that $\mathcal{A}(x,y)$ is constant on each element. Let $\mathcal{K}_\Psi$ be defined in* (1.2), (1.3)*. Then, the operation $\mathcal{K}_\Psi\underline{u}$ requires $\mathcal{O}(p^2)$ operations. Moreover, the generation of $\mathcal{K}_\Psi$ requires $\mathcal{O}(p^2)$ operations.*

Proof. The first assertion follows from Theorem 3.6. Concerning the second assertion, the method in order to derive explicit formulas for the nonzero matrix entries of $\hat{K}$, see (3.6), is explained in the appendix. Using (3.10) and (3.15), the assertion follows. □

Summarizing, we have proposed a method, in which the cost for the generation of the stiffness matrix $\mathcal{K}_\Psi$ and for the multiplication $\mathcal{K}_\Psi\underline{u}$ is proportionally to the number of unknowns with respect to the polynomial degree p.

4.2 A simple preconditioner for the interior bubbles

In subsection 4.1, we have investigated the special case of a polygonal bounded domain Ω and piecewise constant coefficients $\mathcal{A}(x, y)$. In the case of curved boundaries, it is well known from the literature to use curved triangles on the boundary of Ω. Then, the mapping $F_s : \hat{\triangle} \mapsto \triangle_s$ is nonlinear on some triangles $\triangle_s$. Hence, the matrix A_s in (3.10) is not constant on $\hat{\triangle}$. So, we are not able to apply Theorem 3.4 and the element stiffness matrix A_s has $\mathcal{O}(p^4)$ matrix entries. Thus, the operation $\mathcal{K}_\Psi \underline{u}$ requires $\mathcal{O}(p^4)$ operations.
However, we can derive a preconditioner for the block of the interior bubbles which is efficient for this case. Corresponding to the partition of basis functions $\Psi = [\Psi_V, \Psi_E, \Psi_I] = [\Psi_C, \Psi_I]$, i.e. $C = V \cup E$, let

$$\mathcal{K}_\Psi = \begin{bmatrix} \mathcal{K}_C & \mathcal{K}_{CI} \\ \mathcal{K}_{IC} & \mathcal{K}_I \end{bmatrix}$$

be the block structure of the stiffness matrix, see (1.4).
Now, we will derive a preconditioner of the form (1.5) for the matrix $\mathcal{K}_\Psi$. By (1.3), the global stiffness matrix is the result of assembling local stiffness matrices $\tilde{K}_s$, i.e. $\mathcal{K}_\Psi = \sum_{s=1}^{nel} R_s^T \tilde{K}_s R_s$. In the following, we investigate the preconditioner

$$\mathcal{C}_0 = \sum_{s=1}^{nel} L_s C_0 L_s^T, \tag{4.2}$$

where

$$C_0 = \int_{\hat{\triangle}} (\nabla \tilde{\Phi}(x, y))^T \nabla \tilde{\Phi}(x, y) \, \mathrm{d}(x, y).$$

In this preconditioner, the stiffness matrix for the Laplacian on the reference element is assembled on each element. Then, it can be shown, see e.g. [20], that $\kappa({\mathcal{C}_0}^{-\frac{1}{2}} \mathcal{K}_\Psi {\mathcal{C}_0}^{-\frac{1}{2}}) = \mathcal{O}(1)$ under the assumption that the angles of all triangles are distinct from 0 and π. According to (1.4), we consider a block decomposition of $\mathcal{C}_0$, i.e.

$$\mathcal{C}_0 = \begin{bmatrix} \mathcal{C}_C & \mathcal{C}_{CI} \\ \mathcal{C}_{IC} & \mathcal{C}_I \end{bmatrix}. \tag{4.3}$$

By (4.2), the matrix $\mathcal{C}_I$ is a block diagonal matrix, one block corresponds to the interior bubbles of one element. Due to (3.7), (4.2), (3.14) and (3.13), each block is equal to the matrix $\hat{K}_{I,(I)}$, i.e. $\mathcal{C}_I = \text{blockdiag}\left[\hat{K}_{I,(I)}\right]_{s=1}^{nel}$. By Theorem 3.3, the matrix $\hat{K}_{I,(I)}$ has a special nonzero pattern which is displayed in Figure 1. This pattern is similar to a stencil structure. The maximal bandwidth is $2p - 2$. So, the Cholesky factorization can be computed in $\mathcal{O}(p^4)$ operations. Using a reordering of the unknowns, i.e. the method of minimal degree or the method of nested disection, [13], this arithmetical cost can be reduced to $\mathcal{O}(p^3)$, [14]. Figure 2 displays the nonzero pattern of several Cholesky factors. Due to (1.5), we propose now the preconditioner

$$\mathcal{C}_1 = \begin{bmatrix} I & \mathcal{C}_{CI}\mathcal{C}_I^{-1} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} C_S & \mathbf{0} \\ \mathbf{0} & \mathcal{C}_I \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathcal{C}_I^{-1}\mathcal{C}_{IC} & I \end{bmatrix}, \tag{4.4}$$

where C_S is a preconditioner for the Schur complement and $\mathcal{C}_I$ and $\mathcal{C}_{IC}$ are taken from (4.3). We summarize the above observations in the following theorem.

Theorem 4.2. *Let $\mathcal{C}_1$ be defined via (4.4). Moreover, let C_S be a preconditioner for the Schur complement such that $C_S^{-1}\underline{v}$ requires not more than $\mathcal{O}(p^3)$ operations and*

$$c_1\,(C_S\underline{v}, \underline{v}) \le (S\underline{v}, \underline{v}) \le c_2\,(C_S\underline{v}, \underline{v}) \quad \forall \underline{v}$$

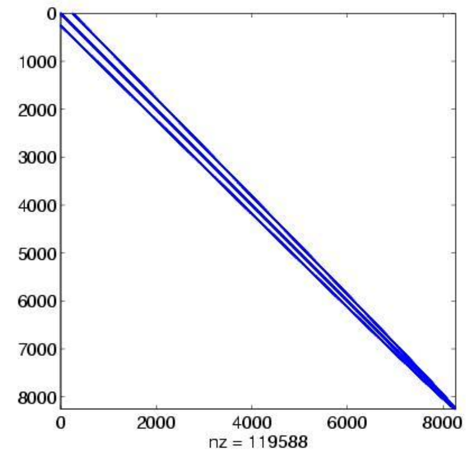

Figure 1: Nonzero pattern of $\hat{K}_{I,(I)}$ for $p = 130$.

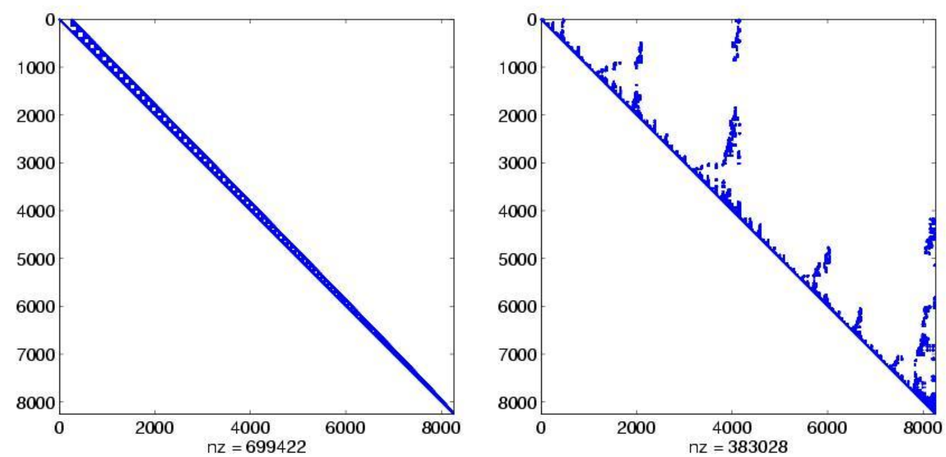

Figure 2: Nonzero pattern of the Cholesky factor of $\hat{K}_{I,(I)}$ for $p = 130$: without permutation of the unknowns (left) and with minimal degree permutation (right).

and some constants c_1, c_2. Then, $\kappa(\mathcal{C}_1{}^{-\frac{1}{2}}\mathcal{K}_\Psi\mathcal{C}_1{}^{-\frac{1}{2}}) = \mathcal{O}(\frac{c_2}{c_1})$. The operation $\mathcal{C}_1^{-1}\underline{u}$ requires $\mathcal{O}(p^3)$ operations.

Remark 4.3. *1. So, we have to use a preconditioner for the Schur-complement in the basis of the integrated Legendre polynomials which requires not more than $\mathcal{O}(p^3)$ operations. Beginning with the Schur complement preconditioners developed in [6], several preconditioners have been proposed in the literature, see e.g., [20], or [22] which is based on the multi-resolution analysis done in [10]. The cost of $C_S^{-1}\underline{v}$ is of order p.*

2. The cost of $\mathcal{C}_1^{-1}\underline{u}$ is $\mathcal{O}(p^3)$ and not $\mathcal{O}(p^2)$, which would be optimal. However, the matrix vector-multiplication requires, in general, $\mathcal{O}(p^4)$ operations. So, this preconditioner is efficient enough. The most expensive operation of the preconditioner (4.4) *is the computation of the Cholesky decomposition for $\hat{K}_{I,(I)}$, which have to be done only one times before starting the pcg-method.*

5 Proof of Theorem 3.3.

In this section, we will prove the main theorem of this paper. We will split the proof into several auxiliary results. In a first lemma, we give a formula for the derivative of the interior bubble functions (3.4).

Lemma 5.1. *Let ϕ_{ij} be defined via* (3.4)*. Then,*

$$\nabla\phi_{ij} = \begin{bmatrix} 2p_{i-1}^0\left(\frac{2x}{1-y}\right)(1-y)^{i-1}\hat{p}_j^{2i-1}(y) \\ p_{i-2}^0\left(\frac{2x}{1-y}\right)(1-y)^{i-1}\hat{p}_j^{2i-1}(y) + \hat{p}_i^0\left(\frac{2x}{1-y}\right)(1-y)^{i}p_{j-1}^{2i-1}(y) \end{bmatrix}. \tag{5.1}$$

Proof. Using (2.2), one easily derives the expression for $\frac{\partial\phi_{ij}}{\partial x}$. In order to compute $\frac{\partial\phi_{ij}}{\partial y}$, we use the product rule and obtain

$$\begin{aligned} \frac{\partial\phi_{ij}}{\partial y}(x,y) &= p_{i-1}^0\left(\frac{2x}{1-y}\right)2x(1-y)^{i-2}\hat{p}_j^{2i-1}(y) \\ &\quad -i\hat{p}_i^0\left(\frac{2x}{1-y}\right)(1-y)^{i-1}\hat{p}_j^{2i-1}(y) \\ &\quad +\hat{p}_i^0\left(\frac{2x}{1-y}\right)(1-y)^{i}p_{j-1}^{2i-1}(y). \end{aligned} \tag{5.2}$$

Using (2.6), we have

$$\frac{2x}{1-y}p_{i-1}^0\left(\frac{2x}{1-y}\right) = \frac{i}{2i-1}p_i^0\left(\frac{2x}{1-y}\right) + \frac{i-1}{2i-1}p_{i-2}^0\left(\frac{2x}{1-y}\right). \tag{5.3}$$

Moreover, by (2.10), one easily derives

$$-i\hat{p}_i^0\left(\frac{2x}{1-y}\right) = -\frac{i}{2i-1}\left[p_i^0\left(\frac{2x}{1-y}\right) - p_{i-2}^0\left(\frac{2x}{1-y}\right)\right]. \tag{5.4}$$

Inserting (5.4) and (5.3) into (5.2) proves the assertion. □

In a second step, we will determine the nonzero structure of several weighted mass matrices $M_1,\ldots,M_8$ with respect to the polynomials (2.1) and (2.2) on the unit interval $I=(-1,1)$. All these matrices have a banded structure and all matrix entries depend on the parameter i.

Lemma 5.2. *Let*

$$M_1 = \left[m_{j,l}^{(1,i)}\right]_{j,l} = \left[\int_{-1}^{1} (1-y)^{2i-2} \hat{p}_j^{2i-3}(y) \hat{p}_l^{2i-1}(y)\, \mathrm{d}y\right]_{j,l}.$$

Then, $m_{j,l}^{(1,i)} = 0$ *if* $j > l+1$ *or* $l > j+3$.

Proof. By (2.14) with $\alpha = \beta = 2i-2$, we can conclude $m_{j,l}^{(1,i)} = 0$ for $j > l+1$. In the case $l > j+3$, we use (2.14) with $\alpha = 2i-2$ and $\beta = 2i-4$. □

Lemma 5.3. *Let*

$$M_2 = \left[m_{j,l}^{(2)}\right]_{j,l} = \left[\int_{-1}^{1} (1-y)^{2i-1} \hat{p}_j^{2i-1}(y) p_l^{2i-3}(y)\, \mathrm{d}y\right]_{j,l}.$$

Then, $m_{j,l}^{(2)} = 0$ *if* $j > l+2$ *or* $l > j+2$.

Proof. The proof is similar to the proof of Lemma 5.1. □

Lemma 5.4. *Let*

$$M_3 = \left[m_{j,l}^{(3)}\right]_{j,l} = \left[\int_{-1}^{1} (1-y)^{2i+1} \hat{p}_j^{2i-1}(y) p_l^{2i+1}(y)\, \mathrm{d}y\right]_{j,l}.$$

Then, $m_{j,l}^{(3)} = 0$ *if* $j > l+4$ *or* $l > j$.

Proof. Using (2.7), we have $m_{j,l}^{(3)} = 0$ for $l > j$. If $j > l+4$, we use (2.14). □

Lemma 5.5. *Let* $j, l \geq 0$ *and*

$$M_4 = \left[m_{j,l}^{(4)}\right]_{j,l} = \left[\int_{-1}^{1} (1-y)^{2i+1} p_j^{2i-1}(y) p_l^{2i-1}(y)\, \mathrm{d}y\right]_{j,l}.$$

Then, $m_{j,l}^{(4)} = 0$ *if* $j > l+2$ *or* $l > j+2$.

Proof. The assertion follows from relation (2.8) with $\alpha = 2i+1$ and $\beta = 2i-1$. □

Lemma 5.6. *Let* $j, l \geq 0$ *and*

$$M_5 = \left[m_{j,l}^{(5)}\right]_{j,l} = \left[\int_{-1}^{1} (1-y)^{2i-1} p_j^{2i-1}(y) p_l^{2i-5}(y)\, \mathrm{d}y\right]_{j,l}.$$

Then, $m_{j,l}^{(5)} = 0$ *if* $j > l$ *or* $l > j+4$.

Proof. Using (2.5), we have

$$\begin{aligned}
p_j^{2i-2}(x) &= \frac{1}{2i+2j-1}\left[(2i-1+j)p_j^{2i-1}(x) - jp_{j-1}^{2i-1}(x)\right], \\
p_j^{2i-3}(x) &= \frac{1}{2i+2j-2}\left[(2i-2+j)p_j^{2i-2}(x) - jp_{j-1}^{2i-2}(x)\right], \\
p_j^{2i-4}(x) &= \frac{1}{2i+2j-3}\left[(2i-3+j)p_j^{2i-3}(x) - jp_{j-1}^{2i-3}(x)\right], \quad \text{and} \\
p_j^{2i-5}(x) &= \frac{1}{2i+2j-4}\left[(2i-4+j)p_j^{2i-4}(x) - jp_{j-1}^{2i-4}(x)\right].
\end{aligned}$$

So, the polynomial $p_j^{2i-5}(x)$ can be represented as a linear combination of $p_j^{2i-1}(x)$, $p_{j-1}^{2i-1}(x)$, $p_{j-2}^{2i-1}(x)$, $p_{j-3}^{2i-1}(x)$ and $p_{j-4}^{2i-1}(x)$. Using (2.7), the assertion follows. $\square$

Lemma 5.7. *Let $j,l \geq 0$ and*

$$M_6 = \left[m_{j,l}^{(6)}\right]_{j,l} = \left[\int_{-1}^{1} (1-y)^{2i} \hat{p}_j^{2i-1}(y) p_l^{2i-1}(y) \, \mathrm{d}y\right]_{j,l}.$$

Then, $m_{j,l}^{(6)} = 0$ if $j > l+3$ or $l > j+1$.

Proof. The assertion follows from relation (2.8) with $\alpha = 2i$ and $\beta = 2i-1$ and relation (2.14) with $\alpha = 2i$ and $\beta = 2i-2$. $\square$

Lemma 5.8. *Let $j,l \geq 0$ and*

$$M_7 = \left[m_{j,l}^{(7)}\right]_{j,l} = \left[\int_{-1}^{1} (1-y)^{2i-1} \hat{p}_j^{2i-1}(y) \hat{p}_l^{2i-1}(y) \, \mathrm{d}y\right]_{j,l}.$$

Then, $m_{j,l}^{(7)} = 0$ if $j > l+2$ or $l > j+2$.

Proof. The assertion follows from (2.14) with $\alpha = 2i-1$ and $\beta = 2i-2$. $\square$

Lemma 5.9. *Let $j,l \geq 0$ and*

$$M_8 = \left[m_{j,l}^{(8)}\right]_{j,l} = \left[\int_{-1}^{1} (1-y)^{2i-2} \hat{p}_j^{2i-1}(y) p_l^{2i-5}(y) \, \mathrm{d}y\right]_{j,l}.$$

Then, $m_{j,l}^{(8)} = 0$ if $j > l+1$ or $l > j+3$.

Proof. The assertion follows from (2.14) with $\alpha = \beta = 2i-2$ and (2.8) with $\alpha = 2i-2$ and $\beta = 2i-5$. $\square$

Now, we consider the matrix

$$\hat{K}_{I,x} = \left[a_{ij,kl}^{(x)}\right]_{(i,j);(k,l)} = \left[\int_{\hat{\triangle}} \frac{\partial \phi_{ij}(x,y)}{\partial x} \frac{\partial \phi_{kl}(x,y)}{\partial x} \, \mathrm{d}(x,y)\right]_{(i,j);(k,l)}. \tag{5.5}$$

Lemma 5.10. *Let $\hat{K}_{I,x}$ be defined via relation* (5.5). *Then, $a_{ij,kl}^{(x)} = 0$ if $i \neq k$ and $|j-l| > 2$.*

Proof. Using (5.5), (5.1) and the substitution $z = \frac{2x}{1-y}$, we obtain

$$\begin{aligned} a_{ij,kl}^{(x)} &= \int_{\hat{\triangle}} 2p_{i-1}^0\left(\frac{2x}{1-y}\right)(1-y)^{i-1}\hat{p}_j^{2i-1}(y) 2p_{k-1}^0\left(\frac{2x}{1-y}\right)(1-y)^{k-1}\hat{p}_l^{2k-1}(y) \, \mathrm{d}(x,y) \\ &= \int_{-1}^{1}\int_{\frac{y-1}{2}}^{\frac{1-y}{2}} p_{i-1}^0\left(\frac{2x}{1-y}\right) p_{k-1}^0\left(\frac{2x}{1-y}\right) 4(1-y)^{i+k-2}\hat{p}_j^{2i-1}(y)\hat{p}_l^{2k-1}(y) \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_{-1}^{1} p_{i-1}^0(z) p_{k-1}^0(z) \, \mathrm{d}z \int_{-1}^{1} 2(1-y)^{i+k-1}\hat{p}_j^{2i-1}(y)\hat{p}_l^{2k-1}(y) \, \mathrm{d}y. \end{aligned}$$

By (2.7), we have $\int_{-1}^{1} p_{i-1}^0(z)p_{k-1}^0(z)\ \mathrm{d}z = \rho_{i-1}^0\delta_{ik}$. Inserting this into the above equation, one easily concludes

$$\begin{aligned}
a_{ij,kl}^{(x)} &= \rho_{i-1}^0\delta_{ik}\int_{-1}^{1} 2(1-y)^{i+k-1}\hat{p}_{j-1}^{2i-1}(y)\hat{p}_{l-1}^{2k-1}(y)\ \mathrm{d}y \\
&= \rho_{i-1}^0\delta_{ik}\int_{-1}^{1} 2(1-y)^{2i-1}\hat{p}_{j}^{2i-1}(y)\hat{p}_{l}^{2i-1}(y)\ \mathrm{d}y \\
&= \rho_{i-1}^0\delta_{ik}2m_{j,l}^{(7)}.
\end{aligned}$$

Using Lemma 5.8, the second integral is zero if $|j-l|>2$. This proves the assertion. $\square$

In a next step, the matrix

$$\hat{K}_{I,y} = \left[a_{ij,kl}^{(y)}\right]_{(i,j);(k,l)}\left[\int_{\hat{\triangle}} \frac{\partial\phi_{ij}(x,y)}{\partial y}\frac{\partial\phi_{kl}(x,y)}{\partial y}\ \mathrm{d}(x,y)\right]_{(i,j);(k,l)} \tag{5.6}$$

is investigated.

Lemma 5.11. *Let $\hat{K}_{I,y}$ be defined via relation* (5.6). *Then, $a_{ij,kl}^{(y)} = 0$ if $|i-k| \notin \{0,2\}$ or $|i+j-k-l|>2$.*

Proof. The proof is similar to the proof of Lemma 5.10. With the substitution $z = \frac{2x}{1-y}$ (Duffy transformation), and relations (5.1) and (5.6), we obtain

$$\begin{aligned}
a_{ij,kl}^{(y)} &= \int_{-1}^{1}\int_{-1}^{1}\left[p_{i-2}^0(z)(1-y)^{i-1}\hat{p}_j^{2i-1}(y) + \hat{p}_i^0(z)(1-y)^i p_{j-1}^{2i-1}(y)\right] \\
&\quad\times\left[p_{k-2}^0(z)(1-y)^{k-1}\hat{p}_l^{2k-1}(y) + \hat{p}_k^0(z)(1-y)^k p_{l-1}^{2k-1}(y)\right]\frac{1-y}{2}\ \mathrm{d}z\ \mathrm{d}y \\
&= \frac{1}{2}\left(\int_{-1}^{1} p_{i-2}^0(z)p_{k-2}^0(z)\ \mathrm{d}z\int_{-1}^{1}(1-y)^{i+k-1}\hat{p}_j^{2i-1}(y)\hat{p}_l^{2k-1}(y)\ \mathrm{d}y\right. \\
&\quad+\int_{-1}^{1} p_{i-2}^0(z)\hat{p}_k^0(z)\ \mathrm{d}z\int_{-1}^{1}(1-y)^{i+k}\hat{p}_j^{2i-1}(y)p_{l-1}^{2k-1}(y)\ \mathrm{d}y \\
&\quad+\int_{-1}^{1} \hat{p}_i^0(z)p_{k-2}^0(z)\ \mathrm{d}z\int_{-1}^{1}(1-y)^{i+k}p_{j-1}^{2i-1}(y)\hat{p}_l^{2k-1}(y)\ \mathrm{d}y \\
&\quad\left.+\int_{-1}^{1} \hat{p}_i^0(z)\hat{p}_k^0(z)\ \mathrm{d}z\int_{-1}^{1}(1-y)^{i+k+1}p_{j-1}^{2i-1}(y)p_{l-1}^{2k-1}(y)\ \mathrm{d}y\right) \\
&:= \frac{1}{2}\left(a_{ij,kl}^{(y,1)} + a_{ij,kl}^{(y,2)} + a_{ij,kl}^{(y,3)} + a_{ij,kl}^{(y,4)}\right).
\end{aligned} \tag{5.7}$$

Now, we start with $a_{ij,kl}^{(y,1)}$. Using (2.7), one derives

$$\begin{aligned}
a_{ij,kl}^{(y,1)} &= \int_{-1}^{1} p_{i-2}^0(z)p_{k-2}^0(z)\ \mathrm{d}z\int_{-1}^{1}(1-y)^{i+k-1}\hat{p}_j^{2i-1}(y)\hat{p}_l^{2k-1}(y)\ \mathrm{d}y \\
&= \rho_{i-2}^0\delta_{i,k}\int_{-1}^{1}(1-y)^{2i-1}\hat{p}_j^{2i-1}(y)\hat{p}_l^{2i-1}(y)\ \mathrm{d}y = \rho_{i-2}^0\delta_{i,k}m_{j,l}^{(7)}.
\end{aligned}$$

By Lemma 5.8, we have

$$a_{ij,kl}^{(y,1)} = 0 \quad \text{if } i \neq k \vee |j-l|>2. \tag{5.8}$$

The next term is $a^{(y,2)}_{ij,kl}$. Using (2.10) with $\alpha = 0$ and (2.7), one obtains

$$
\begin{aligned}
a^{(y,2)}_{ij,kl} &= \int_{-1}^{1} p^0_{i-2}(z)\hat{p}^0_k(z)\,\mathrm{d}z \int_{-1}^{1} 2(1-y)^{i+k}\hat{p}^{2i-1}_j(y)p^{2k-1}_{l-1}(y)\,\mathrm{d}y \\
&= \frac{1}{2k-1}\int_{-1}^{1} p^0_{i-2}(z)(p^0_k(z) - p^0_{k-2}(z))\,\mathrm{d}z \int_{-1}^{1}(1-y)^{i+k}\hat{p}^{2i-1}_j(y)p^{2k-1}_{l-1}(y)\,\mathrm{d}y. \\
&:= \frac{\rho^0_{i-2}}{2k-1}(\delta_{i-2,k} - \delta_{i-2,k-2})t_{i,k,j,l}.
\end{aligned}
$$

So, the factor before $t_{i,k,j,l}$ is zero if $i-2 = k$ or $i = k$. In the case $i-2 = k$, we have

$$
t_{i,i-2,j,l} = \int_{-1}^{1}(1-y)^{2i-2}\hat{p}^{2i-1}_j(y)p^{2i-5}_{l-1}(y)\,\mathrm{d}y = m^{(8)}_{j,l-1}.
$$

By Lemma 5.9, $t_{i,i-2,j,l} = 0$ if $j > l$ or $l > j+4$. In the case $i = k$, one obtains

$$
t_{i,i,j,l} = \int_{-1}^{1}(1-y)^{2i}\hat{p}^{2i-1}_j(y)p^{2i-1}_{l-1}(y)\,\mathrm{d}y = m^{(6)}_{j,l-1}.
$$

By Lemma 5.7, $t_{i,i,j,l} = 0$ if $j > l+2$ or $l > j+2$. Summarizing,

$$
a^{(y,2)}_{ij,kl} = 0 \quad \text{if } (i \neq k \vee |j-l| > 2) \wedge (i-2 \neq k \vee |j+2-l| > 2). \tag{5.9}
$$

By symmetry with respect to i and k, and j and l,

$$
a^{(y,3)}_{ij,kl} = 0 \quad \text{if } (i \neq k \vee |j-l| > 2) \wedge (k-2 \neq i \vee |l+2-j| > 2). \tag{5.10}
$$

The last summand in (5.7) is $a^{(y,4)}_{ij,kl}$. Using (2.10) and (2.7) again, a simple computation gives

$$
a^{(y,4)}_{ij,kl} = \left(\frac{(\rho^0_i + \rho^0_{i-2})\delta_{ik}}{(2i-1)^2} - \frac{\delta_{i,k-2}\rho^0_i + \delta_{i-2,k}\rho^0_k}{(2i-1)(2k-1)}\right)\int_{-1}^{1}(1-y)^{i+k+1}p^{2i-1}_{j-1}(y)p^{2k-1}_{l-1}(y)\,\mathrm{d}y. \tag{5.11}
$$

The first factor of (5.11) is zero if $|i-k| \notin \{0,2\}$. If $i = k$, then

$$
a^{(y,4)}_{ij,il} = \frac{\rho^0_i + \rho^0_{i-2}}{(2i-1)^2}\int_{-1}^{1}(1-y)^{2i+1}p^{2i-1}_{j-1}(y)p^{2i-1}_{l-1}(y)\,\mathrm{d}y = \frac{\rho^0_i + \rho^0_{i-2}}{(2i-1)^2}m^{(4)}_{j-1,l-1}.
$$

By Lemma 5.5, $a^{(y,4)}_{ij,il} = 0$ if $|j-l| > 2$. If $k = i-2$, then

$$
a^{(y,4)}_{i,j;i-2,l} = \frac{-\rho^0_{i-2}}{(2i-1)(2i-5)}\int_{-1}^{1}(1-y)^{2i-1}p^{2i-1}_{j-1}(y)p^{2i-5}_{l-1}(y)\,\mathrm{d}y = \frac{-\rho^0_{i-2}}{(2i-1)(2i-5)}m^{(5)}_{j-1,l-1}.
$$

Using Lemma 5.6, $a^{(y,4)}_{i,j;i-2,l} = 0$ if $|j+2-l| > 2$. By symmetry with respect to i and k, j and l, we have $a^{(y,4)}_{i-2,j;i,l} = 0$ if $|l+2-j| > 2$. Summarizing,

$$
a^{(y,4)}_{ij,kl} = 0 \quad \text{if } (i \neq k \vee |j-l| > 2) \wedge (|i-k| \neq 2 \vee |j+i-k-l| > 2). \tag{5.12}
$$

Inserting (5.8), (5.9), (5.10), and (5.12) into (5.7), we can conclude

$$
a^{(y)}_{ij,kl} = 0 \quad \text{if } |i-k| \notin \{0,2\} \vee |j+i-k-l| > 2),
$$

which proves the assertion. □

Finally, we consider the matrix

$$\hat{K}_{I,xy} = \left[a^{(xy)}_{ij,kl}\right]_{(i,j);(k,l)} = \left[\int_{\hat{\triangle}} \frac{\partial \phi_{ij}(x,y)}{\partial x}\frac{\partial \phi_{kl}(x,y)}{\partial y}\,\mathrm{d}(x,y)\right]_{(i,j);(k,l)}. \tag{5.13}$$

Lemma 5.12. *Let $\hat{K}_{I,xy}$ be defined via relation (5.13). Then, $a^{(xy)}_{ij,kl} = 0$ if $|i-k| \neq 1$ and $|i+j-k-l| > 2$.*

Proof. As in the proof of Lemma 5.11, we obtain

$$\begin{aligned} a^{(xy)}_{ij,kl} &= \int_{-1}^{1} p^0_{i-1}(z)p^0_{k-2}(z)\,\mathrm{d}z \int_{-1}^{1}(1-y)^{i+k-1}\hat{p}^{2i-1}_j(y)\hat{p}^{2k-1}_l(y)\,\mathrm{d}y \\ &\quad + \int_{-1}^{1} p^0_{i-1}(z)\hat{p}^0_k(z)\,\mathrm{d}z \int_{-1}^{1}(1-y)^{i+k}\hat{p}^{2i-1}_j(y)p^{2k-1}_{l-1}(y)\,\mathrm{d}y \\ &:= a^{(xy,1)}_{ij,kl} + a^{(xy,2)}_{ij,kl} \end{aligned}$$

using (5.1) and (5.6). By (2.7), one derives

$$a^{(xy,1)}_{ij,kl} = \rho^0_{i-1}\delta_{i-1,k-2}\int_{-1}^{1}(1-y)^{2i-2}\hat{p}^{2i-1}_j(y)\hat{p}^{2i-3}_l(y)\,\mathrm{d}y = m^{(1)}_{j,l}.$$

By Lemma 5.2, $a^{(xy,1)}_{ij,kl} = 0$ if $i \neq k-1$ or $|l+1-j| > 2$.
Using (2.10) and (2.7), we have

$$a^{(xy,2)}_{ij,kl} = \frac{\rho^0_{i-1}}{2k-1}\left(\delta_{i-1,k} - \delta_{i-2,k-1}\right)\int_{-1}^{1}(1-y)^{i+k}\hat{p}^{2i-1}_j(y)p^{2k-1}_{l-1}(y)\,\mathrm{d}y.$$

So $a^{(xy,2)}_{ij,kl}$ is zero if $|i-k| \neq 1$. In the case $k = i-1$, we conclude

$$a^{(xy,2)}_{i,j;i-1,l} = \frac{\rho^0_{i-1}}{2i-3}\int_{-1}^{1}(1-y)^{2i-1}\hat{p}^{2i-1}_j(y)p^{2i-3}_{l-1}(y)\,\mathrm{d}y = m^{(2)}_{j,l-1}.$$

By Lemma 5.3, $a^{(xy,2)}_{i,j;i-1,l}$ is zero if $|j+1-l| > 2$. If $k-2 = i-1$, i.e. $i+1 = k$, one obtains

$$a^{(xy,2)}_{i,j;i+1,l} = \frac{\rho^0_{i-1}}{2i-3}\int_{-1}^{1}(1-y)^{2i+1}\hat{p}^{2i-1}_j(y)p^{2i+1}_{l-1}(y)\,\mathrm{d}y = m^{(3)}_{j,l-1}.$$

By Lemma 5.4, $a^{(xy,2)}_{i,j;i+1,l}$ is zero if $l-1 < j$ or $l > j-3$. Summarizing $a^{(xy)}_{ij,kl} = 0$ if $|i-k| \neq 1$ or $|i+j-k-l| > 2$. This proves the assertion. $\square$

Summarizing Lemmas 5.10, 5.11 and 5.12, we have shown the following result.

Lemma 5.13. *Let ϕ_{ij}, $i \geq 2$, $j \geq 0$ be defined via (3.4). Then,*

$$a_{ij,kl} = \int_{\hat{\triangle}} (\nabla\phi_{ij}(x,y))^T A \nabla\phi_{kl}(x,y)\,\mathrm{d}(x,y) = 0$$

if $i-k \notin \{-2,-1,0,1,2\}$ or $|i+j-k-l| > 2$.

In a last step, we consider the coupling between the edge block corresponding to e_2 or e_3 and the interior bubbles.
By a simple computation, we obtain

$$\nabla\phi_{e_{2/3},l}(x,y) = \begin{bmatrix} \pm\hat{p}_l^0(y) \\ -\frac{1}{2}\hat{p}_l^0(y) + \frac{1}{2}\left(1 \pm \frac{2x}{1-y}\right)(1-y)p_{l-1}^0(y) \end{bmatrix}. \tag{5.14}$$

Lemma 5.14. *The following assertions are valid for $k = 2, 3$.*

1. *For all $i, j, l \in \mathbb{N}$, we have*
$$\int_{\hat{\triangle}} \frac{\partial\phi_{e_k,l}}{\partial x}\frac{\partial\phi_{ij}}{\partial x}\,\mathrm{d}(x,y) = 0.$$
2. *For $i \geq 4$ or $|l-j-0.5-i| > 2.5$, we have*
$$\int_{\hat{\triangle}} \frac{\partial\phi_{e_k,l}}{\partial y}\frac{\partial\phi_{ij}}{\partial y}\,\mathrm{d}(x,y) = 0.$$
3. *For $i \geq 4$ or $|l-j-0.5-i| > 2.5$, we have*
$$\int_{\hat{\triangle}} \frac{\partial\phi_{e_k,l}}{\partial x}\frac{\partial\phi_{ij}}{\partial y}\,\mathrm{d}(x,y) = 0.$$
4. *For $i \geq 4$ or $|l-j-0.5-i| > 2.5$, we have*
$$\int_{\hat{\triangle}} \frac{\partial\phi_{e_k,l}}{\partial y}\frac{\partial\phi_{ij}}{\partial x}\,\mathrm{d}(x,y) = 0.$$

Proof. The proof of this result is similar as the proof of Lemmas 5.10-5.12. □

Finally, we give the proof of Theorem 3.3:

Proof. The symmetric matrix $\hat{K}$ (3.6) has 9 blocks, where $\hat{K}_{V,(A)} \in \mathbb{R}^{3\times 3}$, $\hat{K}_{V,E,(A)} \in \mathbb{R}^{3\times 3p-3}$, $\hat{K}_{V,E,(A)} \in \mathbb{R}^{3\times(p-1)(p-2)/2}$ and $\hat{K}_{E,(A)} \in \mathbb{R}^{3p-3\times 3p-3}$. So, it suffices to investigate the remaining blocks $\hat{K}_{E,I,(A)}$ and $\hat{K}_{I,(A)}$. For $\hat{K}_{E,I,(A)}$, we can conclude from Lemma 5.14 (for edges e_2 and e_3) and Lemma 5.13 with Remark 3.2 (for edge e_1) that this matrix has $\mathcal{O}(p)$ nonzero matrix entries. For the matrix $\hat{K}_{I,(A)}$, the assertion follows from Lemma 5.13. The special structure of the nonzero matrix entries for $\hat{K}_{I,(I)}$ follows from Lemmas 5.10 and 5.11. □

Remark 5.15. *With the arguments in Lemmas 5.10-5.14, one can prove the relation $\hat{K}_{V,I,(A)} = \mathbf{0}$. Moreover, the matrix $\hat{K}_{V,E,(A)}$ has $\mathcal{O}(1)$ nonzero matrix entries and the matrix $\hat{K}_{E,(A)}$ has $\mathcal{O}(p)$ nonzero matrix entries.*

6 Numerical Experiments

In this section, we will present two numerical experiments. In the first experiment, we determine the arithmetical cost in order to compute the matrix $\hat{K}_{I,(I)}$ (3.7). This matrix has about $15/2\,p^2 \approx 15N$ nonzero matrix entries. The results are compared with the computation of the element matrix stiffness matrix for the Laplacian using sum factorization, [24]. The cost of this algorithm is $\mathcal{O}(p^4)$. In order to get measurable results, we have computed both matrices 100 times. The computations are executed on a Pentium IV, 2400 MHz. Table 1 displays the time in order to compute the matrix block for the interior bubbles using

p	explicit	sum factor.
5	¡0.004	0.016
9	0.004	0.078
13	0.012	0.272
19	0.035	1.411
25	0.059	4.153
31	0.098	9.813
37	0.133	26.265
43	0.180	43.393
49	0.234	83.156
59	0.340	
69	0.473	

Table 1: Time in order to compute the matrix block for the interior bubbles 100 times.

- the basis functions (3.1)-(3.4) and explicit formulas, see the appendix,
- using sum factorization.

From the results, one can see the high speed for the generation of the stiffness matrix if the basis functions (3.1)-(3.4) are used.
In a second example, we measure the quality of a possible preconditioner C_I^{-1} for $\hat{K}_{I,(I)}$ (3.7), see subsection 4.2. First, we determine the number of nonzero matrix entries of the Cholesky factor without permutation, and with the method of minimal degree starting from the first and last column of $\hat{K}_{I,(I)}$ for several polynomial degrees p up to 130.

p	$\hat{K}_{I,(I)}$	Cholesky	Minimal degree with start	
			first row	last row
10	168	188	178	174
20	1082	1992	1713	1777
30	2795	7395	5860	6024
40	5307	18397	14085	14196
50	8619	36999	27187	27343
60	12733	65203	45650	45216
70	17644	105004	71895	68578
80	23354	158404	101478	97745
90	29685	227405	136635	134060
100	37179	314009	186376	176898
110	45291	420211	239246	224198
120	54207	548017	297884	280076
130	63922	699422	383028	343644

Table 2: Nonzero matrix entries for $\hat{K}_{I,(I)}$ and its Cholesky factors.

From the results, one can see that about six times of the memory of the matrix $\hat{K}_{I,(I)}$ is required in order to save its Cholesky factor for $p = 130$.
Finally, we determine maximal and minimal eigenvalue of the matrix $\hat{C}_k^{-1}\hat{K}_{I,(I)}$, where

$$[\hat{C}_k]_{ij} = \begin{cases} [\hat{K}_{I,(I)}]_{ij} & \text{if} \quad |i-j| \le k \\ \left[\hat{K}\right]_{ij} = 0 & \text{else} \end{cases},$$

$k = 0, 2$. Figure 3 displays the increase of maximal and minimal eigenvalue. For $\hat{C}_0$, i.e. it is

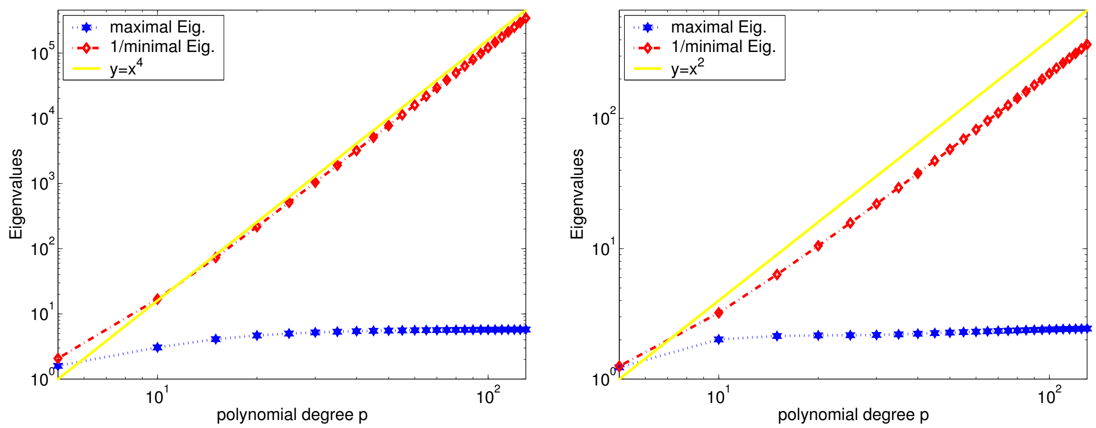

Figure 3: Maximal and minimal eigenvalue of the matrix $\hat{C}_k^{-1}\hat{K}_{I,(I)}$ for $k = 0$ (left) and $k = 2$ (right).

the diagonal part of $\hat{K}_{I,(I)}$, the maximal eigenvalue is bounded whereas the minimal eigenvalue goes to zero about proportionally to $O(p^{-4})$. For $\hat{C}_2$, the minimal eigenvalue goes to zero about proportionally to $O(p^{-2})$. So, this reduces the condition number of the preconditioned system. However, this idea does not yield to an optimal or a suboptimal preconditioner.

7 Open Questions and concluding remarks

Summarizing, the choice of the basis (3.4) has several advantages in order to get sparse system matrices. All orthogonality arguments base on the integral relations (2.8) and (2.14), i.e. a polynomial of degree p is orthogonal to all polynomials of degree $p-k$ in some weighted scalar product. Hence, the results of Theorem 3.3 and 3.4 can be extended to the case of piecewise polynomial coefficients $\mathcal{A}(x,y)$, i.e. $\mathcal{A}(x,y)|_{\triangle_s} \in \mathbb{P}_k$. Then, the matrix $\hat{K}_{I,(\mathcal{A})}$ has about $(5+2k)^2p^2/2$ nonzero matrix elements.
For practical computations, a scaled version of the bubbles should be preferred, i.e.

$$\tilde{\phi}_{ij}(x,y) = \hat{p}_i^0\left(\frac{2x}{1-y}\right)\left(\frac{1-y}{2}\right)^i \hat{p}_j^{2i-1}(y) = \frac{\phi_{ij}(x,y)}{2^i}. \tag{7.1}$$

Then, cf. (2.7) with $\alpha \approx 2i-1$, the nonzero entries of the matrix $\hat{K}_{I,(A)}$ have no factor 4^i which can be very large for $p = 50, \ldots, 100$. For the evaluation of the function values of the shape functions, the recursions (2.6) and (2.11) can be used.
In general, this basis can be used as preconditioner for the block of the interior bubbles $\mathcal{K}_I$, see the Domain Decomposition preconditioner $\mathcal{C}_1$ (4.4). This preconditioner is an optimal preconditioner for $\mathcal{K}_\Psi$. However, the cost in order to compute $\mathcal{C}_1^{-1}\underline{v}$ is not proportionally to the number of unknowns, i.e. $\mathcal{O}(p^2)$ using Cholesky decomposition. Due to this fact, it is important to derive an optimal solver in the case of a sparse stiffness matrix $\mathcal{K}_\Psi$, too. It is a challenge to derive optimal multilevel solvers for $\mathcal{C}_1$ which has been done for quadrilateral elements in [10], [9], [23].
Finally, we would like to mention that the approach can be extended to the 3D case. In the case of the reference tetrahedron with the vertices $(-1,-1,-1)$, $(1,-1,-1)$, $(0,1,-1)$ and $(0,0,1)$ one

has to take the interior bubbles

$$\phi_{ijk}(x,y,z) = \hat{p}_i^0\left(\frac{4x}{1-2y-z}\right)(1-2y-z)^i \hat{p}_j^{2i-1}\left(\frac{2y}{1-z}\right)(1-z)^j \hat{p}_k^{2i+2j-1}(z). \tag{7.2}$$

This will be presented in a forthcoming paper.

A Computation of the nonzero matrix entries of $\hat{K}$

Now, we give the structure of the nonzero entries of $\hat{K}$ (3.5).

A.1 Structure of the matrices M_1 to M_8

Due to the proofs of Lemmas 5.10-5.14, the matrices $M_1, \ldots, M_8$ introduced in Lemmas 5.2-5.9 have to be determined, in order to derive explicit formulas for the nonzero matrix entries of $\hat{K}$. We will demonstrate the idea for the example of the matrix M_3 in Lemma 5.4, i.e.

$$M_3 = \left[m_{j,l}^{(3)}\right]_{j,l} = \left[\int_{-1}^{1} (1-y^{2i+1} \hat{p}_j^{2i-1}(y) p_l^{2i+1}(y)\,\mathrm{d}y\right]_{j,l}. \tag{A.1}$$

The idea is based on the relations (2.10) and (2.5). Due to (2.10), we have

$$\hat{p}_j^{2i-1}(y) = \frac{2(2i+j-1)p_j^{2i-1}(y)}{(2i+2j-1)(2i+2j-2)} + \frac{2(2i-1)p_{j-1}^{2i-1}(y)}{(2i+2j-1)(2i+2j-3)} - \frac{2(j-1)p_{j-2}^{2i-1}(y)}{(2i+2j-2)(2i+2j-3)}.$$

Moreover, by (2.5), one has

$$\begin{aligned}
p_{j-1}^{\alpha-2}(y) &= \frac{(j+\alpha-1)p_{j-1}^{\alpha-1}(y) - (j-1)p_{j-2}^{\alpha-1}(y)}{\alpha+2j-2}, \\
p_{j-1}^{\alpha-1}(y) &= \frac{(j+\alpha)p_{j-1}^{\alpha}(y) - jp_{j-2}^{\alpha}(y)}{\alpha+2j}, \\
p_{j}^{\alpha-2}(y) &= \frac{(j+\alpha-1)p_{j-1}^{\alpha-1}(y) - jp_{j-2}^{\alpha-1}(y)}{\alpha+2j-1},
\end{aligned}$$

i.e.

$$\begin{aligned}
p_j^{\alpha-2}(y) &= \frac{(\alpha+j)(\alpha+j-1)}{(\alpha+2j)(\alpha+2j-1)}p_j^{\alpha}(y) - \frac{2j(\alpha+j-1)}{(\alpha+2j)(\alpha+2j-2)}p_{j-1}^{\alpha}(y) \\
&\quad + \frac{j(j-1)}{(\alpha+2j-1)(\alpha+2j-2)}p_{j-2}^{\alpha}(y).
\end{aligned} \tag{A.2}$$

Using (A.2) for $\alpha = 2i+1$ and j, $j-1$ and $j-2$, one obtains

$$\begin{aligned}
\hat{p}_j^{2i-1}(y) &= \frac{2(2i+j+1)(2i+j)(2i+j-1)}{(2i+2j+1)(2i+2j)(2i+2j-1)(2i+2j-2)}p_j^{2i+1}(y) \\
&\quad +2\frac{(2i-2j+1)(2i+j)(2i+j-1)}{(2i+2j+1)(2i+2j-1)(2i+2j-2)(2i+2j-3)}p_{j-1}^{2i+1}(y) \\
&\quad -\frac{12i(j-1)(2i+j-1)}{(2i+2j)(2i+2j-1)(2i+2j-3)(2i+2j-4)}p_{j-2}^{2i+1}(y) \\
&\quad +\frac{2(j-1)(j-2)(6i+2j-3)}{(2i+2j-1)(2i+2j-2)(2i+2j-3)(2i+2j-5)}p_{j-3}^{2i+1}(y) \\
&\quad -\frac{2(j-1)(j-2)(j-3)}{(2i+2j-2)(2i+2j-3)(2i+2j-4)(2i+2j-5)}p_{j-4}^{2i+1}(y).
\end{aligned} \tag{A.3}$$

Lemma A.1. *Let* M_3 *be defined via* (A.1). *Then,*

$$\begin{aligned}
m^{(3)}_{jj} &= 4^{i+1}\frac{2(2i+j+1)(2i+j)(2i+j-1)}{(2i+2j+2)(2i+2j+1)(2i+2j)(2i+2j-1)(2i+2j-2)},\\
m^{(3)}_{j,j-1} &= 4^{i+1}\frac{2(2i-2j+1)(2i+j)(2i+j-1)}{(2i+2j+1)(2i+2j)(2i+2j-1)(2i+2j-2)(2i+2j-3)},\\
m^{(3)}_{j,j-2} &= -4^{i+1}\frac{12i(j-1)(2i+j-1)}{(2i+2j)(2i+2j-1)(2i+2j-2)(2i+2j-3)(2i+2j-4)},\\
m^{(3)}_{j,j-3} &= 4^{i+1}\frac{2(j-1)(j-2)(6i+2j-3)}{(2i+2j-1)(2i+2j-2)(2i+2j-3)(2i+2j-4)(2i+2j-5)},\\
m^{(3)}_{j,j-4} &= -4^{i+1}\frac{2(j-1)(j-2)(j-3)}{(2i+2j-2)(2i+2j-3)(2i+2j-4)(2i+2j-5)(2i+2j-6)}.
\end{aligned}$$

Proof. The assertion follows from (A.3) and (2.7). □

Finally, the structure of the nonzero entries of the matrices M_1 to M_8:
Let

$$(a)_n = a(a+1)\cdot \ldots \cdot (a+n-1).$$

- Matrix M_1:

$$m^{(1,i)}_{j,j-1} = -4^i\frac{2(j^2+2ji-4j-3+5i-2i^2)}{(2i+2j-6)_5}, \tag{A.4}$$

$$m^{(1,i)}_{jj} = 4^i\frac{2(2i-3)(2i+j-3)}{(2i+2j-5)_5}, \qquad m^{(1,i)}_{j,j-2} = -4^i\frac{2(2i-3)(j-1)}{(2i+2j-7)_5},$$

$$m^{(1,i)}_{j,j-3} = 4^i\frac{(j-1)(j-2)}{(2i+2j-8)_5}, \qquad m^{(1,i)}_{j,j+1} = 4^i\frac{(2i+j-3)(2i+j-2)}{(2i+2j-4)_5}.$$

- Matrix M_2:

$$\begin{aligned}
m^{(2)}_{jj} &= 4^i\frac{4(i-2)(4i^2-8i+2ji-2j+3+j^2)}{(2i+2j-4)_5},\\
m^{(2)}_{j,j-1} &= 4^i\frac{2(j-3+2i)(2j^2+2ji-3j-10i+4+4i^2)}{(2i+2j-5)_5},\\
m^{(2)}_{j,j-2} &= -4^i\frac{2(2i+j-4)_2(j-1)}{(2i+2j-6)_5},\\
m^{(2)}_{j,j+1} &= -4^i\frac{2(j+1)(2j^2-5j+6ji-16i+6+8i^2)}{(2i+2j-3)_5},\\
m^{(2)}_{j,j+2} &= 4^i\frac{2(2i+j-1)(j+1)(j+2)}{(2i+2j-2)_5}.
\end{aligned} \tag{A.5}$$

- Matrix M_4:

$$\begin{aligned}
m^{(4)}_{jj} &= 2^{2i+3}\frac{3j^4+12ij^3-2j^2i-3j^2+20i^2j^2-6ij+16i^3j-4i^2j+8i^4-2i^2+2i-8i^3}{(2i+2j-2)_5},\\
m^{(4)}_{j+2,j} &= m^{(4)}_{j,j+2} = 2^{2i+2}\frac{(j)_2(2i+j-1)_2}{(2i+2j)_5},\\
m^{(4)}_{j+1,j} &= m^{(4)}_{j,j+1} = -2^{2i+3}\frac{(4i^2+4ij+2j^2+2j-1)(2+j)(2i+j+1)}{(2i+2j-1)_5}.
\end{aligned} \tag{A.6}$$

- Matrix M_5:

$$m^{(5)}_{jj} = 4^i \frac{(2i+j-4)_4}{(2i+2j-4)_5}, \qquad m^{(5)}_{j,j+1} = -4^{i+1} \frac{(j+1)(2i+j-3)_3}{(2i+2j-3)_5},$$
$$m^{(5)}_{j,j+2} = 4^i \frac{6(j+1)_2(2i+j-2)_2}{(2i+2j-2)_5}, \qquad m^{(5)}_{j,j+3} = -4^{i+1} \frac{6(j+1)_3(2i+j-1)}{(2i+2j-1)_5},$$
$$m^{(5)}_{j,j+4} = 4^i \frac{6(j+1)_4}{(2i+2j)_5}. \tag{A.7}$$

- Matrix M_6:

$$m^{(6)}_{jj} = 4^i \frac{4(2i+j-1)(2j^2+2ij-j+4i^2-6i)}{(2i+2j-3)_5} \tag{A.8}$$
$$m^{(6)}_{j,j+1} = 4^i \frac{8(2i+j-1)_2(j+1)}{(2i+2j-2)_5}, \qquad m^{(6)}_{j+1,j} = 4^i \frac{8i(j^2+2ij+2-6i+4i^2)}{(2i+2j-2)_5},$$
$$m^{(6)}_{j+2,j} = -4^i \frac{4(j+1)(2j^2+j+6ij+8i^2-4i)}{(2i+2j-1)_5}, \qquad m^{(6)}_{j+3,j} = 4^i \frac{4(j+1)_2(2i+j)}{(2i+2j)_5}.$$

- Matrix M_7:

$$\begin{aligned} m^{(7)}_{jj} &= 4^i \frac{8(4i^2+2ij-8i-2j+3+j^2)}{(2i+2j-4)_5}, \\ m^{(7)}_{j+1,j} = m^{(7)}_{j,j+1} &= 4^i \frac{4(2i-3)(2i-1)}{(2i+2j-3)_5}, \\ m^{(7)}_{j+2,j} = m^{(7)}_{j,j+2} &= -4^i \frac{4(j+1)(2i-1+j)}{(2i+2j-2)_5}. \end{aligned} \tag{A.9}$$

- Matrix M_8:

$$m^{(8)}_{jj} = 4^i \frac{(2i+j-3)_2(2i-2j-5)}{(2i+2j-5)_5}, \tag{A.10}$$
$$m^{(8)}_{j+1,j} = 4^i \frac{(2i+j-4)_3}{(2i+2j-4)_5}, \qquad m^{(8)}_{j,j+1} = -4^i \frac{6(2i+j-3)(i-2)(j+1)}{(2i+2j-4)_5},$$
$$m^{(8)}_{j,j+2} = 4^i \frac{(6i+2j-9)(j+1)_2}{(2i+2j-3)_5}, \qquad m^{(8)}_{j,j+3} = -4^i \frac{(j+1)_3}{(2i+2j-2)_5}.$$

The remaining matrix entries of M_1 to M_8 are zero.

A.2 Structure of the block $\hat{K}_{I,(A)}$

Now, the structure of the nonzero matrix entries of the block of the interior bubbles can be computed via the following relations and (A.4) to (A.10):

$$\begin{aligned}
\int_{\hat{\triangle}} \frac{\partial \phi_{ij}}{\partial x} \frac{\partial \phi_{kl}}{\partial x} &= \frac{4}{2i-1} m^{(7)}_{jl} \delta_{ik} \\
\int_{\hat{\triangle}} \frac{\partial \phi_{ij}}{\partial y} \frac{\partial \phi_{kl}}{\partial y} &= \delta_{ik} \left(\frac{1}{2i-3} m^{(7)}_{jl} + \frac{2m^{(4)}_{j-1,l-1}}{(2i-3)(2i-1)(2i+1)} - \frac{m^{(6)}_{j,l-1} + m^{(6)}_{l,j-1}}{(2i-3)(2i-1)} \right) \\
&\quad + \delta_{i-2,k} \left(-\frac{m^{(5)}_{j-1,l-1}}{(2i-1)(2i-3)(2i-5)} + \frac{m^{(8)}_{j,l-1}}{(2i-3)(2i-5)} \right), \quad i \geq k \\
\int_{\hat{\triangle}} \frac{\partial \phi_{ij}}{\partial y} \frac{\partial \phi_{kl}}{\partial y} &= \int_{\hat{\triangle}} \frac{\partial \phi_{kl}}{\partial y} \frac{\partial \phi_{ij}}{\partial y}, \quad i < k, \\
\int_{\hat{\triangle}} \frac{\partial \phi_{i,j}}{\partial x} \frac{\partial \phi_{i-1,l}}{\partial y} &= 2 \frac{\delta_{i,k-1}}{2i-1} \left(m^{(1,i+1)}_{j,l} - \frac{m^{(3)}_{j,l-1}}{2i+1} \right) + 2 \frac{\delta_{i-1,k} m^{(2)}_{j,l-1}}{(2i-1)(2i-3)},
\end{aligned}$$

where $i, k \geq 2$, $j, l \geq 1$.

A.3 Structure of the blocks $\hat{K}_{IE,(A)}$ and $\hat{K}_{IV,(A)}$

- The entries for the coupling block between the interior bubbles with the edge bubble functions on edge e_1 can be computed via the following relations and (A.4) to (A.10):

$$\begin{aligned}
\int_{\hat{\triangle}} \frac{\partial \phi_{ij}}{\partial x}\frac{\partial \phi_{e_1,l}}{\partial x} &= \frac{4}{2i-1} m_{j,0}^{(7)} \delta_{il}, \\
\int_{\hat{\triangle}} \frac{\partial \phi_{ij}}{\partial x}\frac{\partial \phi_{e_1,k}}{\partial y} &= \delta_{i,k-1} \frac{2}{2i-1} m_{0,j}^{(1)}, \\
\int_{\hat{\triangle}} \frac{\partial \phi_{e_1,i}}{\partial y}\frac{\partial \phi_{kl}}{\partial y} &= \delta_{i,k} \frac{1}{2i-3}\left(m_{0,l}^{(7)} - \frac{m_{0,l-1}^{(6)}}{2i-1}\right) + \delta_{i-2,k} \frac{m_{0,l-1}^{(8)}}{(2i-1)(2i+1)}, \\
\int_{\hat{\triangle}} \frac{\partial \phi_{k,l}}{\partial y}\frac{\partial \phi_{e_1,i}}{\partial x} &= \delta_{i,k-1} \frac{2}{2i-1}\left(m_{0,l}^{(1,i+1)} - \frac{m_{0,l-1}^{(3)}}{2i+1}\right) + \delta_{i-1,k} \frac{2m_{0,l-1}^{(2)}}{(2i-1)(2i-3)},
\end{aligned}$$

  where $i,k \geq 2$ and $l,j \geq 1$.
- The entries for the coupling block between the interior bubbles with the edge bubble functions on edge $e_{2/3}$ can be computed via the following relations:

$$\int_{\hat{\triangle}} \frac{\partial \phi_{ij}}{\partial y}\frac{\partial \phi_{e_{2/3},l}}{\partial x} = \delta_{i2}\left(\int_{-1}^{1} (1-y)^2 \hat{p}_j^3(y)\hat{p}_l^0(y)\,\mathrm{d}y - \frac{1}{3}\int_{-1}^{1} (1-y)^3 p_{j-1}^3(y)\hat{p}_l^0(y)\,\mathrm{d}y\right), \tag{A.11}$$

$$\begin{aligned}
\int_{\hat{\triangle}} \frac{\partial \phi_{ij}}{\partial y}\frac{\partial \phi_{e_{2/3},l}}{\partial y} = {}& -\delta_{i2}\left(\frac{1}{2}\int_{\hat{\triangle}} \frac{\partial \phi_{2j}}{\partial y}\frac{\partial \phi_{e_{2/3},l}}{\partial x}\right. \\
&\left. +\frac{1}{2}\int_{-1}^{1} (1-y)^2 \hat{p}_j^3(y) p_{l-1}^0(y)\,\mathrm{d}y - \frac{1}{6}\int_{-1}^{1} (1-y)^3 p_{j-1}^3(y) p_{l-1}^0(y)\,\mathrm{d}y\right), \\
& +\delta_{i3}\left(\pm\frac{1}{6}\int_{-1}^{1} (1-y)^4 \hat{p}_j^5(y) p_{l-1}^0(y)\,\mathrm{d}y \mp \frac{1}{30}\int_{-1}^{1} (1-y)^5 p_{j-1}^5(y) p_{l-1}^0(y)\,\mathrm{d}y\right).
\end{aligned} \tag{A.12}$$

  Moreover,

$$\int_{\hat{\triangle}} \frac{\partial \phi_{ij}}{\partial x} \nabla \phi_{e_{2/3},l} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

  In order to evaluate the remaining integrals, we explain the method for the first integral in (A.11): The weight function is $(1-y)^2$. So, we transform the functions $\hat{p}_j^3(y)$ and $\hat{p}_j^0(y)$ into the basis $\{p_j^2\}$. For $\hat{p}_j^3(y)$, we can use relation (2.12), whereas $\hat{p}_j^0(y) = \frac{1}{2j-1}\left(p_j^0(y) - p_{j-2}^0(y)\right)$ by relation (2.10) . Using (2.5) for $\alpha = 1,2$ and $j-3,\ldots,j$, we obtain an expression for $\hat{p}_j^0(y)$ in the basis $\{p_j^2\}$. Then, by (2.7) with $\alpha = 2$, the nonzero entries can easily be computed.
- The block of the interior bubbles with the vertex functions is zero, i.e. $\hat{K}_{IV,(A)} = \mathbf{0}$.

A.4 Edge block

Here, we have to distinguish between the edge bubble functions to the three edges e_1, e_2 and e_3. Let

$$\hat{K}_{E,(A)} = \begin{bmatrix} \hat{K}_{e,11} & \hat{K}_{e,12} & \hat{K}_{e,13} \\ \hat{K}_{e,21} & \hat{K}_{e,22} & \hat{K}_{e,23} \\ \hat{K}_{e,31} & \hat{K}_{e,32} & \hat{K}_{e,33} \end{bmatrix}.$$

- Block $\hat{K}_{e,11}$: The matrix entries can be computed via

$$\begin{aligned}
\int_{\hat{\triangle}} \frac{\partial \phi_{e_1,i}}{\partial x}\frac{\partial \phi_{e_1,k}}{\partial x} &= \delta_{ik} \frac{4^i}{(2i-2)(2i-1)}, \\
\int_{\hat{\triangle}} \frac{\partial \phi_{e_1,i}}{\partial y}\frac{\partial \phi_{e_1,k}}{\partial y} &= \delta_{ik} \frac{4^{i-1}}{(2i-2)(2i-3)}, \\
\int_{\hat{\triangle}} \frac{\partial \phi_{e_1,i}}{\partial x}\frac{\partial \phi_{e_1,k}}{\partial y} &= \delta_{i,k-1} \frac{4^i}{(2i-1)(2i-1)}, \quad i,k \geq 2.
\end{aligned}$$

- Blocks $\hat{K}_{e,1,2/3}$:

$$\hat{K}_{e,1,2/3} = \begin{bmatrix} U_{4,5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

with the 4×5 matrix

$$U_{4,5} = c \begin{bmatrix} \frac{4}{3} & -\frac{4}{5} & \frac{4}{15} & -\frac{4}{105} & 0 \\ -\frac{16}{15} & \frac{32}{45} & -\frac{32}{105} & \frac{8}{105} & -\frac{8}{945} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mp b \begin{bmatrix} -\frac{8}{3} & \frac{8}{5} & -\frac{8}{15} & \frac{8}{105} & 0 \\ \frac{8}{5} & 0 & 0 & 0 & 0 \\ -\frac{8}{15} & 0 & 0 & 0 & 0 \\ \frac{8}{105} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Blocks $\hat{K}_{e,2/3,2/3}$: Use

$$\begin{aligned} \int_{\hat{\triangle}} \frac{\partial \phi_{e_k,j}}{\partial x} \frac{\partial \phi_{e_k,l}}{\partial x} &= \frac{4\delta_{jl}}{(2j-3)(2j-1)(2j+1)} - \frac{4\delta_{j+1,l}}{(2j-3)(2j-1)(2j+1)(2j+3)} \\ &\quad - \frac{2\delta_{j+2,l}}{(2j-1)(2j+1)(2j+3)} + \frac{(2j+2)\delta_{j+3,l}}{(2j-1)(2j+1)(2j+3)(2j+5)}, \quad 1 \le j \le l \end{aligned}$$

and symmetry with respect to j and l, $k = 2, 3$. Furthermore,

$$\int_{\hat{\triangle}} \frac{\partial \phi_{e_k,j}}{\partial x} \frac{\partial \phi_{e_{5-k},l}}{\partial x} = -\int_{\hat{\triangle}} \frac{\partial \phi_{e_k,j}}{\partial x} \frac{\partial \phi_{e_k,l}}{\partial x}.$$

Moreover,

$$\begin{aligned} \int_{\hat{\triangle}} \frac{\partial \phi_{e_{2/3},j}}{\partial x} \frac{\partial \phi_{e_k,l}}{\partial y} &= \mp\frac{1}{2} \int_{\hat{\triangle}} \frac{\partial \phi_{e_{2/3},j}}{\partial x} \frac{\partial \phi_{e_k,l}}{\partial x} \pm \frac{1}{2} \int_{-1}^{1} (1-y)^2 \hat{p}_j^0(y) p_{l-1}^0(y)\, \mathrm{d}y, \\ \int_{\hat{\triangle}} \frac{\partial \phi_{e_k,j}}{\partial y} \frac{\partial \phi_{e_{k'},l}}{\partial y} &= -\frac{1}{2} \int_{\hat{\triangle}} \frac{\partial \phi_{e_2,j}}{\partial x} \frac{\partial \phi_{e_{k'},l}}{\partial y} \\ &\quad -\frac{1}{4} \int_{-1}^{1} (1-y)^2 p_{j-1}^0(y) \hat{p}_l^0(y)\, \mathrm{d}y + \frac{1+\delta_{k,k'}}{6} \int_{-1}^{1} (1-y)^2 p_{j-1}^0(y) p_{l-1}^0(y)\, \mathrm{d}y, \end{aligned}$$

where $k, k' \in \{2, 3\}$, $(j, l) \ge 1$.

A.5 Coupling betwwen edge block with Vertex block

A simple computation gives

$$\hat{K}_{V,E,(A)} = \begin{bmatrix} \hat{K}_{V,E1,(A)} & \hat{K}_{V,E2,(A)} & \hat{K}_{V,E3,(A)} \end{bmatrix},$$

where

$$\begin{aligned} \hat{K}_{V,E1,(A)} &= \frac{4b}{3} \begin{bmatrix} 1 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} + \frac{c}{3} \begin{bmatrix} -2 & 0 & \dots & 0 \\ -2 & 0 & \dots & 0 \\ 4 & 0 & \dots & 0 \end{bmatrix}, \\ \hat{K}_{V,E2,(A)} &= \frac{a}{15} \begin{bmatrix} 10 & -5 & 1 & 0 & \dots \\ -10 & 5 & -1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \end{bmatrix} + \frac{b}{15} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ -10 & 5 & -1 & 0 & \dots \\ 10 & -5 & 1 & 0 & \dots \end{bmatrix} \\ &\quad + \frac{c}{60} \begin{bmatrix} -10 & 5 & -1 & 0 & \dots \\ -10 & 5 & -1 & 0 & \dots \\ 20 & -10 & 2 & 0 & \dots \end{bmatrix}, \\ \hat{K}_{V,E3,(A)} &= \frac{a}{15} \begin{bmatrix} -10 & 5 & -1 & 0 & \dots \\ 10 & -5 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \end{bmatrix} + \frac{b}{15} \begin{bmatrix} 10 & -5 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ -10 & 5 & -1 & 0 & \dots \end{bmatrix} \\ &\quad + \frac{c}{60} \begin{bmatrix} -10 & 5 & -1 & 0 & \dots \\ -10 & 5 & -1 & 0 & \dots \\ 20 & -10 & 2 & 0 & \dots \end{bmatrix}. \end{aligned}$$

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References

[1] M. Abramowitz, editor. *Handbook of mathematical functions.* Dover-Publications, 1965.

[2] M. Ainsworth. A preconditioner based on domain decomposition for h-p finite element approximation on quasi-uniform meshes. *SIAM J. Numer. Anal.*, 33(4):1358–1376, 1996.

[3] M. Ainsworth and L. Demkowicz. Explicit polynomial preserving trace liftings on a triangle. Technical Report TICAM Report 03-47, TICAM, November 2003.

[4] M. Ainsworth and B. Gao. An additive Schwarz preconditioner for p-version boundary element approximation of the hypersingular operator in three dimensions. *Numer. Math.*, 85(3):343–366, 2000.

[5] G. E. Andrews, R. Askey, and R. Roy. *Special functions.* Number 71 in Encyclopedia of Mathematics and its applications. Cambridge University Press, 1999.

[6] I. Babuška, A. Craig, J. Mandel, and J. Pitkäranta. Efficent preconditioning for the p-version finite element method in two dimensions. *SIAM J.Numer.Anal.*, 28(3):624–661, 1991.

[7] I. Babuška, M. Griebel, and J. Pitkäranta. The problem of selecting the shape functions for a p-type finite element. *Int. Journ. Num. Meth. Eng.*, 28:1891–1908, 1989.

[8] C. Bernardi, M. Dauge, and Y. Maday. Polynomials in weighted Sobolev spaces: Basics and trace liftings. Technical Report R 92039, Universite Pierre et Marie Curie, Paris, 1993.

[9] S. Beuchler. Multi-grid solver for the inner problem in domain decomposition methods for p-FEM. *SIAM J. Numer. Anal.*, 40(3):928–944, 2002.

[10] S. Beuchler, R. Schneider, and C. Schwab. Multiresolution weighted norm equivalences and applications. *Numerische Mathematik*, 98(1):67–97, 2004.

[11] S. Beuchler and J. Schöberl. Extension operator on tensor product structures in 2d and 3d. Technical Report Report RICAM 2003/01, Johann Radon Institute for Computational and Applied Mathematics, Linz, December 2003.

[12] M. Dubiner. Spectral methods on triangles and other domains. *J. Sci. Computing*, 6:345, 1991.

[13] A. George. Nested dissection of a regular finite element mesh. *SIAM J. Numer. Anal.*, 10:345–363, 1973.

[14] A. George and J.W.-H. Liu. *Computer solution of large sparse positive definite systems.* Prenctice-Hall Inc. Englewood Cliffs. New Jersey, 1981.

[15] B. Guo and W. Cao. A preconditioner for the h-p version of the finite element method in two dimensions. *Numer. Math.*, 75:59–77, 1996.

[16] B. Guo and W. Cao. An iterative and parallel solver based on domain decomposition for the hp-version of the finite element method. *J. Comput. Appl. Math.*, 83:71–85, 1997.

[17] G. Haase, U. Langer, and A. Meyer. Domain decomposition methods with inexact subdomain solvers. *Numerical Linear Algebra with Applications*, 1(1):27–41, 1991.

[18] S. A. Ivanov and V. G. Korneev. On the preconditioning in the domain decomposition technique for the p-version finite element method. Part I. Technical Report SPC 95-35, Technische Universität Chemnitz-Zwickau, December 1995.

[19] S. A. Ivanov and V. G. Korneev. On the preconditioning in the domain decomposition technique for the p-version finite element method. Part II. Technical Report SPC 95-36, Technische Universität Chemnitz-Zwickau, December 1995.

[20] S. Jensen and V. G. Korneev. On domain decomposition preconditioning in the hierarchical $p-$version of the finite element method. *Comput. Methods. Appl. Mech. Eng.*, 150(1–4):215–238, 1997.

[21] G.M. Karniadakis and S.J.Sherwin. *Spectral/HP Element Methods for CFD.* Oxford University Press. Oxford, 1999.

[22] V. Korneev, U. Langer, and L. Xanthis. On fast domain decomposition methods solving procedures for *hp*-discretizations of 3d elliptic problems. *Computational Methods in Applied Mathematics*, 3(4):536–559, 2003.

[23] V. G. Korneev. Почти оптимальныи метод решения задач Дирихле на подобластях декомпосиции иерархическои *hp*-версии. Дифференциальные Уравнения, 37(7):1–15, 2001.

[24] J.M. Melenk, K. Gerdes, and C. Schwab. Fully discrete *hp*-finite elements: Fast quadrature. *Comp. Meth. Appl. Mech. Eng.*, 190:4339–4364, 1999.

[25] R. Munoz-Sola. Polynomial liftings on a tetrahedron and applications to the *h-p* version of the finite element method in three dimensions. *SIAM J. Numer. Anal.*, 34(1):282–314, 1996.

[26] C. Schwab. *$p-$ and $hp-$finite element methods. Theory and applications in solid and fluid mechanics.* Clarendon Press. Oxford, 1998.

[27] P. Solin, K. Segeth, and I. Dolezel. *Higher-Order Finite Element Methods.* Chapman and Hall, CRC Press, 2003.

[28] F.G. Tricomi. *Vorlesungen über Orthogonalreihen.* Springer. Berlin-Göttingen-Heidelberg, 1955.