Solving the Signorini Problem on the Basis of Domain Decomposition Techniques

J. Schöberl[*], Linz

Abstract

Solving the Signorini Problem on the Basis of Domain Decomposition Techniques. The finite element discretization of the Signorini Problem leads to a large scale constrained minimization problem. To improve the convergence rate of the projection method preconditioning must be developed. To be effective, the relative condition number of the system matrix with respect to the preconditioning matrix has to be small, and the applications of the preconditioner as well as the projection onto the set of feasible elements have to be fast computable. In this paper, we show how to construct and analyze such preconditioners on the basis of domain decomposition techniques. The numerical results obtained for the Signorini problem as well as for plane elasticity problems confirm the theoretical analysis quite well.

AMS Subject Classifications: 73T05, 35J85, 65N55, 65F35, 65K10

Key words: contact problem, variational inequality, domain decomposition, preconditioning.

Zusammenfassung

Die Auflösung des Signorini Problems auf der Basis von Gebietszerlegungsmethoden. Die Finite Elemente Diskretisierung des Signorini-Problems führt zu einem restringierten Minimierungsproblem mit vielen Freiheitsgraden. Zur Verbesserung der Konvergenzrate des Projektionsverfahren müssen Vorkonditionierungstechniken entwickelt werden. Um effizient zu sein, muß die relative Konditionszahl der Systemmatrix in Bezug auf die Vorkonditionierungsmatrix klein sein, und die Anwendungen der Vorkonditionierung als auch der Projektion in die zulässige Menge müssen schnell berechenbar sein. In dieser Arbeit wird die Konstruktion und Analyse solcher Vorkonditionierer auf Gebietszerlegungsbasis dargestellt. Numerische Ergebnisse für das Signorini Problem und Kontaktprobleme aus der ebenen Elastizität stimmen mit der theoretischen Analyse gut überein.

[*] This research has been supported by the Austrian Science Foundation - 'Fonds zur Förderung der wissenschaftlichen Forschung (FWF)' - under project grant P10643-TEC.

1 Introduction

The contact problem is an important problem in computational mechanics. An elastic body is deformed due to volume and surface forces, but the body should not penetrate a given, rigid obstacle. This leads to unilateral boundary conditions, called contact conditions. We refer to [28], [7], [23], [22], [17] for mathematical modeling and analysis.

In this paper we are interested in fast numerical algorithms for solving the finite dimensional constrained minimization problem arising from the finite element discretization of the Signorini problem or elastic contact problem [5], [8], [22], [10], [9], [17]. There are classical iterative methods like point projection methods and point over-relaxation methods [8]. These methods suffer from slow convergence rates on fine meshes. Multigrid methods have been successfully applied to obstacle problems with inequality constraints in the whole domain by [15], [25], [19], [20]. Domain decomposition methods for variational inequalities have been investigated in [31], [18], [1]. But these domain decomposition methods differ from our method based on domain decomposition preconditioners. Boundary element methods have also been applied for contact problems in [27], [16], [30].

For the sake of simplicity, we will consider the Signorini Problem for the Poisson equation to develop and analyze the algorithms, but we also provide numerical results for the contact problem in elasticity.

For the Signorini problem, the unknown function u is restricted from below on the Signorini part Γ_C of the boundary $\Gamma = \partial\Omega$. In the classical form, the Signorini problem reads as follows:

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_N, \\ u \geq g, \quad \frac{\partial u}{\partial n} \geq 0, \quad \frac{\partial u}{\partial n}(u-g) &= 0 && \text{on } \Gamma_C. \end{aligned} \tag{1}$$

The domain Ω is supposed to be bounded in $\mathbf{R}^d$, $d = 2$ or 3 with a Lipschitz-continuous boundary $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ such that $meas(\Gamma_D) \neq 0$ and $meas(\Gamma_C) \neq 0$.

The finite element discretization of the weak form of (1) leads to the finite dimensional Constrained Minimization Problem (CMP)

$$\text{Find } u \in K: \quad J(u) = \inf_{v \in K} J(v), \quad \text{with} \quad J(v) := \frac{1}{2} v^T A v - f^T v, \tag{2}$$

or to the equivalent variational inequality

$$\text{Find } u \in K: \quad u^T A (v-u) \geq f^T (v-u) \quad \forall v \in K. \tag{3}$$

The mesh size is denoted by h, the total number of unknowns is N, the number of unknowns on Γ_C is N_C. The solution u, the symmetric and positive definite (spd) system matrix A

and the right-hand side vector f are split into contact boundary components (C) and inner plus Neumann boundary components (I), i.e.

$$u = \begin{pmatrix} u_C \\ u_I \end{pmatrix}, \qquad A = \begin{pmatrix} A_C & A_{CI} \\ A_{IC} & A_I \end{pmatrix}, \qquad f = \begin{pmatrix} f_C \\ f_I \end{pmatrix}. \tag{4}$$

The convex set of feasible functions $K \subset V := \mathbf{R}^N$ is defined by

$$K = \{v \in V : v_C \geq g_C\}, \tag{5}$$

where $v_C \geq g_C$ is meant component-wise.

To solve the finite dimensional constrained minimization problem, the projection method is applied. This method is not very popular in practice, because its convergence rate is very slow on fine meshes, unless good preconditioning is applied. The goal of this paper is to present preconditioning techniques satisfying the following properties:

- The relative condition number of A with respect to C is small,
- the operation $C^{-1} \times v$ is fast executable,
- the projection P with respect to the C energy norm onto K is fast computable,

with the spd preconditioning matrix C. Multi-level preconditioners as well as domain decomposition (DD) preconditioners have been developed for satisfying the first two requirements quite well. The basic concept of Dirichlet DD is the approximative decoupling of the global FE space into inner and extended boundary subspaces. The projection involves only the boundary subspace. To implement this projection, the dual problem is introduced. This enables us to use well-known boundary preconditioners. To avoid two cascaded iterations until convergence, we analyze also the truncated version by means of the Bramble-Pasciak transformation, introduced in [3]. If the underlying components are optimal and a uniformly refined mesh is used, then the complexity is of optimal order for the two as well as three dimensional case. We will combine our algorithms also with nested and adaptive concepts and study the behavior by means of numerical examples.

The rest of the paper is organized as follows. The projection algorithm and its approximative version are presented in Section 2. Convergence in the energy norm is proved. The concepts of additive DD preconditioning is shortly repeated in Section 3. In Section 4, both concepts are combined to develop an optimal preconditioner for the projection algorithm. The truncated variant is analyzed in Section 5. Finally, numerical results are presented in Section 6.

2 The Projection Method

The finite dimensional CMP (2) can be solved by the projection method which reads as:

Algorithm 1 (Projection Method)

```
Choose an arbitrary initial guess $u^1 \in K$.
For $k = 1, 2, \ldots$ do
    $u^{k+1} = P\left(u^k + \tau C^{-1}\left(f - Au^k\right)\right)$.
```

In Algorithm 1, C is the symmetric and positive definite (spd) preconditioning matrix. We assume the spectral equivalence inequalities

$$\underline{\alpha} C \leq A \leq \overline{\alpha} C \tag{6}$$

in the sense of Euclidean inner product, with the positive spectral equivalence constants $\underline{\alpha}$ and $\overline{\alpha}$. In general, these bounds depend on the mesh parameter h. If they are independent of h and the operation $C^{-1} \times v$ is fast executable (i.e. via $O(N)$ arithmetical operations), then we call the preconditioner C asymptotically optimal. Asymptotically optimal preconditioners can be constructed by multi-level techniques as well as by multi-level - DD techniques, see Section 3.

We assume, that the real, positive relaxation parameter τ is chosen such that $\tau \leq 1/\overline{\alpha}$. The operator P is the projection onto K with respect to the energy norm induced by the spd preconditioning matrix C:

$$P : w \in V \to P(w) \in K : \quad \|P(w) - w\|_C \leq \|v - w\|_C, \ \forall v \in K. \tag{7}$$

It is straightforward to show that u^k converges to its limit u in C-energy norm with convergence factor $1 - \tau\underline{\alpha}$ (see, e.g., [9]). A convergence rate estimation in A-energy norm is necessary for the composite algorithm in Section 5. However, it was not available from the literature. It could not be shown that u^k converges monotonically to u in A-norm, but the main theorem of this section provides a monotone decay of the quadratic functional J, and it estimates the convergence rate ρ.

For our application, the exact projection P is too expensive to compute, and so it is replaced by an approximative projection $\tilde{P}$. This leads us to the approximative projection method:

Algorithm 2 (Approximative Projection Method)

```
Choose an arbitrary $u^1 \in K$.
For $k = 1, 2, \ldots$ do
    $\tilde{u}^k = u^k + \tau C^{-1}\left(f - Au^k\right)$,
    $u^{k+1} = \tilde{P}(\tilde{u}^k)$.
```

The following theorem estimates the convergence rate of the approximative projection method, which also includes the exact projection method with $\rho_P = 0$.

Theorem 1 (Energy convergence rate estimate)
Let u^k be the sequence generated by Algorithm 2. The relaxation parameter τ is chosen in the interval $(0, 1/\overline{\alpha}]$. The approximative projection $\tilde{P}$ fulfills

$$\|\tilde{P}(\tilde{u}^k) - \tilde{u}^k\|_C^2 \leq \rho_P \|u^k - \tilde{u}^k\|_C^2 + (1-\rho_P)\,\|P(\tilde{u}^k) - \tilde{u}^k\|_C^2, \tag{8}$$

with $\rho_P \in [0,1)$. Then the estimate

$$J(u^{k+1}) \leq \rho J(u^k) + (1-\rho)\,J(u) \tag{9}$$

holds for every $k \in \mathbf{N}$ with the convergence rate

$$\rho = 1 - \frac{\tau \underline{\alpha}}{2}(1-\rho_P). \tag{10}$$

The error in A-energy norm is bounded by

$$\|u - u^k\|_A^2 \leq 2\rho^{k-1}\left(J(u^1) - J(u)\right). \tag{11}$$

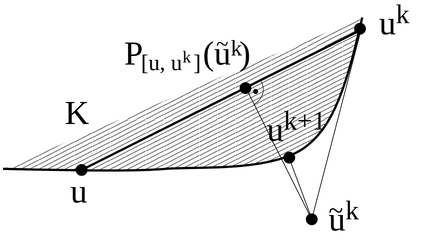
Proof: First, we reduce the approximative projection to the case of an exact projection. We use $\tau^{-1}C - A \geq 0$, the definition of $\tilde{u}^k$ and (8) to obtain

$$\begin{aligned}
J(u^{k+1}) &\leq J(u^{k+1}) + \frac{1}{2}\|u^{k+1} - u^k\|^2_{\frac{1}{\tau}C-A} \\
&= \frac{1}{2\tau}\|u^{k+1} - \tilde{u}^k\|_C^2 - \frac{1}{2\tau}\|u^k - \tilde{u}^k\|_C^2 + J(u^k) \\
&\leq \frac{1}{2\tau}\left(\rho_P\|u^k - \tilde{u}^k\|_C^2 + (1-\rho_P)\|P(\tilde{u}^k) - \tilde{u}^k\|_C^2\right) - \frac{1}{2\tau}\|u^k - \tilde{u}^k\|_C^2 + J(u^k) \\
&= (1-\rho_P)\left[\frac{1}{2\tau}\|P(\tilde{u}^k) - \tilde{u}^k\|_C^2 - \frac{1}{2\tau}\|u^k - \tilde{u}^k\|_C^2 + J(u^k)\right] + \rho_P J(u^k).
\end{aligned}$$

Due to the convexity of K, there holds

$$\|P(\tilde{u}^k) - \tilde{u}^k\|_C \leq \|P_{[u,u^k]}(\tilde{u}^k) - \tilde{u}^k\|_C, \tag{12}$$

with the projection $P_{[u,u^k]}(\tilde{u}^k)$ onto the straight line $[u, u^k]$. This reduces the problem to the plane spanned by $(u, u^k, \tilde{u})$. From the sketch below the next estimates are obvious:



$$\begin{aligned}
&\|P_{[u,u^k]}(\tilde{u}^k) - \tilde{u}^k\|_C^2 - \|u^k - \tilde{u}^k\|_C^2 \\
&= \Big(P_{[u,u^k]}(\tilde{u}^k) - \tilde{u}^k, P_{[u,u^k]}(\tilde{u}^k) - u^k\Big)_C - \Big(\tilde{u}^k - u^k, P_{[u,u^k]}(\tilde{u}^k) - u^k\Big)_C \\
&\leq -\Big(\tilde{u}^k - u^k, P_{[u,u^k]}(\tilde{u}^k) - u^k\Big)_C \\
&= -\max\left\{0, \min\left\{(\tilde{u}^k - u^k, u - u^k)_C, \frac{(\tilde{u}^k - u^k, u - u^k)_C^2}{\|u - u^k\|_C^2}\right\}\right\}. \qquad (13)
\end{aligned}$$

Using the definition of $\tilde{u}^k$, the variational specification (3) of the solution u and the spectral bound $\underline{\alpha}$, we get

$$\begin{aligned}
&\Big(\tilde{u}^k - u^k, u - u^k\Big)_C = \tau(f - Au^k)^T(u - u^k) \\
&= \tau(f - Au)^T(u - u^k) + \tau\|u - u^k\|_A^2 \\
&\geq \tau\underline{\alpha}\|u - u^k\|_C^2. \qquad (14)
\end{aligned}$$

Combining (12), (13), (14) and using $\tau\underline{\alpha} \leq 1$, we get

$$\begin{aligned}
&\|P(\tilde{u}^k) - \tilde{u}^k\|_C^2 - \|u^k - \tilde{u}^k\|_C^2 \\
&\leq -\max\Big\{0, \min\Big\{(\tilde{u}^k - u^k, u - u^k)_C, \tau\underline{\alpha}(\tilde{u}^k - u^k, u - u^k)_C\Big\}\Big\} \\
&= -\tau\underline{\alpha}\Big(\tilde{u}^k - u^k, u - u^k\Big)_C.
\end{aligned}$$

Now, using the last inequality, we continue to estimate the energy functional $J(u^{k+1})$:

$$\begin{aligned}
&J(u^{k+1}) \leq (1 - \rho_P)\left[-\frac{\underline{\alpha}}{2}(\tilde{u}^k - u^k, u - u^k)_C + J(u^k)\right] + \rho_P J(u^k) \\
&= (1 - \rho_P)\left[-\frac{\tau\underline{\alpha}}{2}(f - Au^k)^T(u - u^k) + J(u^k)\right] + \rho_P J(u^k) \\
&= (1 - \rho_P)\left[\left(1 - \frac{\tau\underline{\alpha}}{2}\right) J(u^k) + \frac{\tau\underline{\alpha}}{2}J(u) - \frac{\tau\underline{\alpha}}{4}\|u - u^k\|_A^2\right] + \rho_P J(u^k) \\
&\leq \left[1 - \frac{\tau\underline{\alpha}}{2}(1 - \rho_P)\right] J(u^k) + \frac{\tau\underline{\alpha}}{2}(1 - \rho_P)J(u) \\
&= \rho J(u^k) + (1 - \rho)J(u).
\end{aligned}$$

We use this result to estimate the error in A-energy norm by

$$\begin{aligned}
\|u - u^k\|_A^2 &= 2\Big(J(u^k) - J(u) - (Au - f)^T(u^k - u)\Big) \\
&\leq 2\left(J(u^k) - J(u)\right) \leq 2\rho^{k-1}\left(J(u^1) - J(u)\right). \qquad (15)
\end{aligned}$$

□

For practical computation we need a computable estimate for the iteration error. We get the same error estimator as for iterative methods for linear systems.

Corollary 1
The sequence u^k is generated by Algorithm 1 or by Algorithm 2. Then the iteration error is bounded by

$$\|u - u^{k+1}\|_A^2 \leq \frac{\rho}{1-\rho}(u^{k+1} - u^k)^T(2f - Au^k - Au^{k+1}), \tag{16}$$

with ρ from (10).

Proof: From (15), we get

$$\|u - u^k\|_A^2 \leq 2(J(u^k) - J(u)),$$

and by Theorem 1, we have

$$\begin{aligned}(1-\rho)(J(u^{k+1}) - J(u)) &\leq \rho(J(u^k) - J(u^{k+1})) \\ &= \frac{\rho}{2}(u^{k+1} - u^k)^T(2f - Au^k - Au^{k+1}),\end{aligned}$$

that completes the proof. □

3 Domain Decomposition Preconditioning

In this section we present the approximative additive domain decomposition (DD) preconditioner introduced in [13]. The domain decomposition strategy provides us with a splitting of the FE - space V into inner (V_I) and extended coupling boundary $(\widetilde{V}_C)$ subspaces, which is useful for parallel computing. In our application the coupling boundary is replaced by the contact boundary. By combination with multi-level techniques we gain asymptotically optimal preconditioners.

We recall the splitting of V into the boundary subspace V_C and inner subspace V_I given in (4). These two subspaces are far away from being orthogonal with respect to energy norm. To get an idea about orthogonality we have to go back to the Sobolev space $H^1(\Omega)$. The two subspaces $H_0^1(\Omega)$ and the space of the harmonic functions are orthogonal with respect to the H^1-half norm. The space V_I is an approximation to $H_0^1(\Omega)$. To approximate the harmonic subspace one takes

$$\widetilde{V}_C := \{Ev_C : v_C \in V_C\},$$

where $E = (I_C, E_I^T)^T$ is a computable operator which approximates the discrete harmonic extension of a given boundary function. E is bounded by c_E in the sense of

$$\|Ev_C\|_A \leq c_E \inf_{\substack{w \in V \\ w_C = v_C}} \|w\|_A = c_E \|v_C\|_{S_C}$$

with the Boundary-Schur-Complement $S_C = A_C - A_{CI}A_I^{-1}A_{IC}$.

Note, the best extension operator ($c_E = 1$) would be the solution of the Dirichlet problem with given boundary values v_C. But we need a fast executable extension operator with small constant c_E.

On both sub-spaces V_I and $\tilde{V}_C$ we need spd preconditioners C_I and C_C, respectively, for which we assume the spectral inequalities

$$\underline{\alpha}_C C_C \leq E^T A E \leq \overline{\alpha}_C C_C \quad \text{and} \quad \underline{\alpha}_I C_I \leq A_I \leq \overline{\alpha}_I C_I.$$

Because $E^T A E$ is spectrally equivalent (with constants 1 and c_E^2) to the Schur complement S_C, we call C_C Schur complement preconditioner.

Using these components, we can define the approximative additive domain decomposition preconditioner by

$$C^{-1} = \begin{pmatrix} I_C & 0 \\ E_I & I_I \end{pmatrix} \begin{pmatrix} C_C^{-1} & 0 \\ 0 & C_I^{-1} \end{pmatrix} \begin{pmatrix} I_C & E_I^T \\ 0 & I_I \end{pmatrix}, \tag{17}$$

for which the spectral inequalities

$$\underline{\alpha} C \leq A \leq \overline{\alpha} C$$

hold with the constants

$$\underline{\alpha} = \left(1 - \sqrt{1 - c_E^{-2}}\right) \min\left\{\underline{\alpha}_C, \underline{\alpha}_I\right\} \quad \text{and} \quad \overline{\alpha} = \left(1 + \sqrt{1 - c_E^{-2}}\right) \max\left\{\overline{\alpha}_C, \overline{\alpha}_I\right\}.$$

If the components E, C_I and C_C are asymptotically optimal, then the preconditioner C is as well. For the inner preconditioner a symmetric multigrid preconditioner [2], [21] may be used. For 2D, the transformation to the hierarchical basis [34] gives simple Schur complement preconditioners [13], [29] and extension operators [14], which are optimal up to logarithmic factors. Optimal components for 2D and 3D are constructed by multi-level techniques [4], see [32] for the Schur complement preconditioner and [26] for the extension operator. In [11] additional smoothing improves the extension constant c_E.

All these components have optimal arithmetic complexity, i.e. the operations E, E^T and C_I^{-1} need $O(N)$ operations, while the application of C_C^{-1} needs $O(N_C)$ operations only. The precise analysis of the overall operator C is given in [13], [6]. In [12] it is shown, how a symmetric multiplicative Schwarz preconditioner fits into the framework of additive Schwarz preconditioners.

4 The Projection Method with DD Preconditioning

In this section we will apply the DD preconditioner for the projection method. The first two requirements, namely condition numbers independent of h and fast execution of the preconditioning operation are fulfilled for this preconditioner. To construct the projection we use the basis transformation matrix

$$T = \begin{pmatrix} I_C & 0 \\ E_I & I_I \end{pmatrix} \tag{18}$$

and express the solution u by $u = T\hat{u}$. Therefore, $\hat{u}$ is the solution of the CMP

$$\inf_{\hat{v}\in\hat{K}} \hat{J}(\hat{v})$$

with $\hat{K} = T^{-1}K$ and

$$\hat{J}(\hat{v}) = J(T\hat{v}) = \frac{1}{2}\hat{v}^T \underbrace{T^T A T}_{\hat{A}} \hat{v} - \underbrace{f^T T}_{\hat{f}^T} \hat{v}.$$

If C is a preconditioner for A, then also $\hat{C} = T^T C T$ is one for $\hat{A}$ with the same bounds. For the DD-preconditioner (17), the transformed $\hat{C}$ has the block diagonal structure

$$\hat{C} = \begin{pmatrix} C_C & 0 \\ 0 & C_I \end{pmatrix}.$$

Because v_I can be chosen arbitrarily in a linear space, the set $\hat{K}$ reduces to K:

$$\hat{K} = T^{-1}K = \left\{ \begin{pmatrix} I_C & 0 \\ -E_I & I_I \end{pmatrix} \begin{pmatrix} v_C \\ v_I \end{pmatrix} : v_C \geq g \right\} = K.$$

Now, we apply the projection method to the transformed system:

$$\hat{u}^{k+1} = \hat{P}\left(\hat{u}^k + \tau\hat{C}^{-1}\left(\hat{f} - \hat{A}\hat{u}^k\right)\right).$$

Using the original quantities A, u and f this iteration can be rewritten as

$$u^{k+1} = T\hat{P}\left[T^{-1}u^k + \tau\hat{C}^{-1}T^T\left(Au^k - f\right)\right] = T\hat{P}\tilde{u}^k. \tag{19}$$

The projection involves only boundary components, which are decoupled from the inner components by the inner product induced by $\hat{C}$. Therefore it can be applied just on the boundary, the inner components stay unchanged. Applying the projection means to solve the problem

$$\text{Find}\, u_C \geq g_C : \quad \|u_C - \tilde{u}_C^k\|_{C_C}^2 = \inf_{v_C \geq g_C} \|v_C - \tilde{u}_C^k\|_{C_C}^2. \tag{20}$$

This CMP can be solved by the projection method with Euclidean inner product and pointwise projection

$$u_C^{k,i+1} = P\left(u_C^{k,i} + \tau_2 C_C\left(\tilde{u}_C^k - u_C^{k,i}\right)\right). \tag{21}$$

In DD, we need a fast implementation of the action $C_C^{-1} \times v$. That is ensured by the multi-level Schur complement preconditioner mentioned above. However, for this choice, the operation $C_C \times v$ cannot be performed explicitly. By dualizing the boundary CMP (20) we get the operator C_C into the computable direction. We rewrite the Kuhn Tucker conditions for the CMP, and arrive at the equivalent complementary problem [8]

$$\begin{aligned} C_C u_C + p &= C_C \tilde{u}_C^k, \\ u_C \geq g,\ \ p \leq 0,\ \ p^T(u_C - g) &= 0. \end{aligned}$$

After defining $v := u_C - g$ and exchanging primal and dual variables one can write

$$\begin{aligned} C_C^{-1}\,(-p) + (-v) &= g - \tilde{u}_C^k, \\ (-p) \geq 0, \ \ (-v) \leq 0, \ \ v^T p &= 0. \end{aligned}$$

Now this complementary problem can be rewritten as a CMP in the dual variables as

$$\min_{q \leq 0} \frac{1}{2} q^T C_C^{-1} q - q^T \left(g - \tilde{u}_C^k \right). \tag{22}$$

After calculating p by the projection method, one gets u_C by $u_C = \tilde{u}_C^k - C_C^{-1} p$. Now we can state the whole algorithm developed so far:

Algorithm 3 (Inner-outer projection algorithm)

```
$u^0 = 0$
for $k = 0, 1, 2, ...$
    $w^k = C^{-1}(f - Au^k)$
    $\tilde{u}^k = u^k + \tau w^k$
    $p^0 = 0$
    for $i = 0, 1, 2, ...$
        $p^{i+1} = P\left(p^i + \tau_2 \left(g - \tilde{u}^k - C_C^{-1} p^i\right)\right)$
    $u^{k+1} = \tilde{u}^{k+1} - E C_C^{-1} p^{\infty}$
```

The computational effort is $n_o c_o + n_o n_i c_i$, with the iteration numbers n_o and n_i for outer and inner iterations and the costs per iteration step c_o and c_i, respectively. The number of outer iterations n_o is optimal $O(1)$, n_i is $O(\kappa(C_C))$, which is $O(h^{-1})$ for two as well as three dimensional problems. The costs in the outer iteration are optimal $O(N)$, and the costs per inner iteration are $O(N_C)$. The drawback of Algorithm 3 is the inner iteration, which has to be performed until some convergence criterion is fulfilled. To overcome this disadvantage we look for a better initial value p^0 and perform just a fixed number of iterations. We will analyze this truncated version in the next section by means of the Augmented Lagrangian formulation.

5 Augmented Lagrangian Formulation

The dualization of the large CMP (2) gives

$$\min_{q \in \mathbf{R}^{N_C} : q \leq 0} \frac{1}{2} q^T \mathcal{I}_C^T A^{-1} \mathcal{I}_C q - q^T \mathcal{I}_C^T \left(A^{-1} f + \mathcal{I}_C g \right), \tag{23}$$

where $\mathcal{I}_C$ is the injection of R^{N_C} into R^N. This functional and its derivative are not fast computable, because both need the inverse of A. Using the Augmented Lagrangian technique, we can change it into something computable. Indeed, adding some convex function in v and q the minimum of which in v equals 0 for every fixed q, we obtain the CMP

$$\inf_{\substack{q\in\mathbf{R}^{N_C}:q\leq 0\\ v\in\mathbf{R}^N}} \frac{1}{2}\|Av+\mathcal{I}_C q-f\|^2_{C^{-1}-A^{-1}}+\frac{1}{2}q^T\mathcal{I}_C^T A^{-1}\mathcal{I}_C q-q^T\mathcal{I}_C^T\left(A^{-1}f+\mathcal{I}_C g\right), \tag{24}$$

where C is some properly scaled spd preconditioner for A such that $C^{-1}-A^{-1}$ is spd as well, i.e. $C<A$. The CMP (24) is obviously equivalent to (23), and it can be handled easier than (23), as we will see later on.

5.1 The Bramble Pasciak Transformation

Let us consider now the symmetric, but indefinite system

$$\begin{pmatrix} A & \mathcal{I}_C \\ \mathcal{I}_C^T & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix}=\begin{pmatrix} f \\ g \end{pmatrix}, \tag{25}$$

arising, e.g., from the weak formulation of the Dirichlet boundary conditions as equality constraints to the energy functional (see, e.g., [9]). In [3], it is shown how to change (25) into an spd system provided that some preconditioner C for A is available such that

$$0<A-C\leq\alpha A \tag{26}$$

holds with some $\alpha\in(0,1)$. C may be one of the DD preconditioners (see Section 3) with a proper scaling factor. Multiplying (25) from the left by the matrices

$$\begin{pmatrix} A-C & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ \mathcal{I}_C^T & -I \end{pmatrix}\begin{pmatrix} C^{-1} & 0 \\ 0 & I \end{pmatrix}, \tag{27}$$

we arrive at the spd system

$$\mathcal{A}\mathcal{U}=\mathcal{F} \tag{28}$$

with

$$\mathcal{A}=\begin{pmatrix} AC^{-1}A-A & (AC^{-1}-I)\,\mathcal{I}_C \\ \mathcal{I}_C^T\,(C^{-1}A-I) & \mathcal{I}_C^T C^{-1}\mathcal{I}_C \end{pmatrix},\ \mathcal{U}=\begin{pmatrix} u \\ p \end{pmatrix},\ \mathcal{F}=\begin{pmatrix} (AC^{-1}-I)\,f \\ \mathcal{I}_C^T C^{-1}f-g \end{pmatrix}. \tag{29}$$

In [3], the third multiplication is not applied explicitly. However, it is hidden in the inner product. The main result in [3] was the proof of the spectral equivalence of $\mathcal{A}$ to the block diagonal matrix

$$\begin{pmatrix} A-C & 0 \\ 0 & \mathcal{I}_C^T A^{-1}\mathcal{I}_C \end{pmatrix},$$

with the spectral equivalence constants

$$\underline{\alpha}=\left(1+\alpha/2+\sqrt{\alpha+\alpha^2/4}\right)^{-1}\quad\text{and}\quad\overline{\alpha}=(1+\sqrt{\alpha})/(1-\alpha).$$

If we replace the Schur complement $\mathcal{I}_C^T A^{-1} \mathcal{I}_C$ by the Schur complement $\mathcal{I}_C^T C^{-1} \mathcal{I}_C$, then we get some additional factors into the spectral equivalence constants. The following theorem provides a direct estimation:

Theorem 2
Let C be a preconditioner to A fulfilling (26). Then the block diagonal matrix

$$\mathcal{C} = \begin{pmatrix} \frac{1}{1-\alpha}(A - C) & 0 \\ 0 & \mathcal{I}_C^T C^{-1} \mathcal{I}_C \end{pmatrix} \tag{30}$$

is spectrally equivalent to the block matrix $\mathcal{A}$, defined in (29), with spectral equivalence constants

$$\underline{\alpha} = 1 - \sqrt{\alpha} \quad \text{and} \quad \overline{\alpha} = 1 + \sqrt{\alpha}. \tag{31}$$

Proof: First we show that the inequality

$$(A - C)C^{-1}A \leq \frac{1}{1-\alpha}(A - C) \tag{32}$$

is valid. Indeed, because the matrices involved are spd, the spectral radius $\rho(C^{-1}A)$ is equal to the Rayleigh quotient. Therefore, we can write:

$$\max_{\substack{u \in \mathbf{R}^N \\ u \neq 0}} \frac{((A - C)C^{-1}Au, u)}{((A - C)u, u)} = \max_{\substack{u \in \mathbf{R}^N \\ u \neq 0}} \frac{(C^{-1}Au, u)_{A-C}}{(u, u)_{A-C}} = \rho(C^{-1}A) =$$

$$\max_{\substack{u \in \mathbf{R}^N \\ u \neq 0}} \frac{(Au, u)}{(Cu, u)} = \max_{\substack{u \in \mathbf{R}^N \\ u \neq 0}} \frac{(Au, u)}{(Au, u) - ((A - C)u, u)} \leq \frac{1}{1-\alpha}.$$

The last estimate follows directly from (26). Further, we will use below the inequality

$$2(u, v) \leq \varepsilon \|u\|^2 + \varepsilon^{-1}\|v\|^2, \quad \forall \varepsilon > 0, \tag{33}$$

which is a consequence of the Cauchy-Schwarz inequality and Young's inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2 \quad \forall a, b \in \mathbf{R}$, $\forall \varepsilon > 0$.

Let us now derive the upper bound $\overline{\alpha}$ in the spectral equivalence inequalities. Using inequalities (32) and (33), we can now estimate

$$\begin{aligned}
&(\mathcal{A}\mathcal{U}, \mathcal{U}) = \\
&= ((A - C)C^{-1}Au, u) + 2((A - C)u, \mathcal{I}_C p)_{C^{-1}} + (C^{-1}\mathcal{I}_C p, \mathcal{I}_C p) \\
&\leq ((A - C)C^{-1}Au, u) + \varepsilon^{-1}\|(A - C)u\|_{C^{-1}}^2 + \varepsilon\|\mathcal{I}_C p\|_{C^{-1}}^2 + (C^{-1}\mathcal{I}_C p, \mathcal{I}_C p) \\
&= ((A - C)C^{-1}Au, u) + \varepsilon^{-1}((A - C)C^{-1}(A - C)u, u) + (1 + \varepsilon)\,(C^{-1}\mathcal{I}_C p, \mathcal{I}_C p) \\
&= (1 + \varepsilon^{-1})\,((A - C)C^{-1}Au, u) - \varepsilon^{-1}((A - C)u, u) + (1 + \varepsilon)\,(C^{-1}\mathcal{I}_C p, \mathcal{I}_C p) \\
&\leq \left[(1 + \varepsilon^{-1})(1 - \alpha)^{-1} - \varepsilon^{-1}\right]\,((A - C)u, u) + (1 + \varepsilon)\,(C^{-1}\mathcal{I}_C p, \mathcal{I}_C p) \\
&= \left[(1 + \varepsilon^{-1})(1 - \alpha)^{-1} - \varepsilon^{-1}\right]\,(1 - \alpha)\,\left(\frac{1}{1-\alpha}(A - C)u, u\right) + (1 + \varepsilon)\,(C^{-1}\mathcal{I}_C p, \mathcal{I}_C p).
\end{aligned}$$

Now we choose $\varepsilon = \sqrt{\alpha}$ as the positive solution of the equation

$$\left[(1+\varepsilon^{-1})(1-\alpha)^{-1} - \varepsilon^{-1}\right](1-\alpha) = (1+\varepsilon),$$

that proves the upper bound $1+\sqrt{\alpha}$.

For the lower bound, we use again the inequalities (33) and (32) with $\varepsilon \in (0,1)$ in advance:

$$\begin{aligned}
&(\mathcal{A}\mathcal{U},\mathcal{U}) \geq \\
&\quad \geq \ ((A-C)C^{-1}Au,u) - \varepsilon^{-1}((A-C)C^{-1}(A-C)u,u) + (1-\varepsilon)\,(C^{-1}\mathcal{I}_C p, \mathcal{I}_C p) \\
&\quad = \ (1-\varepsilon^{-1})\,((A-C)C^{-1}Au,u) + \varepsilon^{-1}((A-C)u,u) + (1-\varepsilon)\,(C^{-1}\mathcal{I}_C p, \mathcal{I}_C p) \\
&\quad \geq \ \left[(1-\varepsilon^{-1})(1-\alpha)^{-1} + \varepsilon^{-1}\right](1-\alpha)\left(\frac{1}{1-\alpha}(A-C)u,u)\right) + (1-\varepsilon)\,(C^{-1}\mathcal{I}_C p, \mathcal{I}_C p).
\end{aligned}$$

Note that we have used the fact $1-\varepsilon^{-1} < 0$ for $\varepsilon \in (0,1)$. Indeed, we find that $\varepsilon = \sqrt{\alpha} \in (0,1)$ is again the positive solution of the equilibration equation

$$\left[(1-\varepsilon^{-1})(1-\alpha)^{-1} + \varepsilon^{-1}\right](1-\alpha) = (1-\varepsilon).$$

Therefore, we obtain the left-hand side of the spectral equivalence inequalities:

$$(\mathcal{A}\mathcal{U},\mathcal{U}) \geq (1-\sqrt{\alpha})\,(\mathcal{C}\mathcal{U},\mathcal{U})\,.$$

□

Let us finally reformulate Theorem 2 for an unscaled preconditioner C satisfying the usual spectral equivalence inequalities

$$\underline{\gamma}C < A \leq \overline{\gamma}C,$$

with positive spectral equivalence constants $\underline{\gamma} < \lambda_{\min}(C^{-1}A)$ and $\overline{\gamma} \geq \lambda_{\max}(C^{-1}A)$. Then the rescaled preconditioner $\hat{C} = \underline{\gamma}C$ obviously fulfills (26) with $\alpha = 1-\underline{\gamma}/\overline{\gamma}$. Now Theorem 2 immediately shows that the block matrix

$$\mathcal{A} = \begin{pmatrix} \underline{\gamma}^{-1}AC^{-1}A - A & \left(\underline{\gamma}^{-1}AC^{-1} - I\right)\mathcal{I}_C \\ \mathcal{I}_C^T\left(\underline{\gamma}^{-1}C^{-1}A - I\right) & \underline{\gamma}^{-1}\mathcal{I}_C^T C^{-1}\mathcal{I}_C \end{pmatrix} \tag{34}$$

is spectrally equivalent to the preconditioner

$$\mathcal{C} = \begin{pmatrix} \overline{\gamma}/\underline{\gamma}(A-\underline{\gamma}C) & 0 \\ 0 & \underline{\gamma}^{-1}\mathcal{I}_C^T C^{-1}\mathcal{I}_C \end{pmatrix} \tag{35}$$

with the spectral equivalence constants

$$\underline{\alpha} = 1 - \sqrt{1-\underline{\gamma}/\overline{\gamma}} \quad \text{and} \quad \overline{\alpha} = 1 + \sqrt{1-\underline{\gamma}/\overline{\gamma}}. \tag{36}$$

Note that $\overline{\alpha} \leq 2$ and $\underline{\alpha} > 0.5\underline{\gamma}/\overline{\gamma}$. Therefore, the relative condition number $\kappa(\mathcal{C}^{-1}\mathcal{A}) := \lambda_{\max}(\mathcal{C}^{-1}\mathcal{A})/\lambda_{\min}(\mathcal{C}^{-1}\mathcal{A})$ of the block system is less than four times as worse as the relative condition number $\kappa(C^{-1}A)$ of A and C, i.e. $\kappa(\mathcal{C}^{-1}\mathcal{A}) \leq 4\kappa(C^{-1}A)$. The upper bound 2 for $\overline{\alpha}$ provides a simple choice of the relaxation parameter needed in the Richardson iteration.

Lemma 1
The Schur complement $\mathcal{I}_C^T C^{-1} \mathcal{I}_C$ of the DD - preconditioner (17) is exactly C_C^{-1}.

Proof: By multiplication.

5.2 The Approximative Augmented Projection Algorithm

The matrix of the quadratic term of the functional (24) with an unscaled preconditioner C is exactly the matrix $\mathcal{A}$ in (34). The vector of the linear term is

$$\mathcal{F} = \begin{pmatrix} \left(\underline{\gamma}^{-1} A C^{-1} - I\right) f \\ \underline{\gamma}^{-1} \mathcal{I}_C^T C^{-1} f - g \end{pmatrix}. \tag{37}$$

One can try to use the projection algorithm with inner product (35). We apply one preconditioned Richardson step for the linear system $\mathcal{A}\mathcal{U} = \mathcal{F}$, i.e.

$$\tilde{\mathcal{U}}^k = \mathcal{U}^k + \tau \mathcal{C}^{-1} \left[\mathcal{F} - \mathcal{A}\mathcal{U}\right].$$

With the notations

$$\begin{array}{rcl} w_u^k &=& C^{-1}(f - Au^k - \mathcal{I}_C p^k), \\ w_p^k &=& \mathcal{I}_C w_u^k - \underline{\gamma}(g - \mathcal{I}_C^T u^k), \end{array}$$

the Richardson step simplifies to

$$\begin{pmatrix} \tilde{u}^k \\ \tilde{p}^k \end{pmatrix} = \begin{pmatrix} u^k \\ p^k \end{pmatrix} + \tau \begin{pmatrix} \overline{\gamma}^{-1} w_u^k \\ C_C w_p^k \end{pmatrix}.$$

The restrictions involve only the dual component p and the inner product matrix $\mathcal{C}$ decouples the primal and the dual components, therefore the projection involves only the dual component:

$$\text{Find } p_{ex}^{k+1} \le 0 : \quad \|p_{ex}^{k+1} - \tilde{p}^k\|_{C_C^{-1}}^2 = \inf_{q \le 0} \|q - \tilde{p}^k\|_{C_C^{-1}}^2.$$

By Theorem 1, it is enough to have an approximative projection fulfilling

$$\|p^{k+1} - \tilde{p}^k\|_{C_C^{-1}}^2 \le \rho_P \|p^k - \tilde{p}^k\|_{C_C^{-1}}^2 + (1 - \rho_P)\|p_{ex}^{k+1} - \tilde{p}^k\|_{C_C^{-1}}^2. \tag{38}$$

The approximative projection is implemented by applying n steps of the projection algorithm with Euclidean inner product:

$$\begin{array}{rcl} q^0 &=& p^k \\ q^{i+1} &=& P\left(q^i + \tau_2 C_C^{-1}(\tilde{p}^k - q^i)\right) \\ p^{k+1} &=& q^n. \end{array}$$

It is easily seen, that there is no need to compute $C_C w_p^k$ explicitly. The condition number of the Schur complement preconditioning operation C_C^{-1} is $O(h^{-1})$. By applying Theorem 1 to the inner iteration, we get the estimate

$$\|p^{k+1} - \tilde{p}^k\|_{C_C^{-1}}^2 \leq \rho_i^n \|p^k - \tilde{p}^k\|_{C_C^{-1}}^2 + (1 - \rho_i^n)\|p_{ex}^{k+1} - \tilde{p}^k\|_{C_C^{-1}}^2$$

with the convergence rate $\rho_i = 1 - ch$. The choice $n \geq (ch)^{-1}$ ensures inequality (38) with the h-independent constant $\rho_P \leq e^{-1}$. Theorem 1 proves h-independent convergence rate of the algorithm, which is summarized below.

Algorithm 4 (Approximative Augmented Projection Method)

```
$u^0 = g, p^0 = 0$
for $k = 0, 1, 2, ...$
    $d_u^k = f - Au^k - \mathcal{I}_C p^k$
    $w_u^k = C^{-1} d_u^k$
    $w_p^k = \mathcal{I}_C^T w_u^k - \underline{\gamma}(g_c - \mathcal{I}_C^T u_k)$
    for $i = 0$ to $n - 1$
        $p^{k+\frac{i+1}{n}} = P\left(p^{k+\frac{i}{n}} + \tau_2\left(C_C^{-1} p^k + \tau w_p^k - C_C^{-1} p^{k+\frac{i}{n}}\right)\right)$
    $u^{k+1} = u^k + \tau\overline{\gamma}^{-1} w_u^k$
```

To apply the algorithm, we have to choose some parameters, namely $\underline{\gamma}$ and $\overline{\gamma}$ as close as possible fulfilling the spectral equivalence inequalities $\underline{\gamma}C \leq A \leq \overline{\gamma}C$, τ is calculated via $\tau = \left(1 + \sqrt{1 - \underline{\gamma}/\overline{\gamma}}\right)^{-1}$, τ_2 is set as large as possible such that $\tau_2 C_C^{-1} \leq I$, and n is proportional to the condition number of C_C, e.g. $n = 2^{Multigridlevels}$.

The iteration error can be estimated by Corollary 1, which leads us after some simple calculations to

$$\|u - u^k\|_A^2 + \|p - p^k\|_{C_C^{-1}}^2 \leq c\left[(w_u^k, d_u^k)_{\mathbf{R}^N} + (p^{k+1} - p^k, w_p^k)_{\mathbf{R}^{N_C}}\right], \tag{39}$$

with a constant c only depending on ρ and the spectral equivalence constants of the preconditioner C.

The complexity of Algorithm 4 is $O(n_o c_o + n_o n_i c_i)$. The costs per iteration step c_o and c_i are of optimal order, this means $O(N)$ and $O(N_C)$, respectively. The number of inner iterations n_i is $O(h^{-1})$ for the two dimensional as well as three dimensional case. The number of outer iterations n_o is fixed for a given error ε with $0 < \varepsilon < 1$. This gives the complexity $O((N + h^{-1}N_C)\, ln\, \varepsilon^{-1})$. On a regular refined mesh we have $N_C = O(hN)$ and therefore the complexity reduces to the optimal term $O(N\, ln\, \varepsilon^{-1})$.

5.3 The multiplicative version

Algorithm 4 is an approximative additive Schwarz method with respect to the splitting

$$V_1 = \emptyset \times \mathbf{R}^{N_C}, \qquad V_2 = \mathbf{R}^N \times \emptyset.$$

It is well known for linear problems that the multiplicative method converges approximatively twice as fast as its additive counterpart [24]. Table 2 below shows a similar behavior for the Signorini problem.

With the approximative Projection $\tilde{P}$ the multiplicative method is given by

$$\begin{aligned} \tilde{p}^k &= p^k + (0\ I)\,\mathcal{C}^{-1}\left[\mathcal{F} - \mathcal{A}\begin{pmatrix} u^k \\ p^k \end{pmatrix}\right], \\ p^{k+1} &= \tilde{P}(\tilde{p}^k), \\ u^{k+1} &= u^k + (I\ 0)\,\mathcal{C}^{-1}\left[\mathcal{F} - \mathcal{A}\begin{pmatrix} u^k \\ p^{k+1} \end{pmatrix}\right]. \end{aligned}$$

Because $\mathcal{A} \leq \mathcal{C}$ holds on both subspaces V_1 and V_2, the relaxation parameters could be set to 1. We see that the update for p^{k+1} is equal to the update given by the additive version. The difference between u^{k+1} and u^{k+1}_{ASM} calculated by one additive step from u^k and p^k can be expressed by

$$u^{k+1} - u^{k+1}_{ASM} = (I\ 0)\,\mathcal{C}^{-1}\mathcal{A}\begin{pmatrix} 0 \\ p^k - p^{k+1} \end{pmatrix} = \frac{1}{\overline{\gamma}} E C_C^{-1}(p^{k+1} - p^k). \tag{40}$$

Now we can state the multiplicative algorithm, which is very similar to the inner-outer projection algorithm (Algorithm 3), but the inner iteration is now finite:

Algorithm 5 (Approximative Multiplicative Augmented Projection Method)

```
$u^0 = g, p^0 = 0$
for $k = 0, 1, 2, ...$

    $d_u^k = f - Au^k - \mathcal{I}_C p^k$
    $w_u^k = C^{-1} d_u^k$
    $w_p^k = \mathcal{I}_C^T w_u^k - \underline{\gamma}(g_c - \mathcal{I}_C^T u_k)$
    for $i = 0$ to $n - 1$

        $p^{k+\frac{i+1}{n}} = P\left(p^{k+\frac{i}{n}} + \tau_2\left(C_C^{-1} p^k + w_p^k - C_C^{-1} p^{k+\frac{i}{n}}\right)\right)$

    $u^{k+1} = u^k + \overline{\gamma}^{-1}\left(w_u^k + E C_C^{-1}(p^k - p^{k+1})\right)$
```

6 Numerical Results

To give a numerical verification of the theory, the algorithms above have been implemented and have been applied to equation (1) with $\Omega = (0,1)^2$, $\Gamma_D = [0,1] \times \{1\}$, $\Gamma_N = \{0,1\} \times (0,1)$, $\Gamma_C = [0,1] \times \{0\}$, with source term $f = -1$ and restriction $g = \chi_{[0.25,0.75]}(x_1)$.

The numerical examples have been carried out within the C++ finite element code FE++ on a Sun Ultra 1 with 170 MHz. We used a sequence of hierarchically refined triangular meshes for FE discretization. The level 5 mesh is drawn in Figure 1.

level	$\underline{\gamma}C \le A \le \overline{\gamma}C$			$\underline{\gamma}_C I \le C_C^{-1} \le \overline{\gamma}_C I$		
	N	$\underline{\gamma}$	$\overline{\gamma}$	N_C	$\underline{\gamma}_C$	$\overline{\gamma}_C$
2	9	0.969	1.458	3	0.649	3.851
3	25	0.864	1.732	5	0.552	7.031
4	81	0.814	1.863	9	0.500	13.084
5	289	0.747	1.956	17	0.500	25.077
6	1089	0.693	2.018	33	0.500	49.055
7	4225	0.666	2.065	65	0.500	97.035
8	16641	0.653	2.101	129	0.500	193.02

Table 1: Preconditioner and eigenvalue bounds

level	N	N_C	additive		multiplicative		Ritz functional	
			its	time[s]	its	time[s]	$J(u_h)$	$J(u_h) - J^*$
2	9	3	73	0.08	26	0.05	-	-
3	25	5	91	0.22	36	0.13	0.9180	0.0714
4	81	9	84	0.47	31	0.27	0.8851	0.0385
5	289	17	80	1.38	29	0.86	0.8663	0.0197
6	1089	33	76	4.70	26	2.91	0.8566	0.0100
7	4225	65	75	18.5	23	10.5	0.8516	0.0050
8	16641	129	74	81.7	23	47.7	0.8491	0.0025

Table 2: Algorithms 4 (additive) and 5 (multiplicative) and Ritz functional

As preconditioner C, a symmetric, multiplicative DD preconditioner with a V-2-2 multigrid cycle, a multi-level extension with 2 smoothing steps and a boundary multi-level preconditioner for the Schur complement has been used. The spectral equivalence constants $\underline{\gamma}$ and $\overline{\gamma}$ as well as the eigenvalue bounds $\underline{\gamma}_C$ and $\overline{\gamma}_C$ for the matrix C_C^{-1} have been calculated by the Lanczos method and are given in Table 1. Based on these calculations we have chosen a priori the bounds $\underline{\gamma} = 0.65$, $\overline{\gamma} = 2.2$, $\tau_2 = 2^{-level}$ and $n = 2^{level}$.

We applied Algorithms 4 and 5 to reduce the squared error according to the estimator (39) by a factor of 10^{-8}. The iteration numbers and the CPU times are shown in Table 2. The iteration numbers are bounded by a constant, and the CPU times grow proportional to the number of unknowns.

In our example, the FE solution u_h fulfills the restrictions of the continuous problem. Therefore, the discretization error in energy norm may be estimated as $\|u^* - u_h\|^2 \le 0.5(J(u_h) - J(u^*))$, where u^* is the solution of the continuous problem in weak form. Table 2 contains also the minima of the Ritz functional and the differences to the extrapolated limit $J^* = 0.84657$. The estimated error shows the expected convergence rate $\|u^* - u_h\|^2 = O(h)$ for a solution $u \in H^{1.5}(\Omega)$.

Next, we implemented a nested iterative scheme. On level 3 the problem was solved

level	N	N_C	$J(u_h)$	$J(u_h) - J^*$	solver time[sec]	total time[s]
3	25	5	0.9180	0.0714	0.11	0.12
4	81	9	0.8909	0.0443	0.13	0.17
5	289	17	0.8706	0.0240	0.23	0.41
6	1089	33	0.8588	0.0122	0.58	1.25
7	4225	65	0.8527	0.0061	2.01	4.56
8	16641	129	0.8496	0.0030	8.42	18.5
9	66049	257	0.8481	0.0015	35.2	75.3
10	263169	513	0.8473	0.0007	144.2	305

Table 3: Nested iteration

nearly exactly. On the refined levels we used the prolongated approximative solution of the previous level as initial guess and applied just a fixed number of three steps of the multiplicative algorithm. The prolongation for the primal component u_h was done naturally by linear interpolation. For the dual component p, which is a functional in $H^{-1/2}(\Gamma_C)$, the $L_2(\Gamma_C)$ representative in V_{h_C} was calculated and extended naturally. By the nested method we got nearly the same upper bound for the error within the (accumulated) solver times and total times, including mesh refinement, assembling and solving, shown in Table 3.

Because of the singularities in the solution, an adaptive refined mesh has large advantages with respect to memory requirements and CPU time. For linear problems, the residual error estimator [33] provides one upper and lower bound for the discretization error in energy norm. Without claiming to compute upper and lower bounds to the error, we apply the residual error estimator to the Signorini problem. We compute for each element T the element contribution

$$\eta_T^2 = h_T^2 \|f\|_{0,T}^2 + 0.5 \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h} h_E \|[n_E \nabla u_h]_E\|_{0,E}^2 + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_N} h_E \|n_E \nabla u_h\|_{0,E}^2, \tag{41}$$

where h_T is the longest edge of the triangle T, h_E is the length of the edge E, n_E is the outer normal to the edge, and $[.]_E$ denots the jump accross the edge. $\mathcal{E}(T)$ denotes the 3 edges of an the triangle T, $\mathcal{E}_h$ contains all inner edges of the triangulation and $\mathcal{E}_N$ contains all edges on Γ_N and all edges on Γ_C, on which not both nodes are in contact. All elements T fulfilling $\eta_T \geq 0.5 \max_\tau \eta_\tau$ are marked for refinement. Mesh refinement is done by regular (red) subdivison of marked elements plus forming the conforming (green) closure [33]. Irregular refined elements are removed before further refinement takes place. We calculated the 'error estimator' $\eta^2 = \sum_T \eta_T^2$ and compared it to the error bound $J(u_h) - J^*$. Again, we used a nested solver with three iterations per level. Table 4 shows the results. Although the time complexity is not $O(N)$ anymore, the growing of CPU time is moderate. The adaptive refined mesh at level 11 is given Figure 1.

As announced in the introduction, we have also numerical results for the plane elasticity problem. For the first example, we set

$$\Omega = [0,1] \times [0.05, 1.05], \qquad V = \left\{ v \in \left[H^1(\Omega)\right]^2 : v = 0 \text{ on } \{0\} \times [0.05, 1.05] \right\},$$

level	N	N_C	$J(u_h)$	$J(u_h) - J^*$	$N(J(u_h) - J^*)$	$\eta^2/(J(u_h) - J^*)$	T[sec]
3	25	5	0.95503	0.10847	2.71	3.7	0.1
6	65	14	0.86489	0.01831	1.19	17.8	0.3
9	173	28	0.85287	0.00630	1.09	24.3	0.8
12	423	54	0.84889	0.00232	0.98	27.4	2.1
15	1087	94	0.84741	0.00084	0.91	28.8	5.8
18	2177	123	0.84700	0.00043	0.94	29.4	14.9
21	4441	183	0.84677	0.00020	0.87	30.7	32.0
24	8136	248	0.84668	0.00011	0.88	31.0	71.3
27	15275	336	0.84662	0.00005	0.83	31.9	152.3
30	26170	437	0.84660	0.00003	0.85	33.1	311.4

Table 4: Adaptive refinement

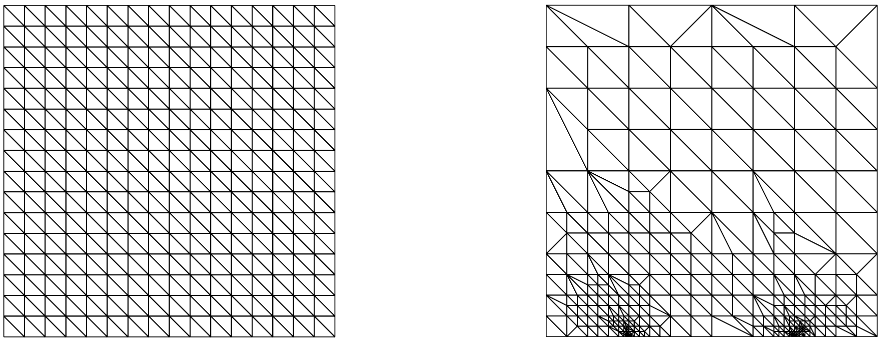


Figure 1: Uniform and adaptive grid with 289 and 311 Nodes, respectively

level	N	N_C	$J(u_h)$	$J(u_h) - \widetilde{J(u)}$	total time[sec]
2	9	3	-0.0059705	1.23E-3	0.27
3	25	5	-0.0066809	5.25E-4	0.38
4	81	9	-0.0070115	1.94E-4	0.64
5	289	17	-0.0071444	6.10E-5	1.56
6	1089	33	-0.0071879	1.75E-5	5.13
7	4225	65	-0.0072006	4.8E-6	20.1
8	16641	129	-0.0072041	1.3E-6	87.2
9	66049	257	-0.0072051	3.0E-7	365.8
∞			$\sim$ -0.0072054		

Table 5: Nested iteration for elasticity problem

for the second one, we use the circle

$$\Omega = C((0.5, 0.5), \sqrt{0.5}), \qquad V = \left[H^1(\Omega)\right]^2.$$

For both examples, we want to find the displacement field minimizing the functional

$$J(v) = \frac{1}{2}\int\limits_{\Omega} \left[2\mu\,|\varepsilon(v)|^2 + \lambda\,(\mathrm{div} v)^2\right]\,\mathrm{dx} - \int\limits_{\Omega} f^T v\,\mathrm{dx}$$

over the set $K = \{v \in V : x_2 + v_2(x) \geq 0 \;\; \forall x \in \partial\Omega\}$, with the linearized strain tensor $\varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i})$, the parameters $\lambda = \mu = 1$ and the volume force $f = (0, -0.2)^T$.

The corresponding bilinear form is continuous and, due to Korn's inequalities, also elliptic modulo the rigid body motions. Therefore a block diagonal preconditioner consisting of two copies of preconditioners for the Laplace equation can be used. We used two times the DD - preconditioner for the Laplacian.

The injection of the boundary space into the global space is now replaced by the injection in normal direction. In the contact zone the normal direction is $(0, -1)$.

We applied the nested multiplicative approximative algorithm with the parameters $\underline{\lambda} = 0.4$, $\overline{\lambda} = 4$, and 12 iterations per level. For the first example, we got the discretization errors shown in Table 5. We observe from Table 5, that the squared error $\|u - u_h\|_1^2 \leq ch^2$, which is the optimal convergence rate for smooth solutions. Therefore an uniformly refined mesh leading to an assymtotically optimal algorithm can be used. We got similar results for the second elasticity example. Both deformed bodies are drawn in Figure 2. For the dual component p the $L_2(\Gamma)$ representative is calculated. It is the normal traction, which is also drawn in Figure 2.

Acknowledgements

I would like to thank Prof. Ulrich Langer for his inspiration to this work and for his clear presentations of domain decomposition techniques. I am also very grateful to the anonymous referees for their valuable comments and suggestions which led to a substantial improvement of the original paper.

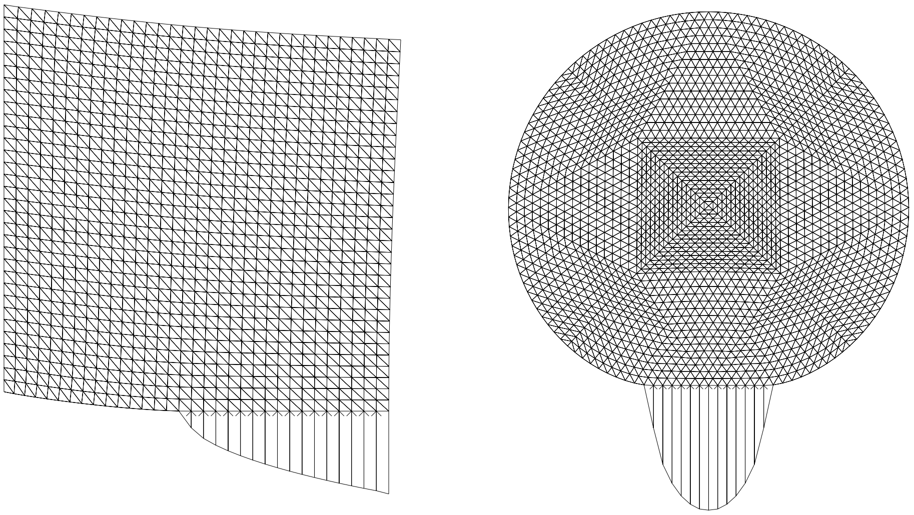


Figure 2: Elasticity problems

References

[1] M. Benassi and R. E. White. Parallel numerical solution of variational inequalities. *SIAM J. Numer. Anal.*, 31(3):813–830, 1994.

[2] D. Braess and W. Hackbusch. A new convergence proof for the multigrid method including the V-cycle. *SIAM J. Numer. Anal.*, 20(5):967–975, 1983.

[3] J. H. Bramble and J. E. Pasciak. A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. *Mathematics of Computation*, 50(181):1–17, 1988.

[4] J. H. Bramble, J. E. Pasciak, and J. Xu. Parallel multilevel preconditioners. *Mathematics of Computation*, 55(191):1–22, July 1990.

[5] F. Brezzi, W. W. Hager, and P. A. Raviart. Error estimates for the finite element solution of variational inequalities. *Numer. Math.*, 28:431–443, 1977.

[6] H. Cheng. *Iterative Solution of Elliptic Finite Element Problems on Partially Refined Meshes and the Effect of Using Inexact Solvers.* PhD thesis, New York University, Courant Institute of Mathematical Sciences, New York, 1993.

[7] G. Fichera. Boundary value problems of elasticity with unilateral constraints. In *Handbuch der Physik - Encyclopedia of Physics*, volume VI a/2 Festkörpermechanik II. Springer, Berlin, 1972.

[8] R. Glowinski, J.-L. Lions, and R. Trémolières. *Numerical analysis of variational inequalities.* North-Holland Publishing Company, Amsterdam - New York - Oxford, 1981.

[9] C. Großmann and H. G. Roos. *Numerik partieller Differentialgleichungen.* Teubner, Stuttgart, 1994.

[10] J. Gwinner. Finite-element convergence for contact problems in plane linear elastostatics. *Quart. of appl. Math.*, L(1):11–25, March 1992.

[11] G. Haase. Hierarchical extension operators plus smoothing in domain decomposition preconditioners. *Appl. Num. Math.*, 23(3):327–346, 1997.

[12] G. Haase and U. Langer. The non-overlapping domain decomposition multiplicative Schwarz method. *Int. J. Comp. Math.*, 44:223–242, 1992.

[13] G. Haase, U. Langer, and A. Meyer. The approximate Dirichlet domain decomposition method. Part I: An algebraic approach. PartII: Applications to 2nd-order elliptic boundary value problems. *Computing*, 47:137–151, 153–167, 1991.

[14] G. Haase, U. Langer, A. Meyer, and S. V. Nepomnyaschikh. Hierarchical extension operators and local multigrid methods in domain decomposition preconditioners. *East-West J. Numer. Math.*, 2(3):173–193, 1994.

[15] W. Hackbusch and H. Mittelmann. On multi-grid methods for variational inequalities. *Numer. Math.*, 42:65–76, 1983.

[16] H. Han. A direct boundary element method for Signorini problems. *Math. Comp.*, 55(191):115–128, 1990.

[17] J. Haslinger, I. Hlaváček, and Nečas. Numerical methods for unilateral problems in solid mechanics. In *Handbook of Numerical Analysis*, volume 2. North Holland, 1996.

[18] K.-H. Hoffmann and J. Zou. Parallel algorithms of Schwarz variant for variational inequalities. *Numer. Funct. Anal. and Optimiz.*, 13:449–462, 1992.

[19] R. H. W. Hoppe. Multigrid algorithms for variational inequalities. *SIAM J. Numer. Anal.*, 24(5):1046–1065, 1987.

[20] R. H. W. Hoppe and R. Kornhuber. Adaptive multilevel methods for obstacle problems. *SIAM J. Numer. Anal.*, 31(2):301–323, 1994.

[21] M. Jung, U. Langer, A. Meyer, W. Queck, and M. Schneider. Multigrid preconditioners and their applications. In G. Telschow, editor, *Third Multigrid Seminar, Biesenthal 1988*, pages 11–52. Karl-Weierstrass-Institut, Berlin, 1989.

[22] N. Kikuchi and J. T. Oden. *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods.* SIAM, Philadelphia, 1988.

[23] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and theri applications.* Academic Press, New York, 1980.

[24] P. Lions. On the Schwarz alternating method I. In R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, editors, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, pages 1 – 42. SIAM, 1988.

[25] J. Mandel. A multilevel iterative method for symmetric, positive definite linear complementarity problems. *Appl. Math. Optim.*, 11:77 – 95, 1984.

[26] S. V. Nepomnyaschikh. Optimal multilevel extension operators. *Preprint SPC 95_3, Technische Universität Chemnitz-Zwickau, Fakultät für Mathematik*, 1995.

[27] H. Schmitz, G. Schneider, and W. Wendland. Boundary element methods for problems involving unilateral boundary conditions. In P. Wriggers and W. Wagner, editors, *Nonlinear computational mechanics–state of the art.* Springer, Berlin Heidelberg New York, 1991.

[28] A. Signorini. Sopra alcune questioni di elastostatica. *Atti Soc. Ial. Progr. Sci.*, 1933.

[29] B. F. Smith and O. B. Widlund. A domain decomposition algorithm using a hierarchical basis. *SIAM J. Sci. Stat. Comput.*, 11(6):1212–1220, 1990.

[30] W. Spann. On the boundary element method for the Signorini problem of the Laplacian. *Numer. Math.*, 65:337 – 356, 1993.

[31] L. Tao, C.-B. Liem, and T.-M. Shih. Parallel algorithms for variational inequalities based on domain decomposition. *System Science and Mathematical Sciences*, 4(4):341–348, 1991.

[32] C. H. Tong, T. F. Chan, and C. C. J. Kuo. A domain decomposition preconditioner based on a change to a multilevel nodal basis. *SIAM J. Sci. Stat. Comp.*, 12(6):1486–1495, 1991.

[33] R. Verfürth. *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques.* Wiley-Teubner, 1996.

[34] H. Yserentant. On the multi-level splitting of finite element spaces. *Numer. Math*, 49:379–412, 1986.