# Additive Schwarz Preconditioning for $p$-Version Triangular and Tetrahedral Finite Elements

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**Abstract**

This paper analyzes two-level Schwarz methods for matrices arising from the $p$-version finite element method on triangular and tetrahedral meshes. The coarse level consists of the lowest order finite element space. On the fine level, we investigate several decompositions with large or small overlap leading to optimal or close to optimal condition numbers. The analysis is confirmed by numerical experiments for a simple model problem and an elasticity problem on a complex geometry. High Order Finite Element Method, Preconditioning.

## 1 Introduction

High order finite element methods ($hp$-FEM) and the closely related spectral element method can lead to very high accuracy and are thus attracting increasing attention in many fields of computational science and engineering. The monographs (SB91; BS94; Sch98; KS99; SDR04; SSD03) on $p$- and $hp$-FEM as well as (GO77; Ors80; CHQZ86; BM92; BM97; DFM02) on spectral and spectral element methods give a broad overview of theoretical and practical aspects of these high order methods.

As the problem size increases (due to small mesh-size $h$ and high polynomial order $p$), the solution of the arising linear system of equations becomes more and more the time-dominating part. Here, iterative solvers can reduce the total simulation time. We consider preconditioners that are based on domain decomposition methods (DW90; GO95; SBG96; TW04; Qua99). In the setting of non-overlapping Schwarz methods (also known as sub-structuring methods) the basic concept is to consider each each high order element (or, more generally, patch of high order elements) as an individual subdomain. Such methods were studied, for example, in (Man90; BCM91; Ain96a; Ain96b; Cas97; Bic97; GC98; SC01; Mel02). For subdomain with tensor product structure arising typically in the spectral element method, efficient preconditioners for the subdomain problems have been developed. In connection with the spectral element method we note here the finite-difference preconditioning in Gauß-Lobatto points (known as "Deville-Mund" preconditioner) and refer to

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(TW04, Chap. 7), (DFM02) for a more detailed discussion; in the context of the *hp*-FEM, we mention the recent publications (KJ99; BSS04; BS05).

In the present work, we study overlapping Schwarz preconditioners with generous overlap on shape regular meshes consisting of triangles (in 2D) and tetrahedra (in 3D). The condition numbers are shown to be bounded uniformly in the mesh size $h$ and the polynomial order $p$. To our knowledge, this is a new result for tetrahedral meshes and generalizes the corresponding work (Pav94) for rectangles/hexahedra to the case of triangles/tetrahedra. We remark that the optimality of the preconditioner has already been conjectured in (PC00) based on numerical evidence. The main difference between the present work and (Pav94) lies in the construction of an $H^1$-stable operator that localizes a piecewise polynomial to a patch. The key ingredients of (Pav94) are $L^2$- and $H^1$-stability properties (on spaces of polynomials) of the tensor product Gauß-Lobatto interpolation operator. For tensorial elements these desirable properties follow from corresponding ones in 1D (see, e. g., (EM06, Lemma 4.1) for a precise statement). For triangles and tetrahedra, an analogous $H^1$-stable interpolation operator is not known and our construction is therefore based on a local averaging operator in the spirit of the Clément quasi-interpolant. Our local averaging operator takes the form of an explicit decomposition of a global finite element function into a coarse grid part and local contributions associated with the vertices, edges, faces, and elements of the mesh. The idea of the construction has already been presented in (SMPZ05).

For the spectral element method, a complete theory of overlapping Schwarz methods is available, covering both the case of generous and small overlap (see, e. g., (TW04, Chap. 7.3), (DFM02)). The use of small overlap is well-established in spectral element methods since one can motivate the choice of overlap and analyze its effect by means of tensor product Gauß-Lobatto grids on each element; in contrast, corresponding tools do not seem to be available at present for the *hp*-FEM based on non-tensorial elements such as triangles and tetrahedra.

The rest of the paper is organized as follows: In Section 2 we state the problem and formulate the main results. We prove the 2D case in Section 3 and extend the proof to the 3D situation in Section 4. Finally, in Section 5 we give numerical results for several versions of the analyzed preconditioners.

## 2 Definitions and Main Result

We consider the Poisson equation on the polyhedral domain $\Omega$ with homogeneous Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega$, and Neumann boundary conditions on the remaining part $\Gamma_N$. With the subspace $V := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, the bilinear-form $A(\cdot,\cdot) : V \times V \to \mathbb{R}$ and the linear-form $f(\cdot) : V \to \mathbb{R}$ are defined as

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad f(v) = \int_\Omega f v \, dx;$$

the weak formulation reads:

$$\text{find } u \in V \text{ such that} \qquad A(u,v) = f(v) \qquad \forall v \in V. \tag{1}$$

We assume that the domain $\Omega$ is subdivided into straight-sided triangular or tetrahedral elements. In general, constants in the estimates depend on the shape of the elements, but they do not depend on the local mesh-size. We define

$$\begin{aligned} \text{the set of vertices} \qquad & \mathscr{V} = \{V\}, \\ \text{the set of edges} \qquad & \mathscr{E} = \{E\}, \\ \text{the set of faces (3D only)} \qquad & \mathscr{F} = \{F\}, \\ \text{the set of elements} \qquad & \mathscr{T} = \{T\}. \end{aligned}$$

We define the sets $\mathscr{V}_f, \mathscr{E}_f, \mathscr{F}_f$ of *free* vertices, edges, and faces not completely contained in the Dirichlet boundary. The high order finite element space is

$$V_p = \{v \in V : v|_T \in P^p \ \forall T \in \mathscr{T}\},$$

where $P^p$ is the space of polynomials of total order $\leq p$. As usual, we choose a basis consisting of lowest order affine-linear functions associated with the vertices, and of edge-based, face-based, and cell-based bubble functions. The Galerkin projection onto $V_p$ leads to a large system of linear equations, which shall be solved with the preconditioned conjugate gradient (PCG) iteration.

This paper is concerned with the analysis of additive Schwarz preconditioning. The basic method is defined by the following space splitting. In Section 5 we will consider several cheaper versions resulting from our analysis. The coarse subspace is the global lowest order space

$$V_0 := \{v \in V : v|_T \in P^1 \ \forall T \in \mathscr{T}\}.$$

For each inner vertex we define the vertex patch

$$\omega_V = \bigcup_{T \in \mathscr{T} : V \in T} T$$

and the vertex subspace

$$V_V = \{v \in V_p : v = 0 \text{ in } \Omega \setminus \omega_V\}.$$

For vertices $V$ not lying on the Neumann boundary, this definition coincides with $V_p \cap H_0^1(\omega_V)$. The additive Schwarz preconditioning operator $C^{-1} : V_p^* \to V_p$ is defined by

$$C^{-1} d = w_0 + \sum_{V \in \mathscr{V}} w_V$$

with $w_0 \in V_0$ such that

$$A(w_0, v) = \langle d, v \rangle \qquad \forall v \in V_0,$$

and $w_V \in V_V$ satisfies

$$A(w_V, v) = \langle d, v \rangle \qquad \forall v \in V_V.$$

This method is very simple to implement for the $p$-version method using a hierarchical basis. The low-order block requires the inversion of the submatrix according to the vertex basis functions. The high order blocks are block-Jacobi steps, where the blocks contain all vertex, edge, face, and cell unknowns associated with mesh entities containing the vertex $V$.

The rate of convergence of the PCG iteration can be bounded by means of the spectral bounds for the quadratic forms associated with the system matrix and the preconditioning matrix. The main result of this paper is to prove optimal results for the spectral bounds:

THEOREM 2.1 The constants $\lambda_1$ and $\lambda_2$ of the spectral bounds

$$\lambda_1 \langle Cu, u \rangle \leq A(u, u) \leq \lambda_2 \langle Cu, u \rangle \qquad \forall u \in V_p$$

are independent of the mesh-size $h$ and the polynomial order $p$.

The proof is based on the additive Schwarz theory, which allows us to express the $C$-form by means of the space decomposition:

$$\langle Cu, u \rangle = \inf_{\substack{u = u_0 + \sum_V u_V \\ u_0 \in V_0, u_V \in V_V}} \|u_0\|_A^2 + \sum \|u_V\|_A^2.$$

The existence of a constant $\lambda_2$ that is independent of $h$ and $p$ follows a standard argument (known as "coloring argument") since $\max_{V \in \mathscr{V}} \operatorname{card}\{V' \in \mathscr{V} \mid \omega_{V'} \cap \omega_V \neq \emptyset\}$ is bounded by a constant that depends solely on the shape regularity of the mesh. In the core part of this paper, we construct an explicit and stable decomposition of $u$ into subspace functions. Section 3 introduces the decomposition for the case of triangles, in Section 4 we prove the results for tetrahedra.

# 3 Subspace splitting for triangles

In this section, we give the proof of Theorem 2.1 for triangles. The case of tetrahedra is postponed to Section 4.

The strategy of the proof is the following: First, we subtract a coarse grid function to eliminate the $h$-dependence. By stepwise elimination, the remaining function is then split into sums of vertex-based, edge-based and inner functions. For each partial sum, we give the stability estimate. This stronger result contains Theorem 2.1, since we can choose corresponding vertices for the edge and inner contributions (see also Section 5).

## 3.1 Coarse grid contribution

In the first step, we subtract a coarse grid function:

LEMMA 3.1 For any $u \in V_p$ there exists a decomposition

$$u = u_0 + u_1 \tag{2}$$

such that $u_0 \in V_0$ and

$$\|u_0\|_A^2 + \|\nabla u_1\|_{L_2}^2 + \|h^{-1} u_1\|_{L_2}^2 \preceq \|u\|_A^2.$$

*Proof.* We choose $u_0 = \Pi_h u$, where $\Pi_h$ is the Clément-operator (Cle75). The norm bounds are exactly the continuity and approximation properties of this operator. $\square$

From now on, $u_1$ denotes the second term in the decomposition (2).

## 3.2 Vertex contributions

In the second step, we subtract functions $u_V$ to eliminate vertex values. Since vertex interpolation is not a bounded operator in $H^1$, we cannot use it. Instead, we construct a new averaging operator mapping into a larger space.

In the following, let $V$ be a vertex not on the Dirichlet boundary $\Gamma_D$, and let $\varphi_V$ be the piecewise linear basis function associated with this vertex. Furthermore, for $s \in [0,1]$ we define the level sets

$$\gamma_V(s) := \{y \in \omega_V : \varphi_V(y) = s\},$$

and write $\gamma_V(x) := \gamma_V(\varphi_V(x))$ for $x \in \omega_V$. For internal vertices $V$, the level set $\gamma_V(0)$ coincides with the boundary $\partial\omega_V$ (cf. Figure 1). The space of functions being constant on these sets reads

$$S_V := \{w \in L_2(\omega_V) : w|_{\gamma_V(s)} = \text{const}, s \in [0,1] \text{ a. e.}\};$$

its finite dimensional counterpart is

$$S_{V,p} := S_V \cap V_p = \text{span}\{1, \varphi_V, ..., \varphi_V^p\}.$$

We introduce the *spider averaging operator*

$$\left(\Pi^V v\right)(x) := \frac{1}{|\gamma_V(x)|} \int_{\gamma_V(x)} v(y)\, dy, \qquad \text{for } v \in L_2(\omega_V).$$

To satisfy homogeneous boundary conditions, we add a correction term as follows (see Figure 2)

$$\left(\Pi_0^V v\right)(x) := \left(\Pi^V v\right)(x) - (\Pi^V v)|_{\gamma_V(0)}(1 - \varphi_V(x)).$$

LEMMA 3.2 The averaging operators fulfill the following algebraic properties

(i)

$$\Pi^V V_p = S_{V,p},$$

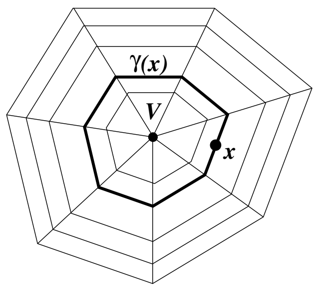


Figure 1: The level sets $\gamma_V(x)$

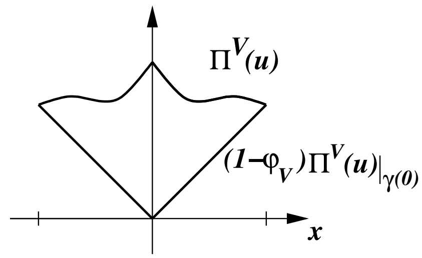


Figure 2: Construction of $\Pi_0^V$

(ii)
$$\Pi_0^V V_p = S_{V,p} \cap V_V,$$

(iii) if $u$ is continuous at $V$, then
$$(\Pi^V u)(V) = \Pi_0^V u(V) = u(V).$$

The proof follows immediately from the definitions.

We denote the distance to the vertex $V$ and the minimal distance to any vertex in $\mathscr{V}$ by
$$r_V(x) := |x - V| \qquad \text{and} \qquad r_{\mathscr{V}}(x) := \min_{V \in \mathscr{V}} r_V(x).$$

LEMMA 3.3 The averaging operators satisfy the following norm estimates

(i)
$$\|\Pi^V u\|_{L_2(\omega_V)} \preceq \|u\|_{L_2(\omega_V)}$$

(ii)
$$\|\nabla \Pi^V u\|_{L_2(\omega_V)} \preceq \|\nabla u\|_{L_2(\omega_V)}$$

(iii)
$$\|r_V^{-1}\{u - \Pi^V u\}\|_{L_2(\omega_V)} \preceq \|\nabla u\|_{L_2(\omega_V)}$$

(iv)
$$\|\nabla\{\varphi_V u - \Pi_0^V u\}\|_{L_2(\omega_V)} \preceq \|\nabla u\|_{L_2(\omega_V)}$$

(v)
$$\|r_{\mathscr{V}}^{-1}\{\varphi_V u - \Pi_0^V u\}\|_{L_2(\omega_V)} \preceq \|\nabla u\|_{L_2(\omega_V)}$$

*Proof.* We parametrize the patch $\omega_V$ by
$$F_V : \gamma_V(0) \times [0,1] \to \omega_V : (y,s) \mapsto y + s(V - y).$$

Splitting the patch into elements, and applying elementwise transformation rules, one proves
$$\int_{\omega_V} \left|f(x)\right| dx \simeq h_V \int_0^1 \int_{\gamma_V(0)} \left|f(F_V(y,s))\right| (1-s)\, dy\, ds,$$
where $h_V := \operatorname{diam}\{\omega_V\}$.

(i) Using $\gamma_V(F_V(y,s)) = \gamma_V(s)$ together with standard inequalities we derive

$$\begin{aligned}
\|\Pi^V u\|^2_{L_2(\omega_V)} &\simeq h_V \int_0^1 \int_{\gamma_V(0)} \left|(\Pi^V u)(F_V(y,s))\right|^2 (1-s)\, dy\, ds \\
&= h_V \int_0^1 \int_{\gamma_V(0)} \left| \frac{1}{|\gamma_V(s)|} \int_{\gamma_V(s)} u(x)\, dx \right|^2 (1-s)\, dy\, ds \\
&\leq h_V \int_0^1 \int_{\gamma_V(0)} \frac{1}{|\gamma_V(s)|} \int_{\gamma_V(s)} u^2(x)\, dx\, (1-s)\, dy\, ds \\
&= h_V \int_0^1 \int_{\gamma_V(0)} \frac{1}{|\gamma_V(0)|} \int_{\gamma_V(0)} u^2(F_V(x,s))\, dx\, (1-s)\, dy\, ds \\
&= h_V \int_0^1 \int_{\gamma_V(0)} u^2(F_V(x,s))\, dx\, (1-s)\, ds \\
&\simeq \int_{\omega_V} u^2(x)\, dx.
\end{aligned}$$

(ii) To verify the estimate for the $H^1$-semi-norm, we rewrite the pointwise gradient:

$$\begin{aligned}
(\nabla \Pi^V u)(x) &= \nabla \left( \frac{1}{|\gamma_V(x)|} \int_{\gamma_V(x)} u(y)\, dy \right) \\
&= \nabla \left( \frac{1}{|\gamma_V(0)|} \int_{\gamma_V(0)} u(F_V(y, \varphi_V(x))\, dy \right) \\
&= \frac{1}{|\gamma_V(0)|} \int_{|\gamma_V(0)|} \frac{d\,(u \circ F_V)}{ds}(y, \varphi_V(x))\, \nabla \varphi_V(x)\, dy \\
&= \frac{1}{|\gamma_V(0)|} \int_{|\gamma_V(0)|} (\nabla u)\big(F_V(y, \varphi_V(x))\big) \cdot (V - y)\, \nabla \varphi_V(x)\, dy.
\end{aligned}$$

Forming the absolute values allows us to estimate

$$\begin{aligned}
|\nabla \Pi^V u|(x) &\leq \frac{1}{|\gamma_V(0)|} \int_{\gamma_V(0)} \left|(\nabla u)(F_V(y, \varphi_V(x)))\right| |V - y| |\nabla \varphi_V|\, dx \\
&\preceq \frac{1}{|\gamma_V(0)|} \int_{\gamma_V(0)} \left|(\nabla u)(F_V(y, \varphi_V(x)))\right| h_V\, h_V^{-1}\, dy \\
&= \frac{1}{|\gamma_V(x)|} \int_{\gamma_V(x)} |(\nabla u)(y)|\, dy \\
&= (\Pi^V |\nabla u|)\,(x).
\end{aligned}$$

The rest follows from the $L_2$-estimate (i) applied to $|\nabla u|$.

(iii) On the manifold $\gamma_V(0)$ there holds the Poincaré inequality

$$\int_{\gamma_V(0)} \left| u(x) - \frac{1}{|\gamma_V(0)|} \int_{\gamma_V(0)} u(y)\, dy \right|^2 dx \preceq h_V^2 \int_{\gamma_V(0)} |\nabla u|^2\, dx.$$

Using $r_V(x) \simeq (1-\varphi_V(x))h_V$ we derive

$$\begin{aligned}
&\int_{\omega_V} \frac{1}{r_V^2}\left(u-\Pi^V u\right)^2 dx \\
&\quad \simeq \quad h_V \int_0^1 \int_{\gamma_V(0)} \frac{1}{r_V^2}\Big(u(F_V(y,s)) - \frac{1}{|\gamma_V(0)|}\int_{\gamma_V(0)} u(F_V(x,s))\,dx\Big)^2 (1-s)\,dy\,ds \\
&\quad \preceq \quad h_V \int_0^1 \int_{\gamma_V(0)} \frac{h_V^2}{r_V^2}\left|\nabla_y u(F_V(y,s))\right|^2 (1-s)\,dy\,ds \\
&\quad \simeq \quad h_V \int_0^1 \int_{\gamma_V(0)} \frac{1}{(1-s)^2}\left|(\nabla u)(F_V(y,s))\frac{\partial F_V}{\partial y}\right|^2 (1-s)\,dy\,ds \\
&\quad = \quad h_V \int_0^1 \int_{\gamma_V(0)} \left|(\nabla u)(F_V(y,s))\right|^2 (1-s)\,dy\,ds \\
&\quad \simeq \quad \|\nabla u\|^2_{L_2(\omega_V)}.
\end{aligned}$$

(iv) Since $\varphi_V 1 = \Pi_0^V 1$, we can subtract the mean value $\bar{u} := \frac{1}{|\omega_V|}\int_{\omega_V} u(x)\,dx$:

$$\begin{aligned}
\|\nabla\{\varphi_V u - \Pi_0^V u\}\| &= \|\nabla\{\varphi_V(u-\bar{u}) - \Pi_0^V(u-\bar{u})\}\| \\
&\leq \|\nabla\{\varphi_V(u-\bar{u})\}\| + \|\nabla\Pi^V(u-\bar{u}) - \Pi^V(u-\bar{u})|_{\gamma_V(0)}\nabla(1-\varphi_V)\| \\
&\preceq \|(\nabla\varphi_V)(u-\bar{u})\| + \|\varphi_V\nabla u\| + \|\nabla u\| + \left|\Pi^V(u-\bar{u})|_{\gamma_V(0)}\right| \|\nabla\varphi_V\| \\
&\preceq h^{-1}\|u-\bar{u}\| + \|\nabla u\| \\
&\preceq \|\nabla u\|.
\end{aligned}$$

We have used (ii) and the trace inequality for

$$\begin{aligned}
\left|\Pi^V(u-\bar{u})|_{\gamma_V(0)}\right| &= \frac{1}{|\gamma_V(0)|}\left|\int_{\gamma_V(0)} u-\bar{u}\,dx\right| \leq \\
&\leq |\gamma_V(0)|^{-1/2}\,\|u-\bar{u}\|_{L_2(\gamma_V(0))} \preceq \|\nabla(u-\bar{u})\| + h^{-1}\|u-\bar{u}\|.
\end{aligned} \tag{3}$$

(v) We finally prove $\|r_{\mathscr{V}}^{-1}\{\varphi_V u - \Pi_0^V u\}\|_{L_2(\omega_V)} \preceq \|\nabla u\|_{L_2(\omega_V)}$. From the definition of $r_{\mathscr{V}}$, we get

$$\left\|\frac{1}{r_{\mathscr{V}}}\{\varphi_V u - \Pi_0^V u\}\right\| \simeq \left\|\frac{1}{r_V}\{\varphi_V u - \Pi_0^V u\}\right\| + \sum_{V'\in\omega_V\setminus\{V\}} \left\|\frac{1}{r_{V'}}\{\varphi_V u - \Pi_0^V u\}\right\|.$$

We bound the first term as follows:

$$\begin{aligned}
&\left\|\frac{1}{r_V}\{\varphi_V u - \Pi_0^V u\}\right\|_{L_2(\omega_V)} \\
&\quad = \quad \left\|\frac{1}{r_V}\left\{(1-(1-\varphi_V))u - \Pi^V u + (1-\varphi_V)(\Pi^V u)|_{\gamma_V(0)}\right\}\right\| \\
&\quad = \quad \left\|\frac{1}{r_V}\left\{(u-\Pi^V u) - (1-\varphi_V)(u-\bar{u}) + (1-\varphi_V)\big((\Pi^V u)|_{\gamma_V(0)} - \bar{u}\big)\right\}\right\| \\
&\quad \preceq \quad \left\|\frac{1}{r_V}(u-\Pi^V u)\right\| + \left\|\frac{1-\varphi_V}{r_V}(u-\bar{u})\right\| + \left\|\frac{1-\varphi_V}{r_V}\big((\Pi^V u)|_{\gamma_V(0)} - \bar{u}\big)\right\| \\
&\quad \preceq \quad \|\nabla u\| + h^{-1}\|u-\bar{u}\| + h^{-1}\left|(\Pi^V u)|_{\gamma_V(0)} - \bar{u}\right| |\omega_V|^{1/2} \\
&\quad \preceq \quad \|\nabla u\|_{L_2(\omega_V)}
\end{aligned}$$

We have used that $(1-\varphi_V)/r_V \simeq h^{-1}$, applied the Poincaré inequality on $\omega_V$, and once again employed (3).

Before treating the second term, we prove the following estimate on a triangle $T$:

$$\int_T \frac{1}{(r_{V'})^2} v^2 \, dx \preceq \|\nabla v\|^2_{L_2(T)}, \tag{4}$$

for functions $v$ vanishing on an edge $E$ containing the vertex $V'$. We transform to the reference triangle $\hat{T} = \{(x_1, x_2) : 0 \le x_2 \le x_1 \le 1\}$ and use Friedrichs' inequality:

$$\begin{aligned} \int_T \frac{1}{(r_{V'})^2} v^2 \, dx &\simeq h^2 \int_0^1 \int_0^{x_1} \frac{1}{h^2 x_1^2} v^2(x_1, x_2) \, dx_2 \, dx_1 \\ &\preceq \int_0^1 \int_0^{x_1} \Big(\frac{\partial v}{\partial x_2}\Big)^2 dx_2 \, dx_1 \preceq \|\nabla v\|^2_{L_2(T)} \end{aligned}$$

Since the function $v := \varphi_V u - \Pi_0^V u$ vanishes on the boundary $\partial \omega_V$, inequality (4) can be applied on each triangle $T \subset \omega_V$:

$$\Big\| \frac{1}{r_{V'}} \{\varphi_V u - \Pi_0^V u\} \Big\|_{L_2(\omega_V)} \preceq \big\| \nabla \{\varphi_V u - \Pi_0^V u\} \big\|_{L_2(\omega_V)}$$

Using (iv) and summing over $V'$, we get the desired estimate. Due to shape regularity this sum is finite.

This finishes the proof of Lemma 3.3 □

The *global spider vertex operator* is

$$\Pi_{\mathscr{V}} := \sum_{V \in \mathscr{V}_f} \Pi_0^V.$$

Obviously, $u - \Pi_{\mathscr{V}} u$ vanishes in any vertex $V \in \mathscr{V}_f$. These well-defined zero vertex values are reflected by the following norm definition:

$$|\!|\!| \cdot |\!|\!|^2 := \|\nabla \cdot\|^2_{L_2(\Omega)} + \|\frac{1}{r_{\mathscr{V}}} \cdot \|^2_{L_2(\Omega)} \tag{5}$$

THEOREM 3.1 Let $u_1$ be as in Lemma 3.1. Then, the decomposition

$$u_1 = \sum_{V \in \mathscr{V}_f} \Pi_0^V u_1 + u_2 \tag{6}$$

is stable in the sense of

$$\sum_{V \in \mathscr{V}_f} \|\Pi_0^V u_1\|_A^2 + |\!|\!| u_2 |\!|\!|^2 \preceq \|u\|_A^2. \tag{7}$$

*Proof.* The vertex terms in equation (7) are bounded by

$$\begin{aligned} \|\Pi_0^V u_1\|_A^2 &= \|\Pi^V u_1 - (\Pi^V u_1)|_{\gamma_V(0)} (1 - \varphi_V)\|_A^2 \\ &\le \|\nabla \Pi^V u_1\|^2_{L_2(\omega_V)} + \big|(\Pi^V u_1)|_{\gamma_V(0)}\big|^2 \|1 - \varphi_V\|_A^2 \\ &\preceq \|\nabla u_1\|^2_{L_2(\omega_V)} + h_V^{-2} \|u_1\|^2_{L_2(\omega_V)}. \end{aligned}$$

We have used that $\|1 - \varphi_V\|_A \simeq 1$. Summing up all terms, one obtains

$$\sum_{V \in \mathscr{V}_f} \|\Pi_0^V u_1\|_A^2 \preceq \|\nabla u_1\|^2_{L_2(\Omega)} + \|h^{-1} u_1\|^2_{L_2(\Omega)} \preceq \|u\|_A^2.$$

To bound the second term, we compare with the partition of unity provided by the hat functions:

$$\begin{aligned}\Big|\Big|\Big| u_1 - \sum_{V\in\mathscr{V}_f} \Pi_0^V u_1 \Big|\Big|\Big|^2 &= \Big|\Big|\Big| \sum_{V\in\mathscr{V}_D} \varphi_V u_1 + \sum_{V\in\mathscr{V}_f} (\varphi_V u_1 - \Pi_0^V u_1) \Big|\Big|\Big|^2 \\ &\preceq \sum_{V\in\mathscr{V}_D} |||\varphi_V u_1|||^2 + \sum_{V\in\mathscr{V}_f} |||(\varphi_V u_1 - \Pi_0^V u_1)|||^2\end{aligned}$$

For $V \in \mathscr{V}_D$, the function $u_1$ vanishes on (at least) one edge of the patch $\omega_V$. Hence, using ideas as in the proof of Lemma 3.3, part (iii) we get $\|\frac{1}{r_V}\varphi_V u_1\|_{L^2(\omega_V)} \preceq \|\nabla(\varphi_V u_1)\|_{L^2(\omega_V)}$. Furthermore, since each element $K$ of the patch $\omega_V$ has an edge on which $\varphi_V u_1$ vanishes, we may employ (4) and reason as in the proof of Lemma 3.3, part (v) to obtain

$$|||\varphi_V u_1|||^2 \preceq \|\nabla(\varphi_V u_1)\|^2_{L_2(\omega_V)} \preceq \|\nabla u_1\|^2_{L_2(\omega_V)};$$

here, the last estimate follows again from the fact that $u_1$ vanishes on an edge in $\omega_V$. □

For the rest of this section, $u_2$ denotes the second term in the decomposition (6).

### 3.3 Edge contributions

As seen in the last subsection, the remaining function $u_2$ vanishes in all vertices. We now introduce an edge-based interpolation operator to carry the decomposition further, such that the remaining function, $u_3$, contributes only to the inner basis functions of each element.

Therefore we need a lifting operator which extends edge functions to the whole triangle preserving the polynomial order. Such operators were introduced in Babuška et al. (BCM91), and later simplified and extended to 3D by Muñoz-Sola (Mun97). The lifting on the reference element $T^R$ with vertices $(-1,0)$, $(1,0)$, $(0,1)$ and edges $E_1^R := (-1,1)\times\{0\}$, $E_2^R$, $E_3^R$ reads:

$$(\mathscr{R}_1 w)(x_1,x_2) := \frac{1}{2x_2}\int_{x_1-x_2}^{x_1+x_2} w(s)\,ds,$$

for $w \in L_1([-1,1])$. The modification by Muñoz-Sola preserving zero boundary values on the edges $E_2^R$ and $E_3^R$ is

$$(\mathscr{R}w)(x_1,x_2) := (1-x_1-x_2)\,(1+x_1-x_2)\left(\mathscr{R}_1 \frac{w}{1-x_1^2}\right)(x_1,x_2).$$

For an arbitrary triangle $T = F_T(T^R)$ containing the edge $E = F_T(E_1^R)$, its transformed version reads

$$\mathscr{R}_T w := \mathscr{R}\big[w\circ F_T\big]\circ F_T^{-1}.$$

The Sobolev space $H_{00}^{1/2}(E)$ on an edge $E = [V_{E,1}, V_{E,2}]$ is defined by its corresponding norm

$$\|w\|^2_{H_{00}^{1/2}(E)} := \|w\|^2_{H^{1/2}(E)} + \int_E \frac{1}{r_{V_E}} w^2\,ds,$$

with

$$r_{V_E} := \min\{r_{V_{E,1}}, r_{V_{E,2}}\}.$$

LEMMA 3.4 The Muñoz-Sola lifting operator $\mathscr{R}_T$ satisfies:

(i) $\mathscr{R}_T$ maps polynomials $w \in P_0^p(E) := \{v \in P^p(E) : v = 0 \text{ in } V_{E,1}, V_{E,2}\}$ into $\{v \in P^p(T) : v = 0 \text{ on } \partial T \setminus E\}$.

(ii) $\mathscr{R}_T$ is bounded in the sense

$$\|\mathscr{R}_T w\|_{H^1(T)} \preceq \|w\|_{H_{00}^{1/2}(E)}.$$

The proof follows from (BCM91) and (Mun97).

We call $\omega_E := \omega_{V_{E,1}} \cap \omega_{V_{E,2}}$ the edge patch. We define an *edge-based interpolation operator* as follows:

$$\begin{aligned} \Pi_0^E : \{v \in V_p : v = 0 \text{ in } \mathscr{V}\} &\to H_0^1(\omega_E) \cap V_p, \\ (\Pi_0^E u)|_T &:= \mathscr{R}_T \operatorname{tr}_E u. \end{aligned} \tag{8}$$

LEMMA 3.5 The edge-based interpolation operator $\Pi_0^E$ defined in (8) is bounded in the $|||\cdot|||$-norm:

$$\|\nabla \Pi_0^E u\|_{L_2(\omega_E)} \preceq |||u|||_{\omega_E}$$

*Proof.* First, we apply Lemma 3.4 on a single triangle $T \subset \omega_E$:

$$\begin{aligned} \|\nabla \Pi_0^E u\|_{L_2(T)}^2 &= \|\nabla \mathscr{R}_T \operatorname{tr}_E u\|_{L_2(T)}^2 \\ &\preceq \|\operatorname{tr}_E u\|_{H^{1/2}(E)}^2 + \int_E \frac{1}{r_{V_E}} (\operatorname{tr}_E u)^2 \, ds. \end{aligned}$$

For the first term, the trace theorem can be applied.

The second term, the weighted $L_2$-norm on the edge, can be bounded by a weighted norm on the triangle. We transform to the reference triangle,

$$\int_E \frac{1}{r_{V_E}} u^2 \, ds = \int_{E_1^R} \frac{1}{r_{V_{E_1^R}}} (u \circ F_T)^2 \, ds,$$

and write $u^R := u \circ F_T$. Due to symmetry, we consider only the right half of the edge $E_1^R$, where $r_{E_1^R} = \frac{1}{1-x_1}$, and finally apply a trace inequality:

$$\begin{aligned} &\int_0^1 \frac{1}{1-x_1} u^R(x_1,0)^2 \, dx_1 \preceq \\ &\preceq \int_0^1 \frac{1}{1-x_1} \int_0^{1-x_1} (1-x_1) \Big[\frac{\partial u^R}{\partial x_2}\Big]^2 + \frac{1}{1-x_1} [u^R]^2 \, dx_2 \, dx_1 \\ &\preceq |||u^R|||_{T^R}^2 \simeq |||u|||_T^2. \end{aligned}$$

□

This leads us immediately to

THEOREM 3.2 Let $u_2$ be as in Theorem 3.1. Then, the decomposition

$$u_2 = \sum_{E \in \mathscr{E}_f} \Pi_0^E u_2 + u_3 \tag{9}$$

satisfies $u_3 = 0$ on $\bigcup_{E \in \mathscr{E}_f} E$ and is bounded in the sense of

$$\sum_{E \in \mathscr{E}_f} \|\nabla \Pi_0^E u_2\|_{L_2}^2 + \|\nabla u_3\|_{L_2}^2 \preceq |||u_2|||^2. \tag{10}$$

## 3.4 Main result

*Proof of Theorem 2.1 for the case of triangles.* Summarizing the last subsections, we have

$$u_1 = u - \Pi_h u, \qquad u_2 = u_1 - \sum_{V \in \mathscr{V}_f} \Pi_0^V u_1, \qquad u_3 = u_2 - \sum_{E \in \mathscr{E}_f} \Pi_0^E u_2,$$

and the decomposition

$$u = \Pi_h u + \sum_{V \in \mathscr{V}_f} \Pi_0^V u_1 + \sum_{E \in \mathscr{E}_f} \Pi_0^E u_2 + \sum_{T \in \mathscr{T}} u_3|_T. \tag{11}$$

is stable in the $\|\cdot\|_A$-norm.

For every edge $E$ or triangle $T$, we can find a vertex $V$ such that the contribution $\Pi_0^E u_2$ or $u_3|_T$ is an element of the space $V_V$. Since for each vertex only finitely many terms appear, we can use the triangle inequality and finally arrive at the desired spectral bound

$$\langle Cu,u\rangle = \inf_{\substack{u=u_0+\sum_V u_V\\ u_0\in V_0, u_V\in V_V}} \|u_0\|_A^2 + \sum_V \|u_V\|_A^2 \preceq \langle Au,u\rangle\,.$$

$\square$

# 4 Subspace splitting for tetrahedra

Most of the proof for the 3D case follows the strategy pursued in Section 3. The only principal difference is the edge interpolation operator, which shall be treated in more detail.

## 4.1 Coarse and vertex contributions

We define the level surfaces of the vertex hat basis functions

$$\Gamma_V(x) := \{y : \varphi_V(y) = \varphi_V(x)\}.$$

As in 2D, we first subtract the coarse grid Clément quasi-interpolant $\Pi_h u$ (cf. Lemma 3.1)

$$u_1 = u - \Pi_h u,$$

and secondly the multi-dimensional vertex interpolant to obtain

$$u_2 = u_1 - \Pi_{\mathscr{V}} u_1,$$

where the definitions of $\Pi^V$, $\Pi_0^V$, $\Pi_{\mathscr{V}}$ are the same as in Section 3, only the level set lines $\gamma_V$ are replaced by the level surfaces $\Gamma_V$. With the same arguments, one easily shows that

$$\sum_{v\in\mathscr{V}_f} \|\Pi_0^V u_1\|_A^2 + \|\nabla u_2\|_{L_2}^2 + \|r_{\mathscr{V}}^{-1} u_2\|_{L_2}^2 \preceq \|u\|_A^2. \tag{12}$$

## 4.2 Edge contributions

Let $F := \{(s,t) : s \geq 0, t \geq 0, s+t \leq 1\}$ be the reference triangle in Figure 3. For $(s,t) \in F$, we define the level lines

$$\gamma_E(s,t) := \{x : \varphi_{V_{E,1}}(x) = s \text{ and } \varphi_{V_{E,2}}(x) = t\},$$

and write

$$\gamma_E(x) := \gamma_E(\varphi_{V_{E,1}}(x), \varphi_{V_{E,2}}(x))$$

for the level line corresponding to a point $x$ in the edge-patch $\omega_E$, see Figure 4.

Define the space of constant functions on these level lines,

$$S_E := \{v : v|_{\gamma_E(x)} = \text{const}\}$$

and its polynomial subspace $S_{E,p} := S_E \cap V_p$. The edge averaging operator into $S_E$ reads

$$\left(\Pi^E v\right)(x) := \frac{1}{|\gamma_E(x)|} \int_{\gamma_E(x)} v(y)\,dy.$$

Furthermore, let $r_{V_E} := \min\{r_{V_{E,1}}, r_{V_{E,2}}\}$, and $r_E(x) := \operatorname{dist}\{x, E\}$.

LEMMA 4.1 The edge-averaging operator satisfies

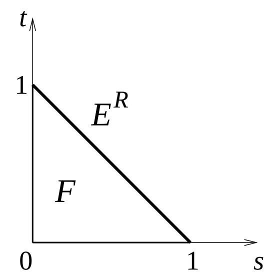


Figure 3: Reference triangle

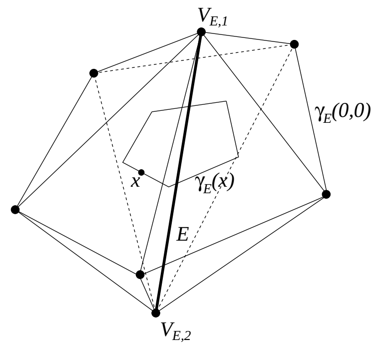


Figure 4: Edge patch

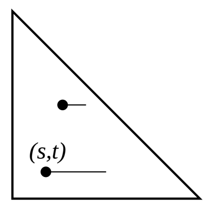


Figure 5: Averaging lines of the smoothing operator $S_s$

(i)

$$\Pi^E V_p = S_{E,p},$$

(ii)

$$(\Pi^E u)(x) = u(x), \qquad \text{for } x \in E,\ u \text{ continuous},$$

(iii)

$$\|\nabla \Pi^E u\|_{L_2(\omega_E)} \preceq \|\nabla u\|_{L_2(\omega_E)},$$

(iv)

$$\|r_{V_E}^{-1} \Pi^E u\|_{L_2(\omega_E)} \preceq \|r_{\mathscr{V}}^{-1} u\|_{L_2(\omega_E)},$$

(v)

$$\|r_E^{-1}(u - \Pi^E u)\|_{L_2(\omega_E)} \preceq \|\nabla u\|_{L_2(\omega_E)},$$

where $u \in H^1(\omega_E)$.

The proof is analogous to the proofs of Lemma 3.2 and Lemma 3.3.

Next, the edge-interpolation operator is modified to satisfy zero boundary conditions on $\partial \omega_E$. By the isomorphism

$$v_F(s,t) := v|_{\gamma_E(s,t)}, \qquad \text{for } v \in S_E, \tag{13}$$

the function space $S_E$ can be identified with a space on the triangle $F$.

LEMMA 4.2 The isomorphism (13) fulfills the following equivalences for functions $v \in S_E$:

(i)

$$\|v\|_{L_2(\omega_E)} \simeq h^{3/2}\, \|r_{E^R}^{1/2} v_F\|_{L_2(F)},$$

(ii)

$$\|\nabla v\|_{L_2(\omega_E)} \simeq h^{1/2}\, \|r_{E^R}^{1/2} \nabla v_F\|_{L_2(F)},$$

(iii)

$$\|r_{V_E}^{-1} v\|_{L_2(\omega_E)} \simeq h^{1/2} \left\| \frac{r_{E^R}^{1/2}}{r_{V_{E^R}}} v_F \right\|_{L_2(F)},$$

(iv)

$$\|r_E^{-1} v\|_{L_2(\omega_E)} \simeq h^{1/2}\, \|r_{E^R}^{-1/2} v_F\|_{L_2(F)},$$

where

$$E^R := \{(s,t) \in F : s+t = 1\},$$
$$r_{E^R}(s,t) := 1-s-t, \quad \text{and} \quad (r_{V_{E^R}})^{-1} := \tfrac{1}{1-s} + \tfrac{1}{1-t}.$$

*Proof.* We parameterize the edge-patch $\omega_E$ by

$$\begin{array}{rcl} F_E : \gamma_E(0,0) \times F & \to & \omega_E \\ (z,(s,t)) & \mapsto & z + s(V_{E,1} - z) + t(V_{E,2} - z). \end{array}$$

Note that functions $v \in S_E$ do not depend on the parameter $z \in \gamma_E(0,0)$ and $v_F(s,t) = (v \circ F_E)(z,s,t)$ for any $z \in \gamma_E(0,0)$. Equivalence (i) holds due to the transformation of the integrals

$$\begin{aligned} \int_{\omega_E} |v|^2\,dx &\simeq h^2 \int_{\gamma_E(0,0)} \int_0^1 \int_0^{1-s} |v \circ F_E|^2 (1-s-t)\,dt\,ds\,dz \\ &\simeq h^3 \int_F |v_F|^2\, r_{E^R}\, d(s,t). \end{aligned}$$

Derivatives evaluate to $\frac{\partial v_F}{\partial s} = (\nabla v) \cdot \frac{\partial F_E}{\partial s} = (\nabla v) \cdot (V_{E,1} - z)$, and thus

$$|(\nabla v) \circ F_E| \simeq h^{-1} |\nabla v_F|.$$

Observing (i), we note that (ii) holds. The equivalences (iii) and (iv) follow from $r_{V_E} \circ F_E \simeq h\, r_{V_{E^R}}$. $\square$

We now modify the function

$$u_F(s,t) := (\Pi^E u_2)|_{\gamma_E(s,t)} \tag{14}$$

to obtain a function $u_{F,00}$ which satisfies zero boundary conditions on the edges $s = 0$ and $t = 0$, and coincides with $u_F$ on the edge $s+t = 1$. This modification is done in such a way that it is continuous in the weighted $H^1$-norm.

First, we define the smoothing operator (cf. Figure 5)

$$(S_s v)(s,t) := \int_0^1 v\big(s + \frac{\tau}{2}(1-s-t), t\big)\,d\tau.$$

Secondly, we modify the operator to obtain

$$(S_{s,0} v)(s,t) := (S_s v)(s,t) - \frac{1-s-t}{1-t} (S_s v)(0,t),$$

which vanishes on the edge $s = 0$.

LEMMA 4.3 The smoothing operator $S_{s,0}$ satisfies

$$S_{s,0} : \{v \in P^p : v(0,1) = v(1,0) = 0\} \to \{v \in P^p : v(1,0) = v(0,\cdot) = 0\}$$

and the estimates

$$\begin{aligned} &\big\|r_{E^R}^{1/2} \nabla (S_{s,0} v)\big\|_{L_2(F)} + \big\|\tfrac{r_{E^R}^{1/2}}{r_{V_{E^R}}} (S_{s,0} v)\big\|_{L_2(F)} + \big\|r_{E^R}^{-1/2} (S_{s,0} v - v)\big\|_{L_2(F)} \\ &\qquad \preceq \big\|r_{E^R}^{1/2} \nabla v\big\|_{L_2(F)} + \big\|\tfrac{r_{E^R}^{1/2}}{r_{V_{E^R}}} v\big\|_{L_2(F)}. \end{aligned} \tag{15}$$

*Proof.* We start with the stated mapping properties of $S_{s,0}$. Let $v$ be a polynomial vanishing in $(0,1)$ and $(1,0)$. Then $S_s v$ is again a polynomial, and $S_s v = v$ on the edge $s+t=1$; hence $S_s v(0,1) = v(0,1) = 0$. Thus, the restriction to the edge $s=0$ is a polynomial in $t$ vanishing at $t=1$. Thus, the factor $1-t$ in the definition of $S_{s,0}$ cancels out, showing that $S_{s,0}v$ is a polynomial.

The stated bounds for $S_{s,0}$ are proved in 2 steps. In the first step, we prove the corresponding estimates for the smoothing operator $S_s$. In the second step, we prove the estimates for the difference $S_{s,0}-S_s$. We start by observing that derivatives of $S_s v$ depend on derivatives of $v$, only:

$$\frac{\partial(S_s v)}{\partial s} = \int_0^1 (\nabla v)\cdot\big(1-\frac{\tau}{2},0\big)\,d\tau, \qquad \frac{\partial(S_s v)}{\partial t} = \int_0^1 (\nabla v)\cdot\big(-\frac{\tau}{2},1\big)\,d\tau.$$

We also observe that on the averaging line $l_{s,t} := [(s,t);(s+\frac{1}{2}(1-s-t),t)]$, we have

$$\min_{(s',t')\in l_{s,t}} r_{E^R} \le \max_{(s',t')\in l_{s,t}} r_{E^R} \le 2\min_{(s',t')\in l_{s,t}} r_{E^R}.$$

The bound for $S_s$ in the weighted $H^1$-semi norm then follows. The approximation property corresponding to the weighted $L_2$-norm follows from Friedrichs' inequality applied on the line $l_{s,t}$.

Having thus proved the bounds for the operator $S_s$, we turn to considering the correction $S_{s,0}-S_s$. The weighted $H^1$-semi norm term for $S_{s,0}-S_s$ is:

$$\begin{aligned}
&\Big\| r_{E^R}^{1/2}\nabla\Big[\tfrac{1-s-t}{1-t}(S_s v)(0,t)\Big]\Big\|_{L_2(F)} \\
\le\;& \Big\| r_{E^R}^{1/2}\Big(\tfrac{-1}{1-t},\tfrac{-s}{(1-t)^2}\Big)(S_s v)(0,t)\Big\|_{L_2(F)} + \Big\| r_{E^R}^{1/2}\tfrac{1-s-t}{1-t}\nabla(S_s v)(0,t)\Big\|_{L_2(F)} \\
\preceq\;& \big\|(1-t)^{-1/2}(S_s v)(0,t)\big\|_{L_2(F)} + \Big\|(1-t)^{1/2}\frac{\partial(S_s v)}{\partial t}(0,t)\Big\|_{L_2(F)} \\
=\;& \big\|(S_s v)(0,t)\big\|_{L_2(0,1)} + \Big\|(1-t)\frac{\partial(S_s v)}{\partial t}(0,t)\Big\|_{L_2(0,1)};
\end{aligned}$$

Likewiwse, $\|\frac{r_{E^R}^{1/2}}{r_{V_{E^R}}}\frac{1-s-t}{1-t}(S_s v)(0,t)\|_{L_2(F)}$ and $\|r_{E^R}^{-1/2}\frac{1-s-t}{1-t}(S_s v)(0,t)\|_{L_2(F)}$ are bounded by these trace norms of $S_s v$. We now bound them by the right-hand side of (15). We start with the $L_2$-norm:

$$\begin{aligned}
\int_0^1 (S_s v)^2(0,t)\,dt &= \int_0^1 \Big[\int_0^1 v\Big(\frac{1-t}{2}\tau,t\Big)d\tau\Big]^2 dt \\
&\le \int_0^1\int_0^1 v^2\Big(\frac{1-t}{2}\tau,t\Big)d\tau\,dt = \int_0^1\int_0^{\frac{1-t}{2}} v^2(s,t)\frac{2}{1-t}\,ds\,dt \\
&\preceq \int_F \frac{r_{E^R}}{r_{V_E}^2} v^2(s,t)\,ds\,dt
\end{aligned} \tag{16}$$

We have substituted $s = \frac{1-t}{2}\tau$, and used that

$$s \le \frac{1-t}{2} \quad \text{implies} \quad \frac{1}{1-t} \le 2\frac{1-s-t}{(1-t)^2} \le 2\frac{r_{E^R}}{r_{V_E}^2}.$$

Similarly, we can bound the weighted $H^1$-norm on the edge by

$$\begin{aligned}
&\int_0^1 (1-t)^2\Big[\frac{\partial(S_s v)}{\partial t}(0,t)\Big]^2 dt = \int_0^1 (1-t)^2\Big[\int_0^1 (\nabla v)\cdot(-\tau/2,1)^T\,d\tau\Big]^2 dt \\
&\preceq \int_0^1 (1-t)^2\int_0^1 \Big|\nabla v\Big(\frac{1-t}{2}\tau,t\Big)\Big|^2 d\tau dt \preceq \int_0^1\int_0^{\frac{1-t}{2}} (1-s-t)|\nabla v|^2\,ds\,dt.
\end{aligned}$$

This concludes the proof. $\square$

In the same manner, we define

$$(S_t u)(s,t) := \int_0^1 u\big(s, t + \frac{\tau}{2}(1-s-t)\big)\, d\tau,$$

and

$$(S_{t,0})(x,y) := (S_t u_{F,0})(s,t) - \frac{1-s-t}{1-s}(S_t u_{F,0})(s,0).$$

These two smoothing operators allow us to define the function

$$u_{F,00} := S_{t,0} S_{s,0} u_F$$

satisfying zero boundary values at both edges $s=0$ and $t=0$.

We define the edge interpolation operator by

$$(\Pi_0^E u_2)(x) := u_{F,00}(\varphi_{V_{E,1}}(x), \varphi_{V_{E,2}}(x)). \tag{17}$$

LEMMA 4.4 For a tetrahedron $T$ of the triangulation let $\mathscr{E}_T$ be the set of its edges. Then, for

$$r_{\mathscr{E}}(x) := \min_{E \in \mathscr{E}} r_E(x)$$

there holds

$$\|r_{\mathscr{E}}^{-1}(v - \sum_{E \in \mathscr{E}_T} \Pi_0^E v)\|_{L_2(T)} \preceq \sum_{E \in \mathscr{E}_T} \|\nabla v\|_{L_2(\omega_E)} + \|r_{\mathscr{V}}^{-1} v\|_{L_2(\omega_E)}. \tag{18}$$

*Proof.* For each $E \in \mathscr{E}_T$ set $T_E := \{x \in T \mid r_E(x) = r_{\mathscr{E}}(x)\}$ and note $T = \cup_{E \in \mathscr{E}_T} T_E$. Then

$$\begin{aligned}
\|r_{\mathscr{E}}^{-1}(v - \sum_{E \in \mathscr{E}_T} \Pi_0^E v)\|_{L_2(T)} &\leq \sum_{E' \in \mathscr{E}_T} \|r_{\mathscr{E}}^{-1}(v - \sum_{E \in \mathscr{E}_T} \Pi_0^E v)\|_{L_2(T_{E'})} \\
\leq \quad & \sum_{E' \in \mathscr{E}_T} \|r_{E'}^{-1}(v - \Pi_0^{E'} v)\|_{L_2(T_{E'})} + \sum_{E' \in \mathscr{E}_T} \|r_{E'}^{-1} \sum_{E \in \mathscr{E}_T : E \neq E'} \Pi_0^E v\|_{L_2(T_{E'})} \\
\leq \quad & \sum_{E' \in \mathscr{E}_T} \|r_{E'}^{-1}(v - \Pi_0^{E'} v)\|_{L_2(T)} + \sum_{E \in \mathscr{E}_T} \sum_{E' \in \mathscr{E}_T : E' \neq E} \|r_{E'}^{-1} \Pi_0^E v\|_{L_2(T)}.
\end{aligned} \tag{19}$$

The first sum in (19) can be bounded by the right-hand side of (18) in view of Lemmas 4.1, 4.2, 4.3. For the second sum, we observe that each function $\Pi_0^E v$ vanishes on those two faces of $T$ which do not contain the edge $E$, and therefore it vanishes on all edges $E' \neq E$. An embedding theorem then gives $\|r_{E'}^{-1} \Pi_0^E v\|_{L_2(T)} \preceq \|\nabla \Pi_0^E v\|_{L_2(T)}$. Employing again Lemmas 4.1, 4.2, 4.3, we see that the double sum in (19) can also be bounded by the right-hand side of (18). $\square$

Finally, we define the *global edge interpolation operator*

$$\Pi_{\mathscr{E}} := \sum_{E \in \mathscr{E}_f} \Pi_0^E, \tag{20}$$

where $\mathscr{E}_f$ is the set of are all free edges, i. e., those which do not lie completely on the Dirichlet boundary. We obtain

THEOREM 4.1 The decomposition

$$u_2 = \sum_{E \in \mathscr{E}_f} \Pi_0^E u_2 + u_3 \tag{21}$$

fulfills the stability estimate

$$\sum_{E \in \mathscr{E}_f} \|\Pi_0^E u_2\|_A^2 + \|\nabla u_3\|^2 + \|r_{\mathscr{E}}^{-1} u_3\|^2 \preceq \|\nabla u_2\|^2 + \|r_{\mathscr{V}}^{-1} u_2\|^2. \tag{22}$$

Moreover, $u_3 = 0$ on $\bigcup_{E \in \mathscr{E}_f} E$.

*Proof.* Theorem 4.1 follows from Lemmas 4.1, 4.3, and 4.4 using the argument of finite summation. $\square$

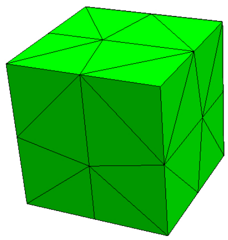

Figure 6: Unstructured mesh of the unit cube, Examples 1–4

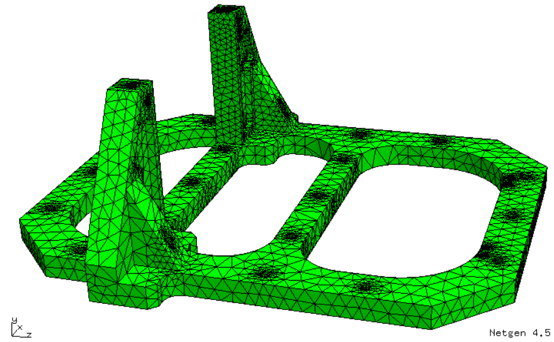


Figure 7: Unstructured mesh of machine frame, Example 5

### 4.3 Main result

*Proof of Theorem 2.1 for the case of tetrahedra.* The interpolation on faces in 3D and its analysis is structurally similar to the procedure in Section 3.3 for the edge interpolation in 2D. One requires, for each face $F$ and its face patch $\omega_F$, the face-lifting operator $\Pi_0^F : H_{00}^{1/2}(F) \to H_0^1(\omega_F)$ as given in (Mun97, Lemma 8) with the appropriate stability properties. Then, we can set

$$u_1 = u - \Pi_h u, \qquad u_2 = u_1 - \sum_{V\in\mathscr{V}_f} \Pi_0^V u_1,$$

$$u_3 = u_2 - \sum_{E\in\mathscr{E}_f} \Pi_0^E u_2, \qquad u_4 = u_3 - \sum_{F\in\mathscr{F}_f} \Pi_0^F u_3,$$

where $\mathscr{F}_f = \{F \in \mathscr{F} : F \not\subset \Gamma_D\}$. As a consequence of the last subsections, the decomposition

$$u = \Pi_h u + \sum_{V\in\mathscr{V}_f} \Pi_0^V u_1 + \sum_{E\in\mathscr{E}_f} \Pi_0^E u_2 + \sum_{F\in\mathscr{F}_f} \Pi_0^F u_3 + \sum_{T\in\mathscr{T}} u_4|_T \tag{23}$$

is stable in the $\|\cdot\|_A$-norm. $\square$

# 5 Numerical results

In this section, we show numerical experiments to illustrate the theory elaborated in the last sections and to get the absolute condition numbers hidden in the generic constants. Furthermore, we study two more preconditioners and apply the method to an elasticity problem on a complex geometry.

First, we consider the $H^1(\Omega)$ inner product $A(u,v) = (\nabla u, \nabla v)_{L_2} + (u,v)_{L_2}$ on the unit cube $\Omega = (0,1)^3$, which is subdivided into 69 tetrahedra, see Figure 6. The polynomial degree $p$ ranges from 2 to 10. The condition numbers of the preconditioned systems are computed by Lanczos' method. In each of our examples, the inner unknowns are eliminated by static condensation.

**Example 1:** The preconditioner is defined by the space decomposition with large overlap of Theorem 2.1:

$$V = V_0 + \sum_{V\in\mathscr{V}} V_V$$

By Theorem 2.1, the condition number is independent of $h$ and $p$. The computed numbers are drawn in Figure 8, labeled 'overlapping V'. The memory requirement of this preconditioner is considerable: For $p = 10$, the memory needed to store the local Cholesky factors is about 4.4 times larger than the memory required for the global matrix.

In Section 2 we have introduced the space splitting into the coarse space $V_0$ and the vertex subspaces $V_V$. However, our proof of Theorem 2.1 involves the finer splitting of a function $u$ into a coarse function, functions in the spider spaces $S_V$, edge-, face-based and inner functions. Other additive Schwarz preconditioners with uniform condition numbers are induced by this finer splitting.

**Example 2:** We decompose the space into the coarse space, the $p$-dimensional spider-vertex spaces $S_{V,0} = \operatorname{span}\{\varphi_V, \ldots, \varphi_V^p\}$, and the overlapping subspaces $V_E$ on the edge patches:

$$V = V_0 + \sum_{V \in \mathscr{V}} S_{V,0} + \sum_{E \in \mathscr{E}} V_E$$

The arguments of the preceding sections show that the condition number is bounded uniformly in $h$ and $p$. The computed values are drawn in Figure 8, labeled 'overlapping E, spider V'. Storing the local factors is now about 80 percent of the memory for the global matrix.

**Example 3:** The interpolation into the spider-vertex space $S_{V,0}$ has two continuity properties: It is bounded in the energy norm, and the interpolation rest satisfies an error estimate in a weighted $L_2$-norm, see Lemma 3.3 and equation (12). Now, we reduce the $p$-dimensional vertex spaces to the spaces spanned by the low energy vertex functions $\varphi_V^{l.e.}$ defined as solutions of

$$\min_{v \in S_{V,0},\, v(V)=1} \|v\|_A^2.$$

These low energy functions can be approximately expressed by the standard vertex functions via $\varphi_V^{l.e.} = f(\varphi_V)$, where the polynomial $f$ solves a weighted 1D problem and can be given explicitly in terms of Jacobi polynomials, cf. (BPP06). The interpolation to the low energy vertex space is uniformly bounded, too. However, the approximation estimate in the weighted $L_2$-norm depends on $p$. The preconditioner is now generated by

$$V = V_0 + \sum_{V \in \mathscr{V}} \operatorname{span}\{\varphi_V^{l.e.}\} + \sum_{E \in \mathscr{E}} V_E.$$

The computed values are drawn in Figure 8, labeled 'overlapping E, low energy V', and show a moderate growth in $p$. Low energy vertex basis functions obtained by orthogonalization on the reference element have also been analyzed in (Bic97; SC01).

**Example 4:** We have also tested the preconditioner without additional vertex spaces, i. e.,

$$V = V_0 + \sum_{E \in \mathscr{E}} V_E.$$

Since vertex values must be interpolated by the lowest order functions, the condition number is no longer bounded uniformly in $p$. The rapidly growing condition numbers are drawn in Figure 9.

**Example 5:** We study an elasticity problem on the complex geometry of a machine frame, cf. Figure 7. The frame is tightened by screws, which leads to Dirichlet boundary conditions in the boreholes. We apply horizontal surface forces at the top of the pillars. The remaining boundary is traction-free. The mesh consists of 45553 tetrahedra, some of them curved. Figure 10 shows the computed von Mises stress in the machine frame. We tested three different preconditioners, see Figure 11. The condition number of the spider vertex preconditioner with the splitting as in Example 2 is proven to be uniformly bounded by a constant. As in Example 3, the low-energy version shows only a small increase in the

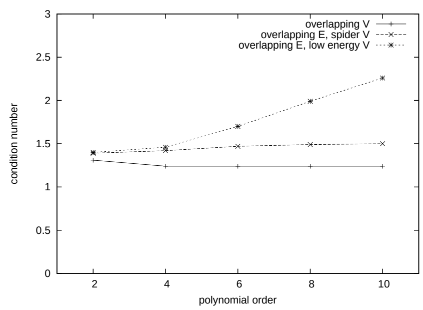


Figure 8: Overlapping blocks

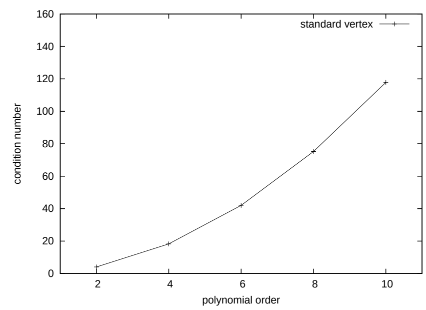


Figure 9: Standard vertex

number of CG iterations. The 'non-overlapping' preconditioner is a standard block Gauss-Seidel smoother where the blocks correspond to the degrees of freedom (dof) of the gobal lowest order spaces, the dofs associated with each edge, and those associated with each face.

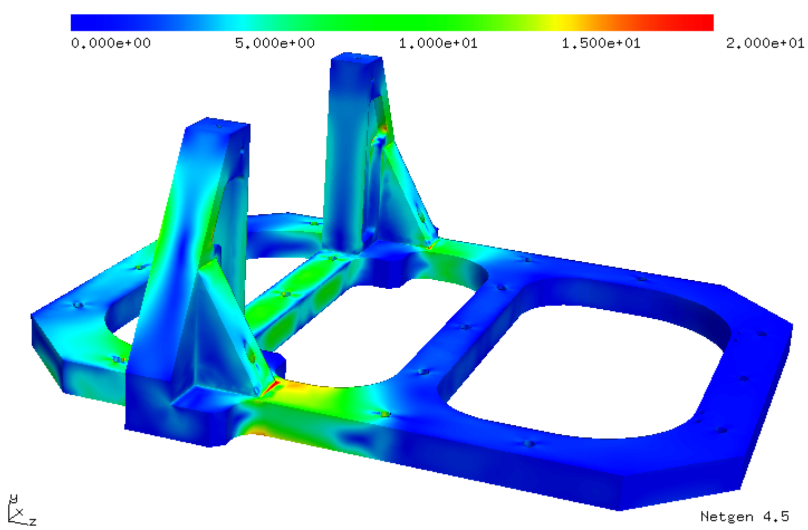


Figure 10: Computed von Mises stress in the machine frame, polynomial order $p = 5$, geometry order 5

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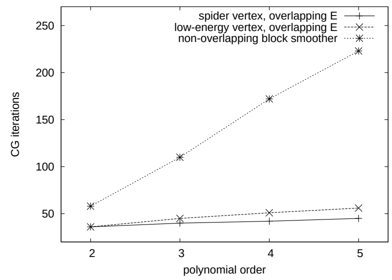


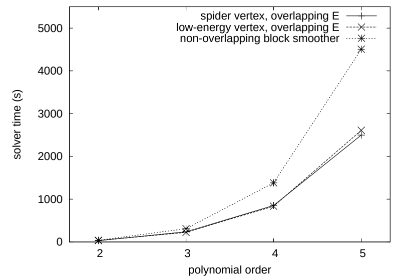


Figure 11: Example 5: Elasticity problem. Left: Iteration number versus polynomial order. Right: Solver time versus polynomial order. Degrees of freedom: $3 \times 81104$ ($p = 2$), $3 \times 250396$ ($p = 3$), $3 \times 566244$ ($p = 4$), $3 \times 1074201$ ($p = 5$).

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