A Saddle Point Variational Formulation for Projection–Regularized Parameter Identification [1]

Barbara Kaltenbacher and Joachim Schöberl

Abstract. This paper is concerned with the ill-posed problem of identifying a parameter in an elliptic equation and its solution applying regularization by projection. As the theory has shown, the ansatz fuctions for the parameter have to be sufficiently smooth. In this paper we show that these — for a practical implementation unrealistic — smoothness assumptions can be circumvented by reformulating the problem under consideration as a mixed variational equation. We prove convergence as the discretization gets finer in the noise free case and convergence as the data noise level δ goes to zero in the case of noisy data, as well as convergence rates under additional smoothness conditions.

Key words. mixed finite elements, parameter identification, regularization

AMS(MOS) Subject Classifications: 35R30, 65N30, 65J15, 65J20, 76S05

1 Introduction

Parameter identification problems play an important role in many applications in science and industry. Here we consider the problem of identifying a distributed parameter $a = a(x)$ in the PDE

$$\begin{aligned} -\bigtriangledown (a \bigtriangledown u) &= f \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

from measurements u^δ of u, which models e.g. the inverse groundwater filtration problem of reconstructing the diffusitivity a of a sediment from measurements of the piezometric head u in the steady state case (see [1] for further applications of (1)). Here Ω is a two dimensional convex domain with Lipschitz boundary, $f \in L^2(\Omega)$, and $g \in H^{\frac{3}{2}}(\Omega)$. We will assume that both this mathematical model and the given data correspond to a physically meaningful setting such that a solution $a^\dagger$ to the unperturbed parameter identification problem exists. If a is known on the boundary $\partial\Omega$ and $\triangle u$ is bounded away from zero then by the basic theory for hyperbolic BVPs (see, e.g., [2]), a is uniquely determined on all of Ω by (1).

For any given $a \in L^\infty(\Omega)$ bounded away from zero, the BVP (1) has a unique weak solution $u \in H^1(\Omega)$; the relation between a and u is obviously nonlinear. On the other hand the inverse problem of identifying the values of a from value (but not derivative) measurements of u is obviously ill-posed due to the required data differentiation, as already the one dimensional case shows (cf. e.g., Section 1.6 in [8]) (cf. [1], [17] for the higher dimensional case) i.e., arbitrarily small noise in the measured data, with some noise level

[1] Supported by the Austrian Science Foundation 'Fonds zur Förderung der wissenschaftlichen Forschung' in the Special Research Program SFB F013 and under grant T7-TEC

δ in

$$\|u^\delta - u\|_{L^2(\Omega)} = \delta \ , \tag{2}$$

can lead to arbitrarily large pointwise deviations in the solution a. Therefore one has to apply some regularization method (cf. e.g. [8], [11], [19], [21], [23], [24], [26]) in order to obtain a reasonable reconstruction.

It is a characteristic feature of ill-posed problems that any solution method converges (if at all) in general arbitrarily slowly and convergence rates can only be obtained under so-called source wise representation conditions on on the difference between an exact solution $a^\dagger$ and an initial guess a_0 used in the approximation method. For a linear ill-posed equation

$$A(a - a_0) = r \tag{3}$$

they read as

$$a^\dagger - a_0 \in \mathcal{R}((A^*A)^\nu) \tag{4}$$

for some positive exponent ν; Since the forward operator A and its adjoint A^* are typically smoothing, (4) can usually be interpreted as additional smoothness (and boundary) conditions.

Here we consider regularization by projection (cf. [7], [8], [9], [10], [12], [16], [22]), that is based on the stabilizing effect of "coarse" discretization. For its convergence analysis the choice of the ansatz functions for the searched for parameter a plays a crucial role: They should be contained in the range of the smoothing operator A^*. In the present parameter identification problem this essentially means that they have to be in $H^{1+2m}(\Omega)$ (where $m \in \{0, 1\}$ is the degree of smoothness up to which the parameter a is supposed to be identified) and fulfill certain (unpleasant) boundary conditions. These are requirements that would make an implemetation with e.g. finite elements extremely complicated.

Now, an important application of mixed finite elements is the weakening of smoothness conditions when solving higher order partial differential equations. A simple example is the biharmonic equation (cf. example 3.7 in [3], as well as [20] and [5]). (Note that this is a well-posed problem, though.)

In this paper we show that also this idea can also be successfully applied to our linearized (ill-posed!) parameter identification problem. In fact, a reformulation as a saddle point problem makes it possible to preserve the theoretical convergence results of [16] in a discretization with the usual hat functions.

The basic ideas of our reconstruction method are as follows: First of all we rewrite the nonlinear problem of identifying a in (1) as a linear one for $a - a_0$, with some fixed a_0:

$$\begin{aligned} -\bigtriangledown(a_0 \bigtriangledown (u - u_0)) &= \bigtriangledown((a - a_0) \bigtriangledown u^\dagger) && \text{in } \Omega \\ u - u_0 &= 0 && \text{on } \partial\Omega \ , \end{aligned} \tag{5}$$

where u_0 solves

$$\begin{aligned} -\bigtriangledown(a_0 \bigtriangledown u_0) &= f && \text{in } \Omega \\ u_0 &= g && \text{on } \partial\Omega \ . \end{aligned} \tag{6}$$

and $u^\dagger$ is unperturbed data, i.e., the solution of the BVP (1) with the exact searched-for parameter $a = a^\dagger$, and inserting a smoothed version of the noisy data on the right-hand side. The (still ill-posed!) resulting linear problem is then regularized by projecting it onto a finite-dimensional space. This projection can be reformulated as a mixed variational problem. By considering weak solutions of the occurring BVP we arrive at a formulation that requires relatively weak smoothness assumptions on its variables: We show that contiuous piecewise linear ansatz functions yield a discrete inf-sup-condition. On this basis we then can prove convergence of the so-defined regularized approximations to $a^\dagger$ as the noise level and an appropriately chosen noise-dependent mesh size goes to zero. Under additional smoothness assumptions also convergence rates can be proven. Our numerical experiments which we did using the FE-package FEPP that had been developped at the SFB F013 in Linz (see [18]), confirm these theoretical results and show the good performance of the proposed method.

Among the literature on the inverse groundwater filtration problem (that the authors do not attempt to cite in a complete way,) there are some papers which we wolud like to point out since these also strongly rely on some kind of weak formulation of the direct problem: Vainikko [25] combines the weak formulation of a BVP similar to (5) with Tikhonov regularization. In Chavent et al. [4], duality principles are applied to a (Tikhonov type regularized version of) (5) to obtain a still weaker notion of solution and its numerical approximation is, like here, based on mixed finite elements. What the authors consider the main new point in the present paper, is the use of regularization solely by finite dimensional projection, without using additional regularizing terms (as they appear, e.g., in Tikhonov regularization) and therefore possibly more natural and problem-adapted.

In the following we will use the notation c or C, for positive constants (independent of the noise level and any mesh size) that are typically (or have to be sufficiently) "small" or "large" respectively; they can have different values whenever they appear in the text, but they are introduced in such a way that no confusion should occur.

2 Reformulation as a Linear Problem, Data Mollification, and Regularization by Projection

2.1 Reformulation as a Linear Problem

As already mentioned, we rewrite (1) to (5), which, if $u^\dagger$ in the right hand side of the first line of (5) would be known, could be seen as a linear equation

$$\bar{A}s = \bar{r}$$

for the difference $s = a - a_0$ between the searched-for parameter a and the initial guess a_0, where $\bar{r} = u - u_0$ and $\bar{A}$ is the linear operator that maps s to the solution z of

$$\begin{aligned} -\nabla(a_0 \nabla z) &= \nabla(s \nabla u^\dagger) && \text{in } \Omega \\ z &= 0 && \text{on } \partial\Omega\,. \end{aligned} \tag{7}$$

Here the exact data $u^\dagger$ turns up, which we do not know, though. Instead we use our measured data u^δ and replace $u^\dagger$ in (5) by a smoothed version u^δ_{sm} of u^δ such that $\bigtriangledown(s\bigtriangledown u^\delta_{sm})$ is well-defined in an appropriate function space. This yields the linear problem

$$As = r^\delta \tag{8}$$

with perturbed operator $A = A(u^\delta)$ and noisy right-hand side r^δ. Here A maps s to the solution z of

$$\begin{aligned} -\bigtriangledown(a_0 \bigtriangledown z) &= \bigtriangledown(s \bigtriangledown u^\delta_{sm}) && \text{in } \Omega \\ z &= 0 && \text{on } \partial\Omega\ . \end{aligned} \tag{9}$$

whose data

$$r^\delta = u^\delta_{sm} - u_0 \tag{10}$$

can be seen as noisy version of the "exact" right hand side $r := u^\delta_{sm} - D_{a_0}^{-1} D_{a^\dagger}(u^\delta_{sm} - u^\dagger) - u_0$, (where for an $a \in L^\infty(\Omega)$ bounded away from zero, D_a is the differential operator $D_a : H^1_0(\Omega) \to H^{-1}(\Omega)$, $\phi \mapsto -\bigtriangledown(a \bigtriangledown \phi)$,) that would give the exact parameter $a^\dagger$:

$$A(a^\dagger - a_0) = r\ .$$

It is not later than at this point that we have to confine ourselves to appropriate function spaces X, Y — preferably Hilbert spaces — on which the operator $A : X \to Y$ is well-defined. The choice of

$$Y = L^2(\Omega)$$

is already fixed by the fact that the degree of accuracy in the measurements is given in terms of this topology (note that we cannot measure derivatives but only values of u). Since As according to (9) with $\bigtriangledown u^\delta_{sm} \in L^\infty(\Omega)$ is well-defined and in $L^2(\Omega)$ for any $s \in L^2(\Omega)$ as long as

$$a_0 \in L^\infty(\Omega) \text{ with } a_0(x) \geq \underline{a} > 0 \ \ a.e.,$$

(which we assume to hold in the following,) we can choose

$$X = H^m_0(\Omega) \tag{11}$$

for some $m \geq 0$ — we will use $m \in \{0, 1\}$. Since a higher degree of smoothness in the preimage space makes the inverse problem more ill-posed, this can be seen as a possible further advantage of the reformulation (8), (9) as compared to (1), where one would have needed at least L^∞-parameters a in order to obtain a well-defined forward operator $a \mapsto u$, and therefore, to work in a Hilbert space setting, $X = H^m(\Omega)$ with $m > 1$.

2.2 Data Mollification

The data smoothing, that is already part of the regularization of our ill-posed parameter identification problem, is done by applying some smoothing operator Π_{sm} (e.g. a Clément operator, cf. [6], [3]) to u^δ,

$$u^\delta_{sm} := \Pi_{sm} u^\delta\ ,$$

such that

$$\begin{array}{lcl} \|u^\delta_{sm} - u^\dagger\|_{L^2(\Omega)} & \leq & C\delta \\ \|u^\delta_{sm} - u^\dagger\|_{H^1_0(\Omega)} & \leq & C\sqrt{\delta} \\ \| \bigtriangledown u^\delta_{sm}\|_{L^\infty} & \leq & C \\ \|\Pi_{sm}\|_{L^2(\Omega)\to L^2(\Omega)} & \leq & C \end{array} \tag{12}$$

holds.

This can, e.g., be achieved by a Clément operator on a grid with appropriately chosen mesh size: Given a regular triangulation τ_{sm} with mesh size h_{sm} of Ω one can, by local averaging, define an operator Π_{sm} projecting onto the space U_h of piecewise linear functions on this triangulation, such that

$$\|\Pi_{sm}\|_{H^p_0(\Omega)\to H^p_0(\Omega)} \leq C \tag{13}$$

$$\|I - \Pi_{sm}\|_{H^q_0(\Omega)\to H^p_0(\Omega)} \leq Ch^{q-p}_{sm} \tag{14}$$

$$(I - \Pi_{sm})u_h = 0 \quad \forall u_h \in U_h \tag{15}$$

for $p \in \{0, 1\}$, $p \leq q \leq 2$.

By the inverse inequalities

$$\|u_h\|_{H^1_0(\Omega)} \leq Ch^{-1}_{sm}\|u_h\|_{L^2(\Omega)}$$

$$\| \bigtriangledown u_h\|_{L^\infty(\Omega)} \leq Ch^{-2}_{sm}\|u_h\|_{L^2(\Omega)}$$

for $u_h \in U_h$, and using a further projection $\tilde{\Pi} : H^2(\Omega) \cap W^{1,\infty}(\Omega) \to U_h$ (e.g., pointwise interpolation) that satisfies

$$\begin{array}{lcl} \|\tilde{\Pi}\|_{W^{1,\infty}(\Omega)\to W^{1,\infty}(\Omega)} & \leq & C \\ \|I - \tilde{\Pi}\|_{H^2(\Omega)\to L^2(\Omega)} & \leq & Ch^2_{sm} \ , \end{array}$$

one therefore has

$$\begin{array}{lcl} \|u^\delta_{sm} - u^\dagger\|_{L^2(\Omega)} & \leq & \|\Pi_{sm}(u^\delta - u^\dagger)\|_{L^2(\Omega)} + \|(I - \Pi_{sm})u^\dagger\|_{L^2(\Omega)} \\ & \leq & C(\delta + h^2_{sm}\|u^\dagger\|_{H^2(\Omega)}) \\ \|(u^\delta_{sm} - u^\dagger)\|_{H^1_0(\Omega)} & \leq & Ch^{-1}_{sm}|\Pi_{sm}(u^\delta - u^\dagger)|_{L^2(\Omega)} + \|(I - \Pi_{sm})u^\dagger\|_{H^1_0(\Omega)} \\ & \leq & Ch^{-1}_{sm}(\delta + h^2_{sm}\|u^\dagger\|_{H^2(\Omega)}) \\ \| \bigtriangledown u^\delta_{sm}\|_{L^\infty(\Omega)} & = & \| \bigtriangledown \Pi_{sm}(u^\delta - u^\dagger) + \bigtriangledown\Pi_{sm}(I - \tilde{\Pi})u^\dagger + \bigtriangledown\tilde{\Pi}u^\dagger\|_{L^\infty(\Omega)} \\ & \leq & C(h^{-2}_{sm}\|u^\delta - u^\dagger\|_{L^2(\Omega)} + h^{-2}_{sm}\|\Pi_{sm}(I - \tilde{\Pi})u^\dagger\|_{L^2(\Omega)} + \|\tilde{\Pi}u^\dagger\|_{W^{1,\infty}(\Omega)} \\ & \leq & Ch^{-2}_{sm}(\delta + h^2_{sm}\|u^\dagger\|_{H^2(\Omega)\cap W^{1,\infty}(\Omega)}) \ , \end{array} \tag{16}$$

so that under the assumption that

$$u^\dagger \in H^2(\Omega) \cap W^{1,\infty}(\Omega) \ , \tag{17}$$

and with the choice

$$h_{sm} \sim \sqrt{\delta} \tag{18}$$

we have (12).

2.3 Regularization by Projection

For an implementation, any regularization method has to be discretized. On the other hand, discretization, i.e., projection on a finite-dimensional space itself can have a regularizing effect: Finite dimensional problems — though they might be ill-conditioned — are always well-posed in the sense of stable dependence of the bestapproximate solution on the data. As the convergence analysis of projection methods for ill-posed problems — see e.g. Section 3.3 in [8] — has shown, there are good reasons for using regularization by projection in image space rather than in preimage space. Given a family of finite dimensional nested subspaces whose union is dense in $\overline{R(A)}$

$$Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \ldots \qquad \overline{\bigcup_{n\in\mathbb{N}} Y_n} = \overline{R(A)}$$

we consider, instead of

$$As = r \tag{19}$$

the projected equations

$$P_{Y_n} As = P_{Y_n} r$$

(here and in the following P_Z denotes the orthogonal projection onto some linear space Z) and take their bestapproximate solutions, i.e., the minimizers s_n of the constrained minimization problems

$$\begin{array}{l} \frac{1}{2}\|s\|^2 = \min! \\ P_{Y_n} As = P_{Y_n} r \end{array} \tag{20}$$

as approximations for the exact solution $s^\dagger = A^\dagger b$ of (19), that minimizes

$$\begin{array}{l} \frac{1}{2}\|s\|^2 = \min! \\ As = r \end{array} \tag{21}$$

With some basis $\{\psi_1^n, \ldots, \psi_{d(n)}^n\}$ of Y_n, s_n can be rewritten as the (unique) solution of the ($d(n)$-dimensional) problem

$$\begin{array}{ll} \langle As, \psi_j^n\rangle = \langle r, \psi_j^n\rangle & j \in \{1, \ldots, d(n)\} \\ s \in \operatorname{span}\{A^*\psi_1, \ldots, A^*\psi_k\} & \end{array}$$

or as the linear combination

$$s_n = \sum_{j=1}^{d(n)} \alpha_j A^* \psi_j^n \tag{22}$$

whose coefficients solve the linear system

$$M\alpha = \beta$$

with

$$M_{i,j} = \langle A^*\psi_i, A^*\psi_j\rangle \ , \quad \beta_i = \langle A^*\psi_i, r\rangle \ .$$

It is quite straightforward to see that the so defined approximations are just the orthogonal projections of the exact solution onto the finite-dimensional spaces A^*Y_n and therefore converge to $s^\dagger$ as $n \to \infty$.

From the representation (22) it can be seen that the ansatz functions for s_n have to be in the image of Y_n under A^*. Since the adjoint of the operator A as defined in subsection 2.1 is given by

$$A^*\psi = (-\triangle)^{-m} \bigtriangledown u_{sm}^\delta \cdot \bigtriangledown \tilde{\psi} \tag{23}$$

where $-\triangle : H_0^1(\Omega) \cap H^2(\Omega) \to L^2(\Omega)$, $\phi \mapsto -\triangle \phi)$, $m \in \{0,1\}$, and $\tilde{\psi}$ solves

$$\begin{array}{rcll} -\bigtriangledown (a_0 \bigtriangledown \tilde{\psi}) & = & \psi & \text{in } \Omega \\ \tilde{\psi} & = & 0 & \text{on } \partial\Omega \ , \end{array} \tag{24}$$

this means that they have to be in $H^{1+2m}(\Omega)$ and satisfy quite complicated boundary conditions. Given a basis $\{\psi_1^n, \ldots, \psi_{d(n)}^n\}$, the functions $A^*\psi_j^n$ can theoretically be computed explicitely (as it was done for the one-dimensional case in [16]) or numericaly, which, as soon as Ω is a general domain with more complicated geometry becomes much too expensive for a practical implementation, though. In order to avoid this, we will study a weaker formulation of (21):

3 A Mixed Formulation. Convergence Analysis

Using the weak formulation in the definition of the operator A, we can rewrite (21) as

$$\begin{array}{rcl} \frac{1}{2}\|s\|_X^2 & = & \text{min!} \\ \int_\Omega s \bigtriangledown u_{sm}^\delta \bigtriangledown q \; dx & = & -\int_\Omega a_0 \bigtriangledown r \bigtriangledown q \; dx \quad \forall q \in H_0^1 \ . \end{array}$$

With the definitions

$$\begin{array}{rcl} a(s,t) & := & \langle s,t \rangle_X \\ b(s,q) & := & \int_\Omega s \bigtriangledown u_{sm}^\delta \bigtriangledown q \; dx \\ g(q) & := & -\int_\Omega a_0 \bigtriangledown r \bigtriangledown q \; dx \end{array} \tag{25}$$

this formally leads to the variational equation

$$\begin{array}{rcll} a(s,t) + b(t,p) & = & 0 & \forall t \in V \\ b(s,q) & = & g(q) & \forall q \in Q \end{array} \tag{26}$$

for the primal and dual variables

$$s \in V := X \ , \quad p \in Q \ ,$$

respectively. If we define

$$Q := \overline{H_0^1(\Omega)}^Q \quad \|v\|_Q := \| \bigtriangledown u_{sm}^\delta \bigtriangledown v\|_{H^{-m}(\Omega)} \ .$$

obviously the bilinear forms a and b are bounded, a is V-elliptic and b satisfies the continuous inf-sup condition

$$\inf_{p\in Q}\sup_{s\in V}\frac{b(s,p)}{\|s\|_V\|p\|_Q}\geq 1\ .$$

The latter relation, especially its finite-dimensional counterpart, plays an important role in the stability and convergence analysis of the discretized mixed variational problem (26). As stated in the next lemma, with an appropriate choice of the finite dimensional subspaces V_h, Q_h defining the discretization, it can be shown to hold:

Lemma 1. *Let τ_V, τ_Q be regular triangulations of Ω with mesh size h_V, h_Q, respectively, with*

$$\tau_Q\subseteq\tau_V\ \wedge\ \tau_{sm}\subseteq\tau_V\ , \tag{27}$$

where τ_{sm} is the triangulation used for the data mollification, let V_h consist of the union of the continuous piecewise linear elements with homogenuous Dirichlet boundary condition on $\partial\Omega$, with the element bubbles on τ_V and let Q_h be the continuous piecewise linear elements vanishing on $\partial\Omega$. Then there exists a constant c independent of the noise level and of the mesh sizes h_{sm}, h_v, h_Q such that

$$\inf_{p\in Q_h}\sup_{s\in V_h}\frac{b(s,p)}{\|s\|_V\|p\|_Q}\geq c\ .$$

Proof. The proof is based on the standard technique (cf. [3]) of deriving the discrete inf-sup condition from the continuous one by using an appropriate Fortin-projector Π such that

$$\begin{array}{l}\|\Pi\|_{V\to V}\leq C\\ b(s-\Pi s,q_h)=0\quad\forall q_h\in Q_h\ ,\ s\in V\ .\end{array} \tag{28}$$

This is achieved by the decomposition

$$\Pi=\Pi_2(I-\Pi_1)+\Pi_1\ ,$$

where Π_1 is a Clément operator on V_h and Π_2 is a projection on the element bubbles such that for any $s\in V$

$$\|\Pi_2\|_{L^2(\Omega)\to H_0^m}\leq Ch^{-m} \tag{29}$$

$$\int_T(s-\Pi_2 s)\ dx\ =\ 0\quad\forall T\in\tau_V \tag{30}$$

Indeed, using (29) and the properties (13), (14) of the Clément operator, we obtain uniform boundedness of Π. Moreover, since due to our conditions (27) on the triangulations, for any $q_h\in Q_h$, $\nabla u^\delta_{sm}\nabla q_h$ is constant on each $T\in\tau_V$, we have, by (30),

$$b(s-\Pi_2 s,q_h)=\sum_{T\in\tau_V}\int_T(s-\Pi_2 s)\nabla u^\delta_{sm}\nabla q_h\ dx=0\ ,$$

which, by the representation $I - \Pi = (I - \Pi_2)(I - \Pi_1)$ also yields the second equality in (28). Now the discrete inf-sup condition follows immediately from (28) via

$$\inf_{p\in Q_h} \sup_{s\in V_h} \frac{b(s,p)}{\|s\|_V \|p\|_Q} \geq \inf_{p\in Q} \sup_{s\in V} \frac{b(\Pi s,p)}{\|\Pi s\|_V \|p\|_Q} \geq \frac{1}{C} \inf_{p\in Q} \sup_{s\in V} \frac{b(s,p)}{\|s\|_V \|p\|_Q} \ .$$

□

Assuming that

$$(\triangle u^\dagger)(x) \leq -\kappa < 0 \quad a.e \tag{31}$$

we get the following relation between the anisotrope norm $\|\|_Q$ and the L^2-norm on Q_h:

Lemma 2. *Let (31) hold and assume that*

$$\kappa \geq C h_{sm}^{-1} h_Q^{-2} (\delta + h_{sm}^2 \|u^\dagger\|_{H^2(\Omega)}) \tag{32}$$

with some sufficiently large constant C. Then there exists a constant C independent of δ and the mesh sizes such that

$$\|q_h\|_Q \geq c h_Q^m \|q_h\|_{L^2(\Omega)} \quad \forall q_h \in Q_h \ .$$

Proof. Observe that

$$\| \bigtriangledown u_{sm}^\delta \bigtriangledown q_h\|_{H^{-m}(\Omega)} = \sup_{q\in H_0^m(\Omega)} \frac{\int_\Omega \bigtriangledown u_{sm}^\delta \bigtriangledown q_h \ q \ dx}{\|q\|_{H_0^m(\Omega)}} \geq \frac{\int_\Omega \bigtriangledown u_{sm}^\delta \bigtriangledown q_h \ q_h \ dx}{\|q_h\|_{H_0^s(\Omega)}} \ .$$

Integration by parts gives

$$\begin{aligned} \int_\Omega \bigtriangledown u_{sm}^\delta \bigtriangledown q_h \ q_h \ dx &= \int_\Omega \bigtriangledown q_h \bigtriangledown u^\dagger \ q_h \ dx + \int_\Omega \bigtriangledown (u_{sm}^\delta - u^\dagger) \bigtriangledown q_h \ q_h \ dx \\ &= - \int_\Omega \underbrace{q_h \ \bigtriangledown (q_h \bigtriangledown u^\dagger)}_{=\bigtriangledown q_h \bigtriangledown u^\dagger \ q_h + (q_h)^2 \triangle u^\dagger} \ dx + \int_\Omega \bigtriangledown (u_{sm}^\delta - u^\dagger) \bigtriangledown q_h \ q_h \ dx \end{aligned}$$

and therefore

$$\int_\Omega \bigtriangledown u_{sm}^\delta \bigtriangledown q_h \ q_h \ dx \geq \frac{1}{2} (\kappa \|q_h\|_{L^2(\Omega)}^2 - \|u_{sm}^\delta - u^\dagger\|_{H_0^1(\Omega)} \| \bigtriangledown q_h\|_{L^\infty(\Omega)} \|q_h\|_{L^2(\Omega)})$$

Now if the constant C in (32) is larger than the product of the constants C appearing in (16) and in the inverse inequality

$$\| \bigtriangledown q_h\|_{L^\infty(\Omega)} \leq C h_Q^{-2} \|q_h\|_{L^2(\Omega)}$$

then this (together with one more inverse inequality

$$\|q_h\|_{H_0^1(\Omega)} \leq C h_Q^{-1} \|q_h\|_{L^2(\Omega)}$$

for the case $m = 1$) immediately yields the assertion.

□

To apply regularization by projection, we use an a priori (especially independently of the data) fixed sequence $(\tau_Q^l)_{l\in\mathbb{N}}$ of triangulations of Ω with strictly monotonically but not too fast decreasing mesh size h_Q^l

$$1 < \frac{h_Q^{l-1}}{hQ^l} \leq C \quad \forall l \in \mathbb{N}$$

defining the discretization of the dual spaces Q_h, i.e., the projection spaces. A second sequence of triangulations of Ω, $(\tau_V^l)_{l\in\mathbb{N}}$, results from an also a priori fixed refinement of the τ_Q^l such that

$$\tau_Q^l \subseteq \tau_V^l \quad \wedge \quad h_Q^l \sim (h_V^l)^\gamma \quad \forall l \in \mathbb{N} \tag{33}$$

for some appropriately chosen exponent $0 < \gamma \leq \frac{1}{2}$ to be specified later, and defines the discretization for the data mollification and the primal spaces V_h. Choosing the index $l_* = l_*(\delta)$ such that

$$h_V^{l_*-1} \geq C\sqrt{\delta} \geq h_V^{l_*} \ , \tag{34}$$

we set

$$\tau_Q := \tau_Q^{l_*} \ , \quad \tau_V := \tau_{sm} := \tau_V^{l_*} \ ,$$

and define the mollification as described in subsection 2.2, and the spaces V_h, Q_h as in Lemma 1. Note that by (34) we have (18) and therefore (12), and that (up to some constants having to be sufficiently small) also (32) follows.

On the basis of this discretization and by Lemma 1, the following method is well-defined and, as we will show in the remainder of this section, gives a stable and convergent approximation of $s^\dagger$:

Method 1. *Let s_h be given by the primal part of a solution to the finite-dimensional mixed variational problem*

$$\begin{array}{lcll} a(s_h, t_h) \ + \ b(t_h, p_h) & = & 0 & \forall t_h \in V_h \\ b(s_h, q_h) & = & g^\delta(q_h) & \forall q_h \in Q_h. \end{array} \tag{35}$$

(Note that the dual part p_h of a solution to (35) is not necessarily unique.) Here the the bilinear forms a, b are as defined in (25); the linear functional g^δ, given by (25) with r replaced by the noisy data r^δ,

$$g^\delta(q_h) := -\int_\Omega a_0 \bigtriangledown r^\delta \bigtriangledown q_h \ dx \ ,$$

is bounded, but not uniformly bounded as $h_Q \to 0$, which reflects the ill-posedness of the underlying problem.

For fixed finite noise level and mesh sizes this gives in fact a stable reconstruction method:

Theorem 1. *Let (31), (32) hold, let, for some $u^\delta \in L^2(\Omega)$ with (2), $(u^{\delta,i})_{i\in\mathbb{N}}$ be a sequence of data converging to u^δ in $L^2(\Omega)$, and let $(s_h^i)_{i\in\mathbb{N}}$ be defined by (35) with the respective data $u^{\delta,i}$ inserted for u^δ whenever it appears.*
Then the s_h^i converge to s_h in $X = V$ as $i \to \infty$.

Proof. Without loss of generality we can assume that the L^2-difference between the $u^{\delta,i}$ and u^δ is so small that the respective triangulations τ_Q^i, τ_V^i, τ_{sm}^i as defined above coincide with τ_Q, τ_V, τ_{sm}, so that the discretization is independent of i. Then the differences $(d_h^i, e_h^i) := (s_h^i - s_h, p_h^i - p_h)$ are in $V_h \times Q_h$ and solve the following mixed problem:

$$\begin{array}{lclcll} a(d_h^i, t_h) & + & b(t_h, e_h^i) & = & [b - b^i](t_h, p_h^i) & \forall t_h \in V_h \\ b(d_h^i, q_h) & & & = & [b - b^i](s_h^i, q_h) + [g^\delta - g^{\delta,i}](q_h) & \forall q_h \in Q_h. \end{array} \tag{36}$$

Here we assume for technical reasons but w.l.o.g. that the dual solutions p_h^i are those with minimal norm. By the boundedness, ellipticity, and the discrete inf-sup condition for a, b we obtain (cf., e.g., Theorem 4.6 in[13]) the estimates

$$\begin{array}{rcl} \|s_h^i\|_V & \leq & \frac{2}{c} \sup_{q_h \in Q_h} \frac{g^\delta(q_h)}{\|q_h\|_Q} \\ & \leq & \frac{2}{c} \|a_0\|_{L^\infty(\Omega)} (\|u_{sm}^{\delta,i}\|_{H_0^1(\Omega)} + \|u_0\|_{H_0^1(\Omega)}) \sup_{q_h \in Q_h} \frac{\|q_h\|_{H_0^1(\Omega)}}{\|q_h\|_Q} \\ & \leq & C h_Q^{-(1+m)} \\ \|p_h^i\|_Q & \leq & \frac{2}{c^2} \sup_{q_h \in Q_h} \frac{g^\delta(q_h)}{\|q_h\|_Q} \leq C h_Q^{-(1+m)} \ , \end{array} \tag{37}$$

where we have used Lemma 2. The same argument, applied to (36) yields

$$\|d_h^i\|_V \leq \sup_{t_h \in V_h} \frac{[b - b^i](t_h, p_h^i)}{\|t_h\|_V} + 2c \sup_{q_h \in Q_h} \frac{[b - b^i](s_h^i, q_h) + [g^\delta - g^{\delta,i}](q_h)}{\|q_h\|_Q} \ . \tag{38}$$

For any $t_h \in V_h$, $q_h \in Q_h$

$$\begin{array}{rcl} [b - b^i](t_h, q_h) & = & \int_\Omega t_h \bigtriangledown (u_{sm}^\delta - u_{sm}^{\delta,i}) \bigtriangledown q_h \; dx \\ & \leq & \|t_h\|_{L^n(\Omega)} \|u_{sm}^\delta - u_{sm}^{\delta,i}\|_{H_0^1(\Omega)} \| \bigtriangledown q_h\|_{L^{\frac{2n}{n-2}}(\Omega)} \\ & \leq & C h_{sm}^{-1} h_Q^{-2} |\log h_V|^{m/2} \|t_h\|_V \|q_h\|_Q \|u^{\delta,i} - u^\delta\|_{L^2(\Omega)} \ , \end{array}$$

where we have set $n := 2$ if $m = 0$ and $n = \infty$ if $m = 1$, applied the inverse inequality (cf., e.g., [13])

$$\|t_h\|_{L^\infty(\Omega)} \leq C \sqrt{\log h_V} \|t_h\|_{H_0^1(\Omega)} \ ,$$

and used the boundedness of the data smoothing operator acting from $L^2(\Omega)$ into itself (see the last line in (12)). Similarly we get

$$[g^\delta - g^{\delta,i}](q_h) \leq \|a_0\|_{L^\infty(\Omega)} \|u_{sm}^\delta - u_{sm}^{\delta,i}\|_{H_0^1(\Omega)} \|q_h\|_{H_0^1(\Omega)} \leq C h_{sm}^{-1} h_Q^{-(1+m)} \|q_h\|_Q \|u^{\delta,i} - u^\delta\|_{L^2(\Omega)} \ .$$

Inserting this and (37) into (38), we obtain

$$\|d_h^i\|_V \leq C h_{sm}^{-1} h_Q^{-(3+m)} |\log h_V|^{m/2} \|u^{\delta,i} - u^\delta\|_{L^2(\Omega)}$$

and therefore convergence of d_h^i to zero in V as $i \to \infty$.

□

In the following theorem we formulate our main result, concerning the convergence of the regularized aprroximations

$$a_h := a_0 + s_h \tag{39}$$

to a solution $a^\dagger$ of the parameter identification problem (1), as the noise level and the appropriately chosen mesh sizes go to zero:

Theorem 2. *Let (31), (32), and*

$$\bigtriangledown u^\dagger \in L^\infty$$

hold and let, in (33), $\gamma \leq \frac{1}{2}$ *if* $m = 0$, $\gamma < \frac{1}{2}$ *if* $m = 1$.
Then (39) with (35), and the data mollification as well as the discretization as described in subsection 2.2 and after Lemma 2, defines a regularization method for our parameter identification problem, i.e., for any noisy data u^δ *with (2)*

$$a_h = a_h(u^\delta, \delta) \to a^\dagger \text{ in } X \text{ as } \delta \to 0 \ .$$

Proof. $\bar{s} = a^\dagger - a_0$ is the primal part of a solution of

$$\begin{array}{lllll} \bar{a}(\bar{s}, t) & + & \bar{b}(t, \bar{p}) & = & 0 \qquad \forall t \in \bar{V} \\ \bar{b}(\bar{s}, q) & & & = & \bar{g}(q) \quad \forall q \in \bar{Q}, \end{array} \tag{40}$$

where

$$\begin{array}{lll} \bar{a}(s,t) & := & \langle s,t\rangle_X \\ \bar{b}(s,q) & := & \int_\Omega s \bigtriangledown u^\dagger \bigtriangledown q \; dx \\ \bar{g}(q) & := & -\int_\Omega a_0 \bigtriangledown \bar{r} \bigtriangledown q \; dx \ , \end{array}$$

$$\bar{V} := X \ , \quad \bar{Q} := \overline{H_0^1(\Omega)}^{\bar{Q}} \ , \quad \|v\|_{\bar{Q}} := \| \bigtriangledown u^\dagger \bigtriangledown v\|_{H^{-m}(\Omega)} \ ,$$

and $\bar{r} = u^\dagger - u_0$. Indeed, since $\bar{a}$ and $\bar{b}$ are bounded and satisfy $\bar{V}$-ellipticity and inf-sup-condition, respectively, and also $\bar{g}$ is bounded:

$$\bar{g}(q) = -\int_\Omega a_0 \bigtriangledown u^\dagger \bigtriangledown q \; dx + \underbrace{\int_\Omega a_0 \bigtriangledown u_0 \bigtriangledown q \; dx}_{=\int_\Omega a^\dagger \bigtriangledown u^\dagger \bigtriangledown q \; dx} \leq \|a_0 - a^\dagger\|_X \|q\|_{\bar{Q}} \ ,$$

(40) is solvable. From (35) and (40) we can now conclude that for any pair $(\tilde{s}_h, \tilde{p}_h) \in V_h \times Q_h$

$$\begin{array}{lllll} a(s_h - \tilde{s}_h, t_h) & + & b(t_h, p_h - \tilde{p}_h) & = & \bar{a}(\bar{s} - \tilde{s}_h, t_h) + \bar{b}(t_h, \bar{p} - \tilde{p}_h) \quad \forall t_h \in V_h \\ b(s_h - \tilde{s}_h, q_h) & & & = & \bar{b}(\bar{s} - \tilde{s}_h, q_h) + [g^\delta - \bar{g}](q_h) \quad \forall q_h \in Q_h. \end{array} \tag{41}$$

The standard estimate for solutions of mixed variational problems that we have already used in the proof of Theorem 1 (cf. e.g. Theorem 4.6 in [13]) now, together with the ellipticity of a and Lemma 1 yields

$$\|s_h - \tilde{s}_h\| \leq \sup_{t_h \in V_h} \frac{\bar{a}(\bar{s} - \tilde{s}_h, t_h) + \bar{b}(t_h, \bar{p} - \tilde{p}_h)}{\|t_h\|_V} + \frac{2}{c} \sup_{q_h \in Q_h} \frac{\bar{b}(\bar{s} - \tilde{s}_h, q_h) + [g^\delta - \bar{g}](q_h)}{\|q_h\|_Q} \ ,$$

and therefore, using the triangle inequality

$$
\begin{aligned}
\|\bar{s} - s_h\| &\leq \|\bar{s} - \tilde{s}_h\| + \|s_h - \tilde{s}_h\| \\
&\leq (3 + C\sqrt{\delta}^{2\gamma} h_Q^{-2})\|\bar{s} - \tilde{s}_h\| + C\sqrt{\delta} h_Q^{-(1+m)} + \sup_{t_h \in V_h} \frac{\bar{b}(t_h, \bar{p} - \tilde{p}_h)}{\|t_h\|_V} ,
\end{aligned}
\tag{42}
$$

where we have used the estimates

$$
\begin{aligned}
\bar{b}(\bar{s} - \tilde{s}_h, q_h)/\|q_h\|_Q &\leq \|\bar{s} - \tilde{s}_h\|(1 + \|\nabla(u^\delta_{sm} - u^\dagger)\nabla q_h\|_{H^{-m}(\Omega)}/\|q_h\|_Q) , \\
\|\nabla(u^\delta_{sm} - u^\dagger)\nabla q_h\|_{L^2(\Omega)} &\leq \|\nabla(u^\delta_{sm} - u^\dagger)\|_{L^2(\Omega)}\|\nabla q_h\|_{L^\infty(\Omega)} , \\
\|\nabla(u^\delta_{sm} - u^\dagger)\nabla q_h\|_{H^{-1}(\Omega)} &\leq \|id\|_{H^1_0(\Omega)\to L^{2/(1-2\gamma)}(\Omega)}\|\nabla(u^\delta_{sm} - u^\dagger)\|_{L^{1/\gamma}(\Omega)}\|q_h\|_{H^1_0(\Omega)} \\
&\leq C(\|\nabla(u^\delta_{sm} - u^\dagger)\|_{L^2(\Omega)})^{2\gamma}\|q_h\|_{H^1_0(\Omega)} , \text{ if } \gamma < \tfrac{1}{2} , \\
[g^\delta - \bar{g}](q_h) &\leq \|a_0\|_{L^\infty(\Omega)}\|u^\delta_{sm} - u^\dagger\|_{H^1_0(\Omega)}\|q_h\|_{H^1_0(\Omega)} ,
\end{aligned}
$$

(where we have used the interpolation inequality and (12) to obtain the fourth inequality,) as well as (12) and Lemma 2. Taking the infimum over all $\tilde{s}_h \in V_h$ we get convergence to zero of the first term on the right hand side of (42). The second term is, due to (33), (34), controlled by $C\sqrt{\delta}^{1-\gamma(1+m)}$, which, by our choice of γ, goes to zero as $\delta \to 0$. To estimate the last term on the right hand side of (42) we set $\tilde{p}_h$ equal to the dual part $\bar{p}_h$ of a solution $(\bar{s}_h, \bar{p}_h)$ to the semidiscrete version of (40)

$$
\begin{aligned}
\bar{a}(\bar{s}_h, t) \quad + \quad \bar{b}(t, \bar{p}_h) &= 0 \qquad \forall t \in \bar{V} \\
\bar{b}(\bar{s}_h, q_h) &= \bar{g}(q_h) \quad \forall q_h \in Q_h,
\end{aligned}
$$

so that, by the fact that $\bar{s}_h$ is just the orthogonal projection of $\bar{s}$ on $\bar{A}^* Q_h$, we get

$$
\sup_{t_h \in V_h} \frac{\bar{b}(t_h, \bar{p} - \bar{p}_h)}{\|t_h\|_V} = \sup_{t_h \in V_h} \frac{\bar{a}(\bar{s}_h - \bar{s}, t_h)}{\|t_h\|_V} \leq \|[I - P_{\bar{A}^* Q_h}]\bar{s}\|_X ,
$$

which, since $\bar{s} \in N(\bar{A})^- = \overline{R(\bar{A}^*)}$, and $\overline{\bigcup_{h_Q > 0} Q_h} = N(\bar{A}^*)^-$, goes to zero as $h_Q \to 0$.

□

The following corollary contains a convergence rates result under a source condition (4) with $\nu = \frac{1}{2}$, i.e., under the assumption that

$$
a^\dagger - a_0 = \bar{A}^* \psi \tag{43}
$$

for some $\psi \in Y$. Due to (23) (with u^δ_{sm} replaced by $u^\dagger$, (43) means that the difference $a^\dagger - a_0$ is in $H^{1+2m}(\Omega)$ and satisfies certain boundary conditions.

Corollary 1. *Let the conditions of Theorem 2 and additionally*

$$
\partial\Omega \in C^{1+m} , \quad \nabla a_0 \in L^\infty(\Omega) , \quad \triangle u^\dagger \in L^{2+m\epsilon}(\Omega) \textit{ for some } \epsilon > 0
$$

as well as (43) hold, and let γ be chosen as

$$\gamma = \frac{1}{2(1+m)} \ .$$

Then

$$\|a_h - a^\dagger\|_X = O(\delta^{1/4})$$

in both cases $X = H_0^m(\Omega)$, $m = 0, 1$.

Proof. Condition (43) means that

$$(-\triangle)^m \bar{s} = \nabla u^\dagger \nabla \tilde{\psi} \tag{44}$$

where $\tilde{\psi}$ solves the second order BVP (24) with L^2-right hand side ψ and is therefore in $H^2(\Omega)$. On the other hand, the pair $(\bar{s}, \tilde{\psi})$ solves the mixed variational problem (40) so that we may set $\bar{p} := \tilde{\psi}$. Moreover, (44) implies that $(-\triangle)^m \bar{s} \in H^1(\Omega)$ and therefore $\bar{s} \in H^{1+m}(\Omega)$.

Now we use the estimate (42) in the proof of Theorem 2 where this time we estimate the last term by

$$\sup_{t_h \in V_h} \frac{\bar{b}(t_h, \bar{p} - \tilde{p}_h)}{\|t_h\|_V} \leq \begin{cases} \| \nabla u^\dagger \|_{L^\infty(\Omega)} \|\bar{p} - \tilde{p}_h\|_{H_0^1(\Omega)} & \text{if } m = 0 \\ \sup_{t_h \in V_h} \frac{\|\nabla (t_h \nabla u^\dagger))\|_{L^2(\Omega)}}{\|t_h\|_{H_0^1(\Omega)}} \|\bar{p} - \tilde{p}_h\|_{L^2(\Omega)} & \text{if } m = 1 \end{cases} .$$

Taking the infimum over all $(\tilde{s}_h, \tilde{p}_h) \in V_h \times Q_h$ yields

$$\begin{aligned} \|\bar{s} - s_h\| &\leq C(h_V \|\bar{s}\|_{H^{1+m}(\Omega)} + \sqrt{\delta} h_Q^{-(1+m)} + h_Q^{1+m} \|\bar{p}\|_{H^2(\Omega)}) \\ &\leq C(\sqrt{\delta} + \sqrt{\delta}^{1-\gamma(1+m)} + \sqrt{\delta}^{\gamma(1+m)}) \ , \end{aligned}$$

which, by our choice of γ, yields the assertion.

□

The suboptimality in the convergence rate — the optimal rate under a source condition (4) with $\nu = \frac{1}{2}$ being $O(\sqrt{\delta})$ — has two reasons:
Firstly the anisotropy in the (natural) dual norm leads to a loss of regularity, namely, by regularity theory for hyperbolic problems (cf., e.g., [2]) the lower estimate of $\|\|_Q$ by only an L^2-norm cannot be improved to an estimate by an H^n-norm with $n > 0$, while on the other hand obviously an H^1-norm is needed to estimate $\|\|_Q$ from above.
Secondly, the low order of the ansatz functions in the approximation spaces, that is from a practical point of view hightly recommendable, leads to a saturation effect in some approximation estimates.

4 Numerical Results

To illustrate the theoretical results of the foregoing section, we implemented the proposed Method 1 on basis of the FE-package FEPP that has been developped at the SFB F013 in Linz, and applied it to two different test problems in the following setting:

The domain Ω was chosen to be the unit square, the right hand side f and the boundary conditions g in (1) as

$$f(x,y) :\equiv 1 \ , \quad g(x,y) :\equiv 0 \ ,$$

and the starting parameter a_0 as

$$a_0 :\equiv 1 \ .$$

We define two test examples by the exact solutions

$$a(x,y) := 1 + 1.6x(1-x)y(1-y) \tag{45}$$

$$a(x,y) := 1 + 6.4x(1-x)y(0.5-y)\chi_{[0,0.5]}(y) \tag{46}$$

which we will refer to as the "smooth" and the "non-smooth" case, respectively. The first example is supposed to represent the situation with source condition as in Corollary 1, while in the second one obviously (43) does not hold and only the results of Theorem 2 can be expected to hold. In preimage space X we confined ourselves to the H_0^1-norm (i.e., $m = 1$ in (11), Lemma 2, Theorem 2, and Corollary 1), considering this the more interesting case, especially for comparison with other solution methods for the parameter identification problem under consideration.

Investigating first the noiseless case, where the smoothing procedure can be omitted, we duplicated the search for an optimal relationship between the mesh sizes h_Q and h_V by comparing the resulting errors $\|s_h - s^\dagger\|_{H_0^1}$ for different refinement levels $l_{Q/V}$ ($h_{Q/V} \sim 2^{-l_{Q/V}}$, so that $dim(Q/V_l) = (2^{l-1}-1)^2$, i.e., for $l = 3,4,5,6,7,8$, $dim(Q/V_l) = 9, 225, 961, 3969, 16129$; see Figure 4). In fact, for each fixed Q-level, the decay of the error with growing V-level saturates at refinement levels as proposed by the asymptotics $h_Q \sim (h_V)^\gamma$ (i.e. V-level $\sim \frac{1}{\gamma}\cdot$ Q-level, with $\gamma \leq \frac{1}{2}$ being smaller in the smooth than in the non-smooth case. Plotting, still in the noiseless case, the develpoment of the error for growing Q-level (with optimal V-level) shows, as expected, a higher speed of convergence for the smooth example, namely $O(h_Q^2)$, than for the non-smooth one (see Figure 4).

To define the data smoothing operator Π_{sm} we used, for sake of simplicity, the interpolation operator Π_{Int} on an equidistant grid with grid points N_j. Assuming that pointwise data measurements u_j^δ for $u(N_j)$ with

$$\max_j |u_j^\delta - u(N_j)| \leq \delta \tag{47}$$

one easily sees that the first three inequalities in (12) hold and therefore the convergence (rate) results of Theorem 2 and Corollary 1 can be applied. The stability result Theorem 1 remains valid if convergence of $u^{\delta,i}$ in L^2 is replaced by L^∞-convergence. Note, that the

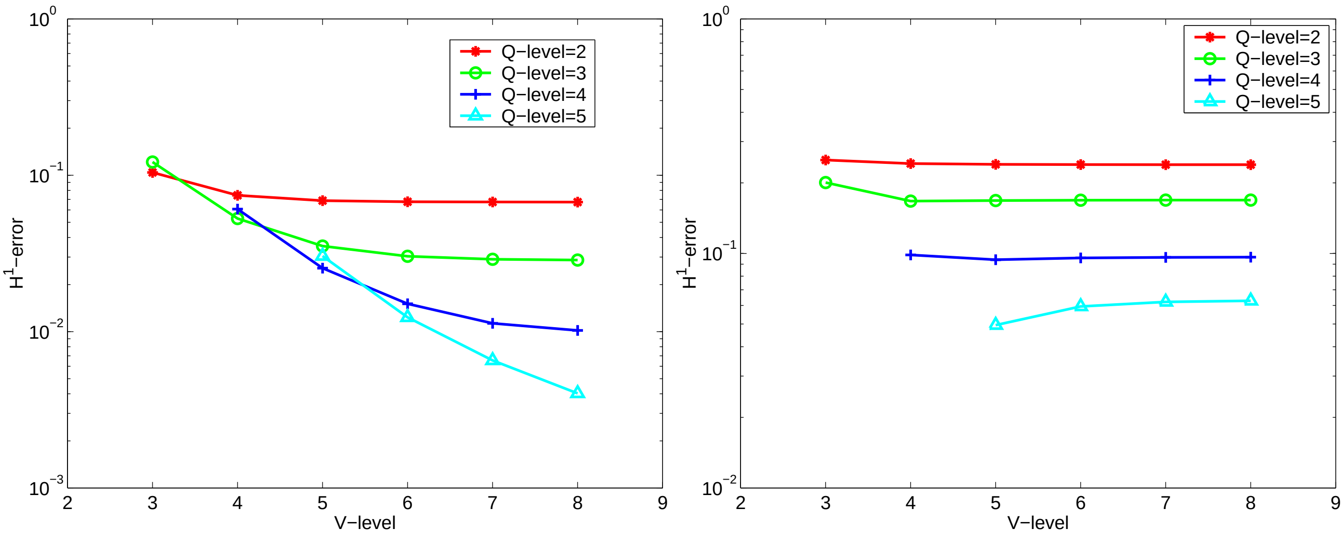

Figure 1: Optimal choice of $h_V = h_{sm}$ for different values of h_Q in the noiseless case, for example (45) (left) and (46) (right)

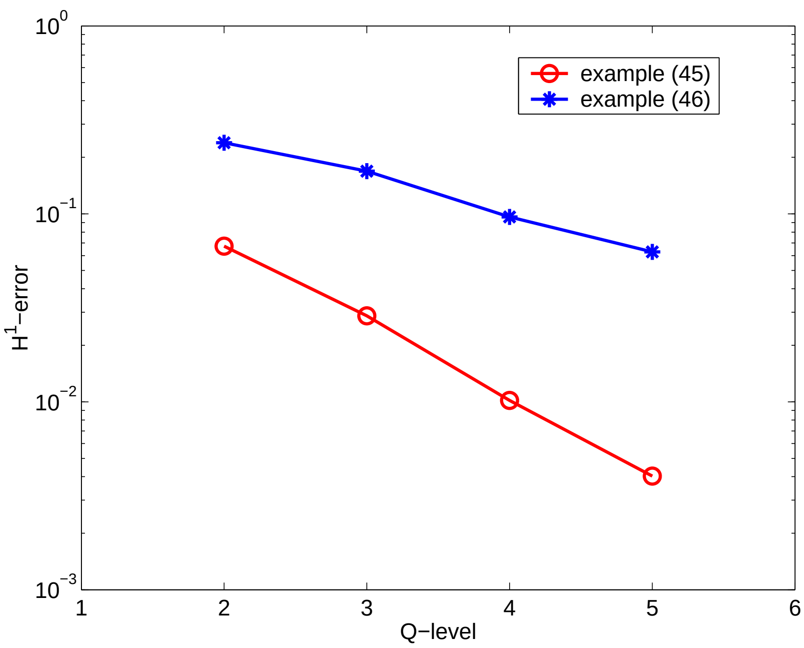

Figure 2: Convergence speed as $h_Q \to 0$ for smooth and non-smooth example, respectively (noiseless case)

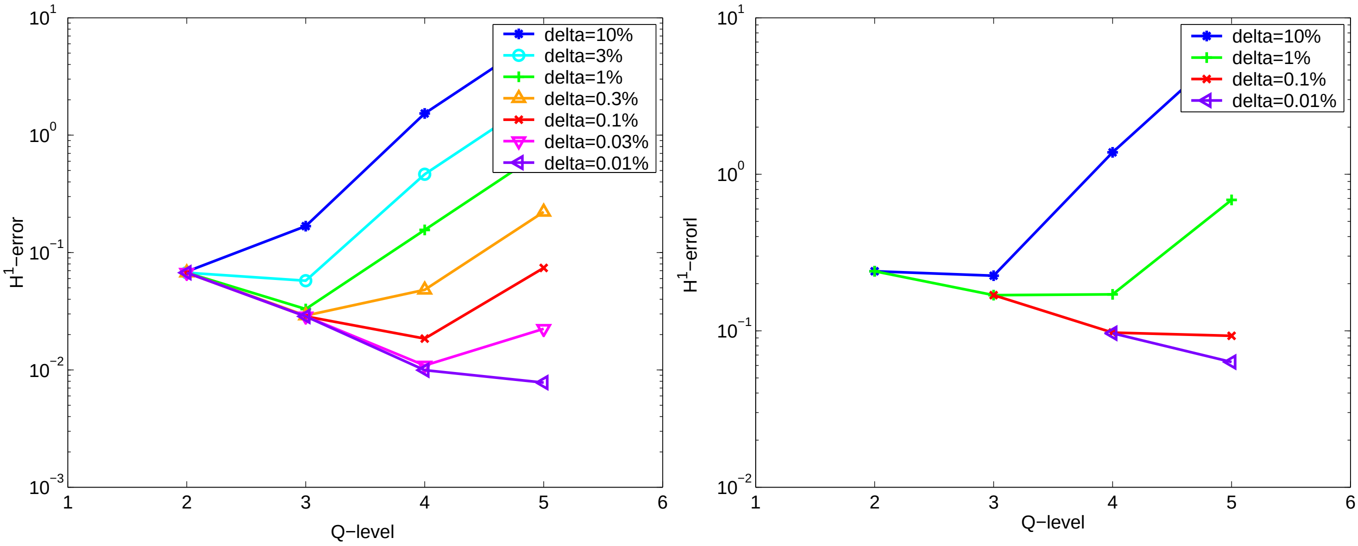

Figure 3: Error norms for different noise levels in dependence of h_Q, for smooth ((45), left) and non-smooth example ((46), right), respectively.

assumption of L^∞-measurements (47) is — although theoretically of course stronger than assuming L^2-measurements (2) — just as natural in applications such as inverse groundwater filtration and therefore not really restrictive.

An observation of the error for for shrinking h_Q (and h_V, h_{sm} chosen according to (33) with fixed $\gamma < \frac{1}{2}$), for several noise levels (see Figure 4) shows that — as usual in ill-posed problem — the decay due to the reduction of the approximation error stops at some "optimal regularization parameter" and then turns to some growth caused by the then dominating propagated data noise.

Optimally chosen Q-levels according to these curves as well as to the theoretical asymptotics proposed in Theorem 2 and Corollary 1 yield error plots suggesting convergence as $\delta \to 0$, at a faster rate in the smooth than in the non-smooth example; in the smooth case the predicted rate $O(\delta^{\frac{1}{4}})$ really seems to be obtained (see Figure 4).

Finally, the left hand picture in Figure 4 shows our reconstruction from data with one per cent noise, corresponding to example (45), with discretization according to Theorem 2 and Corollary 1. The obtained approximation is obviously very close to the exact solution. For comparison, as shown in the right hand picture, due to the instability, a too fine discretization choice, yields a bad reconstruction, even with less data noise.

References

[1] H.T. Banks, K. Kunisch, *Estimation Techniques for Distributed Parameter Systems*, Birkhäuser, Basel Boston Berlin, 1989.

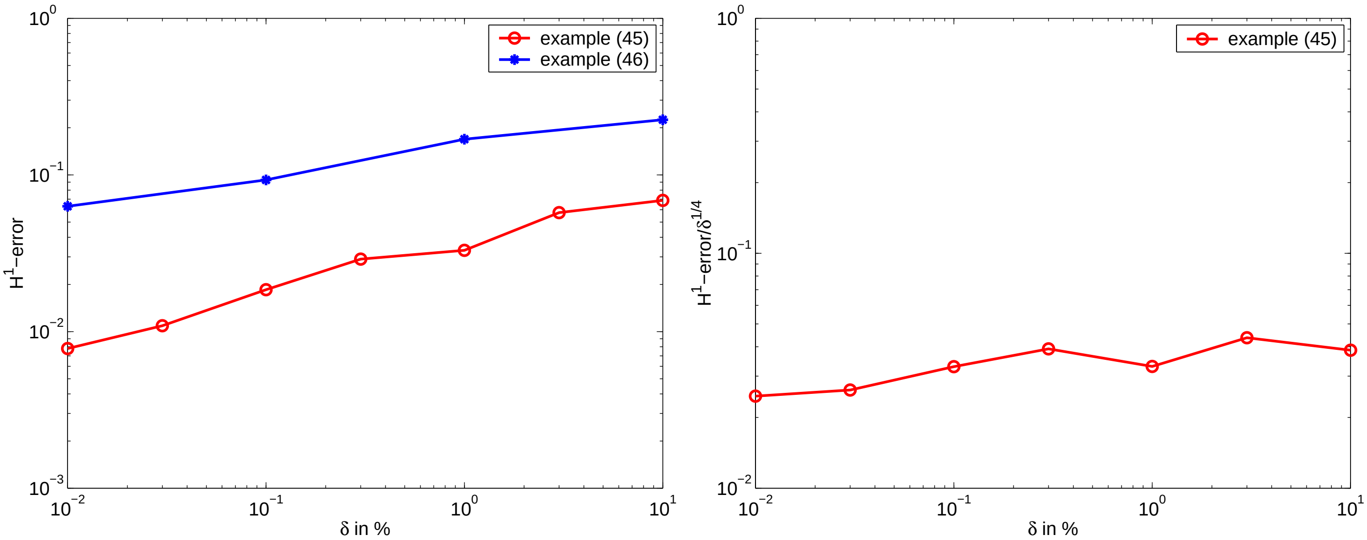

Figure 4: Convergence speed as $\delta \to 0$ for smooth and non-smooth example, respectively; quotient $\|a_h - a\|/\delta^{\frac{1}{4}}$ for example (45).

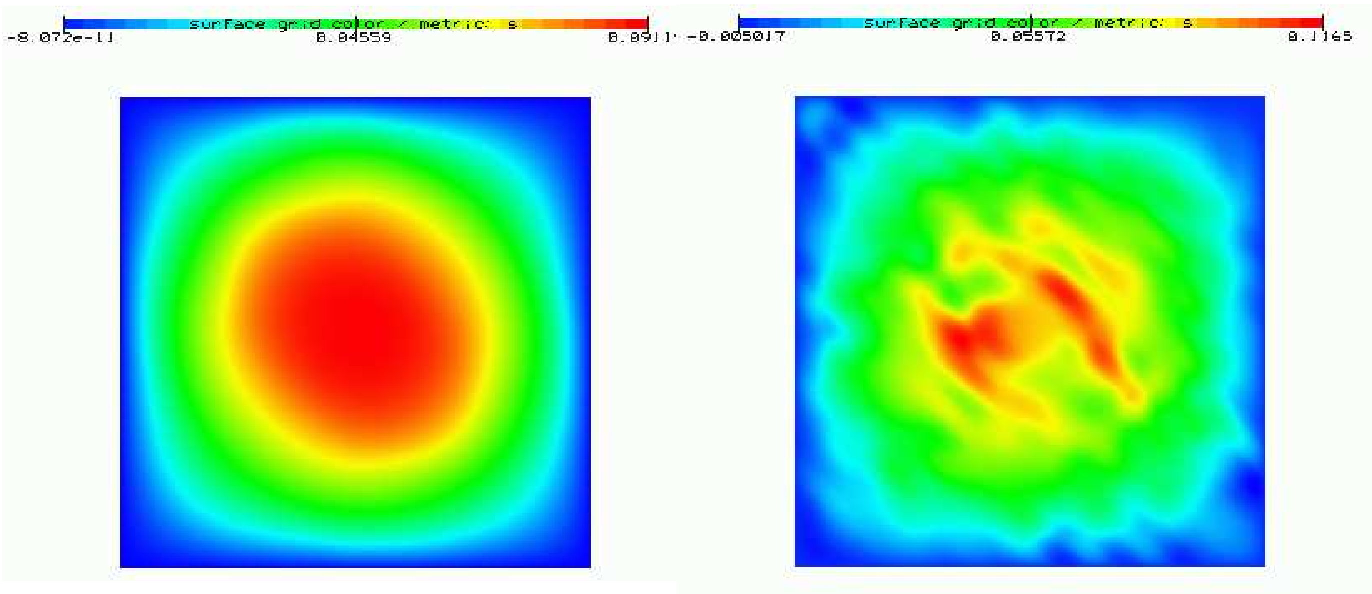

Figure 5: Reconstructions of example (45) with "correct" discretization choice (Q-level=3) and 1 per cent data noise (left) and with "wrong" discretization choice (Q-level=5) and 0.3 per cent data noise (right).

[2] C. Bardos, D. Brezis, H Brezis, *Perturbations singulières et prolonguements maximaux d'opérateurs positives*, Arch. Rat. Mech. Anal. 53 (1973), 69–100.

[3] F. Brezzi, M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer, New York Berlin Heidelberg, 1991.

[4] G. Chavent, K. Kunisch, J.E. Roberts, *Primal-Dual Formulations for Parameter Estimation Problems*, Comp. Appl. Math 18 (1999).

[5] C.G. Ciarlet, P.A. Raviart, *A mixed finite element method for the biharmonic equation*, in Mathematical Aspects of Finite Elements in Partial Differential Equations, C. de Boor, ed., Academic Press, New York, 1974.

[6] P. Clément, *Approximation by finite element functions using local regularization*, R.A.I.R.O. Anal. Numer. 9 (1975), 77–84.

[7] H.W. Engl, *On the convergence of regularization methods for ill-posed operator equations*, in: G. Hämmerlin, K.H. Hoffmann, eds., *Improperly Posed Problems and their Numerical Treatment*, Birkhäuser, Basel, 1983, 81–95.

[8] H.W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996

[9] H.W. Engl, A. Neubauer, *On projection methods for solving linear ill–posed problems*, in: A. Vogel, ed., Model Optimization in Exploration Geophysics, Vieweg, Braunschweig, 1987, 73–92.

[10] C.W. Groetsch, *The Theory of Tikhonov Regularization for Fredholm Integral Equations of the First Kind*, Pitman, Boston, London, Melbourne, 1984.

[11] C.W. Groetsch, *Inverse Problems in Mathematical Sciences*, Vieweg, Braunschweig, 1993.

[12] C.W. Groetsch, A. Neubauer, *Convergence of a general projection method for an operator equation of the first kind*, Houston J. Mathem. 14 (1988), 201–207.

[13] Ch. Grossmann, H.-G. Roos, *Numerik partieller Differentialgleichungen*, Teubner, Stuttgart, 1994.

[14] M. Hanke, *A regularization Levenberg–Marquardt scheme, with applications to inverse groundwater filtration problems*, Inverse Problems 13 (1997), 79–95.

[15] K. Ito, K. Kunisch, *On the injectivity and linearization of the coefficient-to-solution mapping for elliptic boudary value problems*, J.Math.Anal.Appl. 188 (1994), 1040-1066.

[16] B. Kaltenbacher, *A projection-regularized Newton method for nonlinear ill-posed problems and its application to parameter identification problems with finite element discretization*, to appear in SIAM J. Numer. Anal.

[17] C. Kravaris, J.H. Seinfeld, *Identification of parameters in distributed parameter systems by regularization*, SIAM J. Control Optim. 23 (1985), 217-241.

[18] M. Kuhn, U. Langer and J. Schberl, *Scientific computing tools for 3D magnetic field problems*, SFB-Report No. 99-13, Johannes Kepler Universitt Linz, Spezialforschungsbereich F013, 4020 Linz, Austria, 1999, to appear in the MAFELAP 1999 proceedings.

[19] A. K. Louis, *Inverse und schlecht gestellte Probleme*, Teubner, Stuttgart, 1989.

[20] B. Mercier, *Numerical solution of the biharmonic problem by mixed finite elements of class C^0*, Boll.U.M.I. 10 (1974), 133–149.

[21] V.A. Morozov, *Regularization Methods for Ill-Posed Problems*, CRC Press, Boca Raton, 1993.

[22] F. Natterer, *Regularisierung schecht gestellter Probleme durch Projektionsverfahren*, Numer. Math. 28 (1977), 329–341.

[23] F. Natterer, *The Mathematics of Computerized Tomography*, Teubner, Stuttgart, 1986.

[24] A.N. Tikhonov, V.A. Arsenin, *Methods for Solving Ill-Posed Problems*, Nauka, Moskau, 1979.

[25] G. Vainikko, *Identification of filtration coefficient*, in: Ill-posed Problems in Natural Sciences, A.Tikhonov, e.d, VSP/TVP, 1992, 202-213.

[26] G.M. Vainikko, A.Y. Veterennikov, *Iteration Procedures in Ill–Posed Problems*, Nauka, Moscow, 1986. In Russian.