POLYNOMIAL EXTENSION OPERATORS. PART III

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Abstract. In this concluding part of a series of papers on tetrahedral polynomial extension operators, the existence of a polynomial extension operator in the Sobolev space $\boldsymbol{H}(\mathrm{div})$ is proven constructively. Specifically, on any tetrahedron K, given a function w on the boundary of K that is a polynomial on each face, the operator applied to w gives a polynomial of at most the same degree in the tetrahedron. The trace of the normal component of the extension coincides with w. Furthermore, the extension is continuous from $H^{-1/2}(\partial K)$ into $\boldsymbol{H}(\mathrm{div}, K)$.

1. Introduction

This is the final installment of our series of papers [6, 7] devoted to the construction of polynomial extensions on any tetrahedron K. In Part I [6], we constructed an extension operator from $H^{1/2}(\partial K)$ into $H^1(K)$ that preserves polynomials (in the sense made precise there). In Part II [7] we extended our techniques to develop an operator that extends appropriate tangential vector fields on ∂K into $\boldsymbol{H}(\mathbf{curl}, K)$ and preserves polynomials in some sense. This part is devoted to the construction of an $\boldsymbol{H}(\mathrm{div})$ polynomial extension operator. This operator extends functions in $H^{-1/2}(\partial K)$ into $\boldsymbol{H}(\mathrm{div}, K)$ in such a way that if the function to be extended is a polynomial on each face of K, then the extended function is also a polynomial of at most the same degree on K. The overall technique employed here for constructing the extension operator is similar to the previous two parts. However, there are some fresh ingredients that play an important role, such as a commuting volumetric extension operator and an operator used in proving the classical Poincaré lemma in differential geometry. *The main result of this paper is* Theorem 7.1. *The results of this series of papers can be succinctly represented in a commuting diagram given at the end in* (8.1). For the history of the polynomial extension problem and contributions by many authors to it, we refer the reader to the introduction of Part I [6].

The construction of our $\boldsymbol{H}(\mathrm{div})$ extension will be guided, as in the previous cases [6, 7], by a target commutativity property. Namely, the final extension operator $\mathcal{E}_K^{\mathrm{div}}$ should satisfy

$$\mathcal{E}_K^{\mathrm{div}}(\mathrm{curl}_\tau\, \boldsymbol{v}) = \mathbf{curl}\,(\mathcal{E}_K^{\mathrm{curl}} \boldsymbol{v}), \tag{1.1}$$

for all $\boldsymbol{v}$ in the space of traces of $\boldsymbol{H}(\mathbf{curl})$. Here $\mathcal{E}_K^{\mathrm{curl}}$ is the $\boldsymbol{H}(\mathbf{curl})$ polynomial extension operator we constructed in [7] and $\mathrm{curl}_\tau\, \boldsymbol{v}$ denotes the surface curl of $\boldsymbol{v}$. The subscript τ indicates tangential components, and accordingly $\mathbf{grad}_\tau, \mathrm{curl}_\tau$, and div_τ denote tangential gradient, curl, and divergence, on the boundaries of suitable three dimensional domains. For details concerning the definition of differential operators on nonsmooth manifolds, see [3, 4]. So as not to leave any sign ambiguity, let us clarify that if $\boldsymbol{\phi}$ is smooth function on K and $\boldsymbol{n}$ is the unit outward

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normal vector on ∂K, then

$$\operatorname{curl}_\tau(\operatorname{trc}_\tau \boldsymbol{\phi}) = \operatorname{div}_\tau(\boldsymbol{\phi}\times\boldsymbol{n}) = \boldsymbol{n}\cdot\mathbf{curl}\,\boldsymbol{\phi} \tag{1.2}$$

on ∂K. Here trc_τ denotes the tangential trace map. For smooth vector functions $\boldsymbol{\phi}$, the action of trc_τ is defined by

$$\operatorname{trc}_\tau \boldsymbol{\phi} = \big(\boldsymbol{\phi} - (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}\big)\big|_{\partial K}.$$

In contrast, the normal trace operator is defined by

$$\operatorname{trc}_n \boldsymbol{\phi} = (\boldsymbol{\phi}\cdot\boldsymbol{n})\big|_{\partial K}.$$

The extension $\boldsymbol{\mathcal{E}}_K^{\mathrm{div}}$ we shall construct in this paper is a right inverse of trc_n.

Additionally, our extension has a polynomial preservation property important in finite element applications. To describe this property, let $P_\ell(D)$ denote the space of polynomials of degree at most ℓ on any subset D of a Euclidean space (so it could be univariate or multivariate polynomials). Further, let $\boldsymbol{P}_\ell(D)$ denote the set of vector functions on D whose components are in $P_\ell(D)$. Our extension operator $\boldsymbol{\mathcal{E}}_K^{\mathrm{div}} : H^{-1/2}(\partial K) \mapsto \boldsymbol{H}(\mathrm{div})$ is such that if w is a polynomial of degree at most p on each of the faces of the tetrahedron K, then $\boldsymbol{\mathcal{E}}_K^{\mathrm{div}} w$ is in $\boldsymbol{P}_p(K)$. In conforming finite element applications involving $\boldsymbol{H}(\mathrm{div},\Omega)$, the discrete space restricted to an element K is often $\boldsymbol{P}_p(K)$. It could also be the so-called Raviart-Thomas space [12], defined by $\boldsymbol{R}_p(K) = \{\boldsymbol{q}_p + \boldsymbol{x} r_p : \ \boldsymbol{q}_p \in \boldsymbol{P}_P(K),\ r_p \in P_p(K)\}$ where $\boldsymbol{x} = (x,y,z)^t$ is the coordinate vector. Since

$$\operatorname{trc}_n(\boldsymbol{P}_p(K)) = \operatorname{trc}_n(\boldsymbol{R}_p(K))$$

and $\boldsymbol{R}_p(K) \supseteq \boldsymbol{P}_p(K)$, our extension operator has polynomial preservation properties with respect to both the choices, i.e.,

$$\boldsymbol{\mathcal{E}}_K^{\mathrm{div}} : \operatorname{trc}_n(\boldsymbol{P}_p(K)) \mapsto \boldsymbol{P}_p(K) \qquad \text{as well as} \qquad \boldsymbol{\mathcal{E}}_K^{\mathrm{div}} : \operatorname{trc}_n(\boldsymbol{R}_p(K)) \mapsto \boldsymbol{R}_p(K).$$

Hence we anticipate its utility in high order Raviart-Thomas finite elements as well as the high order BDM method [2].

The organization of this paper is as follows. We start by establishing a stable decomposition of the $\boldsymbol{H}(\mathrm{div})$ trace space into a regular part and a surface curl. For this we need an extension of volume data with specific regularity properties. Therefore we will first construct such volumetric extensions. These are generalizations of well known classical extensions [10, 13] and are therefore independently interesting. We use them to establish the stable trace decomposition. Next, in Section 3, we develop an $\boldsymbol{H}(\mathrm{div})$ extension from a plane as the first step towards constructing an extension from the boundary of a tetrahedron. In the succeeding sections, we develop a sequence of correction operators that will progressively help us solve a sequence of simpler problems of increasing complexity, leading to the solution of the full polynomial extension problem from ∂K. These simpler problems are the two-face problem (Section 4), the three face problem (Section 5) and the four-face problem (Section 6). The main theorem is in Section 7, and we conclude in Section 8. An appendix which collects all the proofs of technical lemmas is also included.

2. A decomposition of the trace space

In this section, we show that the trace space of $\boldsymbol{H}(\mathrm{div})$ admit a stable decomposition consisting of two components, one of which is regular, and the other is a surface curl. We begin by developing a volumetric extension which we will need in proving the existence of the stable decomposition. Such extensions are also interesting in their own right.

2.1. **Commuting extensions of volume data.** The purpose of this subsection is to generalize an old technique of [10] to obtain volumetric extensions. Volumetric extensions do not extend traces, but rather extends functions defined on three dimensional domains to larger three dimensional domains. The generalization is aimed at obtaining some specific regularity properties in the spaces

$$\boldsymbol{H}^k(\mathbf{curl}, D) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, D) : \quad \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{H}^k(D)\}$$
$$\boldsymbol{H}^k(\mathrm{div}, D) = \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, D) : \quad \mathrm{div}\,\boldsymbol{v} \in H^k(D)\},$$

normed in the obvious way. We begin by giving commuting generalized reflection operators.

Proposition 2.1. *Suppose D is an open bounded subset of the plane $\mathbb{R}^2$ with Lipschitz boundary. For any $\ell > 0$, consider the three dimensional domains $S = (0,\ell) \times D$ and $\widetilde{S} = (-\ell,\ell) \times D$. Then, for any integer $k \geq 1$, there are volumetric extension operators $\hat{\mathcal{G}}^{\mathrm{grad}}$, $\hat{\mathcal{G}}^{\mathrm{curl}}$, $\hat{\mathcal{G}}^{\mathrm{div}}$, and $\hat{\mathcal{G}}_1$ extending functions on S to $\widetilde{S}$ such that the diagram*

$$\begin{array}{ccccccc} H^k(S) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}^{k-1}(\mathbf{curl}, S) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}^{k-1}(\mathrm{div}, S) & \xrightarrow{\mathrm{div}} & H^{k-1}(S) \\ \big\downarrow \hat{\mathcal{G}}^{\mathrm{grad}} & & \big\downarrow \hat{\mathcal{G}}^{\mathrm{curl}} & & \big\downarrow \hat{\mathcal{G}}^{\mathrm{div}} & & \big\downarrow \hat{\mathcal{G}}_1 \\ H^k(\widetilde{S}) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}^{k-1}(\mathbf{curl}, \widetilde{S}) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}^{k-1}(\mathrm{div}, \widetilde{S}) & \xrightarrow{\mathrm{div}} & H^{k-1}(\widetilde{S}) \end{array} \tag{2.1}$$

commutes. The operators are continuous as maps from and into the above indicated spaces.

Proof. Let u be a function in $C^\infty(\overline{S})$ and α_j be real numbers to be specified shortly. The generalized reflection operator considered in [10, Lemma 3] is

$$\hat{\mathcal{G}}_0 u\,(x,y,z) = \begin{cases} u(x,y,z), & \text{if } x > 0, \\ \displaystyle\sum_{j=1}^{k} \alpha_j\ u(-x/j, y, z), & \text{if } x \leq 0. \end{cases}$$

For this proof, we need an additional operator

$$\hat{\mathcal{G}}_1 u\,(x,y,z) = \begin{cases} u(x,y,z), & \text{if } x > 0, \\ \displaystyle\sum_{j=1}^{k} -(\frac{\alpha_j}{j})\, u(-x/j, y, z), & \text{if } x \leq 0. \end{cases}$$

Observe that if the α_j's satisfy

$$\sum_{j=1}^{k} \alpha_j \frac{1}{(-j)^m} = 1, \tag{2.2}$$

then

$$\lim_{\substack{x\to 0\\ x>0}} \frac{\partial^m}{\partial x^m} \hat{\mathcal{G}}_0 u(x,y,z) - \lim_{\substack{x\to 0\\ x<0}} \frac{\partial^m}{\partial x^m} \hat{\mathcal{G}}_0 u(x,y,z) = 0. \tag{2.3}$$

Considering (2.2) for $m = 0, 1, \dots, k-1$ as a linear system of k equations in the k unknowns α_j, it is easy to see that there is a unique set of α_j's that solves it. Thus (2.3) holds for all $m = 0, 1, \dots, k-1$, and consequently, by standard arguments, $\hat{\mathcal{G}}_0$ extends to

$$\hat{\mathcal{G}}_0 : H^k(S) \mapsto H^k(\widetilde{S}) \tag{2.4}$$

as a continuous map. With the same choice of α_i's, we similarly also have that $\hat{\mathcal{G}}_1$ extends to

$$\hat{\mathcal{G}}_1 : H^{k-1}(S) \mapsto H^{k-1}(\widetilde{S}) \tag{2.5}$$

as a continuous operator.

We define the required volumetric extensions for a scalar function u and a vector function $\boldsymbol{v}$ with components v_1, v_2, v_3 by

$$\hat{\mathcal{G}}^{\text{grad}} u = \hat{\mathcal{G}}_0 u, \qquad \hat{\mathcal{G}}^{\text{curl}} \boldsymbol{v} = \begin{pmatrix} \hat{\mathcal{G}}_1 v_1 \\ \hat{\mathcal{G}}_0 v_2 \\ \hat{\mathcal{G}}_0 v_3 \end{pmatrix}, \qquad \hat{\mathcal{G}}^{\text{div}} \boldsymbol{v} = \begin{pmatrix} \hat{\mathcal{G}}_0 v_1 \\ \hat{\mathcal{G}}_1 v_2 \\ \hat{\mathcal{G}}_1 v_3 \end{pmatrix}.$$

Using the obvious identities

$$\partial_x(\hat{\mathcal{G}}_0 u) = \hat{\mathcal{G}}_1(\partial_x u), \qquad \begin{aligned} \partial_y(\hat{\mathcal{G}}_i u) &= \hat{\mathcal{G}}_i(\partial_y u), \quad \text{for } i = 0 \text{ or } 1, \\ \partial_z(\hat{\mathcal{G}}_i u) &= \hat{\mathcal{G}}_i(\partial_z u), \quad \text{for } i = 0 \text{ or } 1, \end{aligned} \tag{2.6}$$

we immediately verify that the above defined operators satisfy the commutativity properties in (2.1).

To prove the continuity properties asserted in the proposition, first note that the required continuity of $\hat{\mathcal{G}}_1$ is already proved in (2.5). The continuity of $\hat{\mathcal{G}}^{\text{div}}$ follows from the commutativity

$$\operatorname{div} \hat{\mathcal{G}}^{\text{div}} \boldsymbol{w} = \hat{\mathcal{G}}_1 \operatorname{div} \boldsymbol{w}$$

and the continuity of $\hat{\mathcal{G}}_1$ as follows:

$$\| \operatorname{div} \hat{\mathcal{G}}^{\text{div}} \boldsymbol{w} \|_{H^{k-1}(\widetilde{S})} = \| \hat{\mathcal{G}}_1 \operatorname{div} \boldsymbol{w} \|_{H^{k-1}(\widetilde{S})} \le C \| \operatorname{div} \boldsymbol{w} \|_{H^{k-1}(S)}.$$

The continuity of $\hat{\mathcal{G}}^{\text{curl}}$ follows from

$$\mathbf{curl}\, \hat{\mathcal{G}}^{\text{curl}} \boldsymbol{v} = \hat{\mathcal{G}}^{\text{div}} \mathbf{curl}\, \boldsymbol{v}.$$

and the already established continuity of $\hat{\mathcal{G}}^{\text{div}}$. The continuity of $\hat{\mathcal{G}}^{\text{grad}}$ is the same as in (2.4). □

It is obvious from the above proof that one can consider domains more general than $\widetilde{S}$. Indeed, the proof holds for any $\widetilde{S}$ that has reflectional symmetry about a plane. We now use this to generalize the above result to a domain around the unit tetrahedron. Let $\hat{K}$ denote the closed unit tetrahedron with vertices $\hat{\boldsymbol{a}}_0 = (0,0,0)$, $\hat{\boldsymbol{a}}_1 = (1,0,0)$, $\hat{\boldsymbol{a}}_2 = (0,1,0)$, $\hat{\boldsymbol{a}}_3 = (0,0,1)$. Let $\hat{F}_i$ denote the face of $\hat{K}$ opposite to $\hat{\boldsymbol{a}}_i$. We want to find an operator that extends functions defined outside $\hat{K}$ into $\hat{K}$ by combining reflections across the faces $\hat{F}_1$, $\hat{F}_2$, and $\hat{F}_3$. Of course, mere addition of the reflections across each of these faces is insufficient because one application of the generalized reflection across a face alters the values near the remaining faces. We must combine the reflections more carefully.

We now do this for some specific domains we shall need later (although more general domains can be handled equally well), which we describe first. They are convex enlargements of $\hat{K}$

defined separately for each index I in $\{i, ij, ijk\}$, where $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$. Let $\widetilde{\boldsymbol{a}}_{i,j} = 2\hat{\boldsymbol{a}}_j - \hat{\boldsymbol{a}}_i$ and $\widetilde{\boldsymbol{a}}_{i,k} = 2\hat{\boldsymbol{a}}_k - \hat{\boldsymbol{a}}_i$. Define the enlarged domains

$$\widetilde{K}_i = \operatorname{conv}(\hat{K}, \widetilde{\boldsymbol{a}}_{i,j}, \widetilde{\boldsymbol{a}}_{i,k}, -\hat{\boldsymbol{a}}_i), \tag{2.7}$$

$$\widetilde{K}_{ij} = \operatorname{conv}(\widetilde{K}_i, \widetilde{K}_j), \tag{2.8}$$

$$\widetilde{K}_{ijk} = \operatorname{conv}(\widetilde{K}_i, \widetilde{K}_j, \widetilde{K}_k). \tag{2.9}$$

where $\operatorname{conv}(\cdots)$ denotes the convex hull of all its arguments. We want to construct an operator extending functions on $\widetilde{K}_I \setminus \hat{K}$ into $\hat{K}$.

Proposition 2.2. *For any integer $k \geq 1$, and any I in $\{i, ij, ijk\}$, there are continuous volumetric extension operators*

$$\begin{array}{rccll}
\mathcal{G}^{\mathrm{grad}}: & H^k(\widetilde{K}_I \setminus \hat{K}) & \longmapsto & H^k(\widetilde{K}_I) & (\mathcal{G}^{\mathrm{grad}} u|_{\widetilde{K}_I\setminus\hat{K}} = u), \\
\mathcal{G}^{\mathrm{curl}}: & \boldsymbol{H}^{k-1}(\mathbf{curl}, \widetilde{K}_I \setminus \hat{K}) & \longmapsto & \boldsymbol{H}^{k-1}(\mathbf{curl}, \widetilde{K}_I) & (\mathcal{G}^{\mathrm{curl}} \boldsymbol{v}|_{\widetilde{K}_I\setminus\hat{K}} = \boldsymbol{v}), \\
\mathcal{G}^{\mathrm{div}}: & \boldsymbol{H}^{k-1}(\mathrm{div}, \widetilde{K}_I \setminus \hat{K}) & \longmapsto & \boldsymbol{H}^{k-1}(\mathrm{div}, \widetilde{K}_I) & (\mathcal{G}^{\mathrm{div}} \boldsymbol{w}|_{\widetilde{K}_I\setminus\hat{K}} = \boldsymbol{w})
\end{array}$$

satisfying the commutativity properties in (2.1).

Proof. Consider the case $I = 123$. In this case the domain $D = \{(x, y, z) : \ |x| + |y| + |z| \leq 1\}$, formed of eight tetrahedra, is contained in $\widetilde{K}_I$. Let $D_{\pm x} = \{(x, y, z) \in D : \pm x \geq 0\}$, and define $D_{\pm y}$ and $D_{\pm z}$ similarly. Recall the extension $\hat{\mathcal{G}}^{\mathrm{curl}}$ defined in the proof of Proposition 2.1. It is obtained by generalized reflections about the yz-plane. Hence it defines an operator extending functions on D_{-x} into D_{+x}. To distinguish this extension from reflections about other faces, we now call it $\mathcal{G}_x^{\mathrm{curl}}$, i.e.,

$$\mathcal{G}_x^{\mathrm{curl}} \boldsymbol{v}(x, y, z) = \begin{cases} \boldsymbol{v}(x, y, z) & \text{if } (x, y, z) \in \widetilde{K}_I \text{ and } x < 0, \\ \hat{\mathcal{G}}^{\mathrm{curl}} \boldsymbol{v}(x, y, z) & \text{if } (x, y, z) \in D_{+x}. \end{cases}$$

Similarly, we define $\mathcal{G}_y^{\mathrm{curl}}$ and $\mathcal{G}_z^{\mathrm{curl}}$ using generalized reflections across the other faces. All these maps have continuity properties in the $\boldsymbol{H}^{k-1}(\mathbf{curl}, \cdot)$-norm as in Proposition 2.1.

We now use these maps to obtain a continuous extension from $\widetilde{K} \setminus \hat{K}$ into $\hat{K}$. Let $\mathcal{R}_x$, $\mathcal{R}_y$, and $\mathcal{R}_z$ denote the operations of restricting functions to D_{+x}, D_{+y} and D_{+z}, respectively. Then set

$$\mathcal{G}^{\mathrm{curl}} \boldsymbol{u} = \begin{cases} \boldsymbol{u}, & \text{on } \widetilde{K} \setminus \hat{K}, \\ \mathcal{G}_x^{\mathrm{curl}} \mathcal{R}_x \boldsymbol{u} + \mathcal{G}_y^{\mathrm{curl}} \mathcal{R}_y (\boldsymbol{u} - \mathcal{G}_x^{\mathrm{curl}} \mathcal{R}_x \boldsymbol{u}) & \\ \quad + \ \mathcal{G}_z^{\mathrm{curl}} \mathcal{R}_z (\boldsymbol{u} - \mathcal{G}_x^{\mathrm{curl}} \mathcal{R}_x \boldsymbol{u} - \mathcal{G}_y^{\mathrm{curl}} \mathcal{R}_y (\boldsymbol{u} - \mathcal{G}_x^{\mathrm{curl}} \mathcal{R}_x \boldsymbol{u})), & \text{on } \hat{K}. \end{cases}$$

We similarly define $\mathcal{G}^{\mathrm{grad}}$ and $\mathcal{G}^{\mathrm{div}}$. Then the claimed commutativity and continuity properties follow.

The cases of the remaining I in $\{i, ij\}$ are similar and simpler, so we omit the details. □

2.2. **Regular decomposition of traces.** The idea for decomposing the trace space of $\boldsymbol{H}(\mathrm{div}, \Omega)$ is easy to describe. Let Ω be a polyhedral domain with Lipschitz continuous boundary. If $\boldsymbol{w}$ is in $\boldsymbol{H}(\mathrm{div}, \Omega)$, then it is well known that there is a stable decomposition

$$\boldsymbol{w} = \mathbf{curl}\, \boldsymbol{\phi} + \boldsymbol{\theta}$$

with $\boldsymbol{\phi}$ in $\boldsymbol{H}(\mathbf{curl}\,,\Omega)$ and $\boldsymbol{\theta}$ in $\boldsymbol{H}^1(\Omega)$. Taking the normal trace, we find that

$$\begin{aligned}\operatorname{trc}_n \boldsymbol{w} &= \boldsymbol{n}\cdot\mathbf{curl}\,\boldsymbol{\phi} + \boldsymbol{n}\cdot\boldsymbol{\theta}\\ &= \operatorname{curl}_\tau(\operatorname{trc}_\tau \boldsymbol{\phi}) + \operatorname{trc}_n \boldsymbol{\theta}.\end{aligned}$$

Thus we have a decomposition

$$H^{-1/2}(\partial\Omega) = \operatorname{curl}_\tau(\boldsymbol{X}^{-1/2}(\partial\Omega)) + \operatorname{trc}_n(\boldsymbol{H}^1(\Omega)) \tag{2.10}$$

where $\boldsymbol{X}^{-1/2}(\partial\Omega)$ is the range of trc_τ. Note that the results of [3] provide a characterization of $\boldsymbol{X}^{-1/2}(\partial\Omega)$ and in particular show that $\operatorname{curl}_\tau : \boldsymbol{X}^{-1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$ is continuous. Hence the decomposition in (2.10) is stable. We will use the decomposition (2.10) with Ω set to the tetrahedron K.

In fact, it is possible to refine the decomposition (2.10) further and choose the argument of curl_τ to be more regular. First note that the trace space of $\boldsymbol{H}(\mathbf{curl}\,,\Omega)$ admits the decomposition

$$\boldsymbol{X}^{-1/2}(\partial\Omega) = \mathbf{grad}_\tau(H^{1/2}(\partial\Omega)) + \operatorname{trc}_\tau(\boldsymbol{H}^1(\Omega)). \tag{2.11}$$

This was used in Part II [7]. Its proof is simple, once we use the well known regular decomposition of $\boldsymbol{H}(\mathbf{curl}\,,\Omega)$ that asserts that for any $\boldsymbol{v}$ in $\boldsymbol{H}(\mathbf{curl}\,,\Omega)$, there is a unique φ in $H^1(\Omega)$ and $\boldsymbol{\psi}$ in $\boldsymbol{H}^1(\Omega)$ such that

$$\boldsymbol{v} = \mathbf{grad}\,\varphi + \boldsymbol{\psi}$$

is a stable decomposition. Taking trc_τ, we obtain (2.11). Now, using (2.11) in (2.10) and observing that $\operatorname{curl}_\tau \mathbf{grad}_\tau = 0$, we can revise (2.10) to

$$H^{-1/2}(\partial\Omega) = \operatorname{curl}_\tau \operatorname{trc}_\tau(\boldsymbol{H}^1(\Omega)) + \operatorname{trc}_n(\boldsymbol{H}^1(\Omega)). \tag{2.12}$$

The potential space $\operatorname{trc}_\tau(\boldsymbol{H}^1(\Omega))$ is now more regular than in (2.10). It is characterized in [3]. It is (strictly) contained in the space of square integrable tangential vector fields on $\partial\Omega$ whose restrictions to each face Γ of the polyhedral boundary are in $\boldsymbol{H}^{1/2}(\Gamma)$. Clearly this space is more regular than the space $\boldsymbol{X}^{-1/2}(\partial\Omega)$ in (2.10) which is contained only in a Sobolev space of negative order.

One immediate use of the decomposition (2.12) is in defining a restriction operator on $H^{-1/2}(\partial\Omega)$. If F is a face of the polyhedron $\partial\Omega$, then decomposing any g in $H^{-1/2}(\partial\Omega)$ using (2.12) as $g = \operatorname{curl}_\tau \boldsymbol{\phi} + \theta$ with $\boldsymbol{\phi}$ in $\operatorname{trc}_\tau(\boldsymbol{H}^1(\Omega))$ and θ in $\operatorname{trc}_n(\boldsymbol{H}^1(\Omega))$, the restrictions $\boldsymbol{\phi}|_F$ and $\theta|_F$ make sense, so the restriction operator

$$R_F g = \operatorname{curl}_\tau(\boldsymbol{\phi}\big|_F) + \theta|_F \tag{2.13}$$

is well defined and continuous on $H^{-1/2}(\partial\Omega)$.

Not withstanding the fact that (2.12) has a more regular potential than (2.10), in what follows, we shall continue to work with (2.10). This is because $\boldsymbol{X}^{-1/2}(\partial\Omega)$ can be normed by a quotient norm inherited from $\boldsymbol{H}(\mathbf{curl}\,,\Omega)$, as described in [7], while the natural norm that makes $\operatorname{trc}_\tau \boldsymbol{H}^1(\Omega)$ complete [3] seems to be a bit unwieldy for our purposes.

In constructing the extension operator, we shall use the above mentioned facts with Ω set to a tetrahedron K. We first need to make precise the notion of traces that weakly vanish on a subset $S \subseteq \partial K$. Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $H^{-1/2}(\partial K)$ and $H^{1/2}(\partial K)$. Suppose S

has positive boundary measure. Define

$$H^1_{0,S}(K) = \{\psi \in H^1(K) : \operatorname{trc}\psi|_S = 0\},$$
$$\boldsymbol{H}_{0,S}(\mathbf{curl}) = \{\boldsymbol{\phi} \in \boldsymbol{H}(\mathbf{curl}) : \langle \boldsymbol{n}\times\boldsymbol{\psi}, \operatorname{trc}_\tau \boldsymbol{\phi}\rangle = 0 \quad \forall \boldsymbol{\psi} \text{ with components in} H^1_{0,\partial K\setminus S}(K)\},$$
$$\boldsymbol{H}_{0,S}(\operatorname{div}) = \{\boldsymbol{w} \in \boldsymbol{H}(\operatorname{div}) : \langle \psi, \boldsymbol{n}\cdot\boldsymbol{w}\rangle = 0 \quad \text{for all } \psi \in H^1_{0,\partial K\setminus S}(K)\}.$$

Consider the case when S is composed of one or more faces of K. We adopt the notations in the previous parts for faces and vertices of K. E.g., $\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ denote the vertices of K, and F_i denote the face of K opposite to $\boldsymbol{a}_i$. So, we consider the situation when S is one of F_i, $F_{ij} = F_i \cup F_j$, or $F_{ijk} = F_i \cup F_j \cup F_k$. Here and elsewhere the indices i, j, k, l are a permutation of 0,1,2,3. For all subscripts I in the set $\{i, ij, ijk\}$, define the following ranges of the trace maps:

$$H^{1/2}_{0,I}(\partial K) = \operatorname{trc}\ H^1_{0,F_I}(K),$$
$$\boldsymbol{X}^{-1/2}_{0,I}(\partial K) = \operatorname{trc}_\tau \boldsymbol{H}_{0,F_I}(\mathbf{curl}),$$
$$H^{-1/2}_{0,I}(\partial K) = \operatorname{trc}_n \boldsymbol{H}_{0,F_I}(\operatorname{div}).$$

We will often omit the argument ∂K when the space consists of functions defined on the whole boundary ∂K. All these spaces are normed by quotient norms, e.g.,

$$\|\nu\|_{H^{-1/2}_{0,I}(\partial K)} = \inf_{\operatorname{trc}_n(\boldsymbol{w})=\nu} \|\boldsymbol{w}\|_{\boldsymbol{H}(\operatorname{div})}, \tag{2.14}$$

where the infimum runs over all $\boldsymbol{w}$ in $\boldsymbol{H}_{0,F_I}(\operatorname{div})$ such that $\operatorname{trc}_n \boldsymbol{w} = \nu$. The space $H^{1/2}_{0,I}(\partial K)$ and its restrictions to faces $H^{1/2}_{0,I}(F_l)$ featured in Part I [6]. The space $\boldsymbol{X}^{-1/2}_{0,I}(\partial K)$ was important in Part II [7], where we also precisely defined the notion of its restrictions to a face F_l, denoted there by $\boldsymbol{X}^{-1/2}_{0,I}(F_l)$. In the current paper, we will need restrictions of elements in $H^{-1/2}_{0,I}(\partial K)$ to F_l, where the restriction operation is defined as in (2.13). Abbreviating R_{F_l} to R_l, define

$$H^{-1/2}_{0,I}(F_l) = R_l(H^{-1/2}_{0,I}(\partial K)). \tag{2.15}$$

Also let

$$H^{-1/2}(F_l) = R_l(H^{-1/2}(\partial K)). \tag{2.16}$$

Remark 2.1. For a planar domain F, it is standard to denote by $H^{-1/2}(F)$ the dual space of $H^{1/2}_0(F) \equiv \{u \in H^{1/2}(F) : \text{the extension by zero of } u \text{ to } \mathbb{R}^2 \text{ is in } H^{1/2}(\mathbb{R}^2)\}$. Hence it might appear that the redefinition of $H^{-1/2}(F)$ in (2.16) is a high abuse of notation. However, the space $H^{-1/2}(F_l)$ in (2.16) coincides with the dual of $H^{1/2}_0(F_l)$. This follows from well known characterizations of $H^{-1/2}(\partial K)$, for instance, a result in [4, pp. 43] shows that functions in the dual of $H^{1/2}_0(F_l)$ have continuous extensions into $H^{1/2}(\partial K)$.

The following theorem gives the stable decompositions we will need in the analysis of polynomial extensions.

Theorem 2.1. *The spaces $H^{-1/2}_{0,I}(F_l)$ and $H^{-1/2}(F_l)$ admit the stable decompositions*

$$H^{-1/2}_{0,I}(F_l) = \operatorname{curl}_\tau \boldsymbol{X}^{-1/2}_{0,I}(F_l) + H^{1/2}_{0,I}(F_l)$$
$$H^{-1/2}(F_l) = \operatorname{curl}_\tau \boldsymbol{X}^{-1/2}(F_l) + H^{1/2}(F_l),$$

for all indices I in $\{i, ij, ijk\}$.

Proof. Because of the previous discussion, we only need to prove the first decomposition. Moreover, it suffices to prove it for the "reference tetrahedron" $\hat{K}$ introduced earlier, with the index $l = 0$ and $\{i, j, k\}$ a permutation of $\{1, 2, 3\}$. Let $\widetilde{K}_I$ be as defined in (2.7)–(2.9) and let $\widetilde{F}_I$ denote the face of $\widetilde{K}_I$ containing $\hat{F}_0$.

Given ν in $H_{0,I}^{-1/2}(\hat{F}_0)$, there is a $\boldsymbol{w}$ in $\boldsymbol{H}_{0,F_I}(\mathrm{div}, \hat{K})$ such that

$$R_l \,\mathrm{trc}_n(\boldsymbol{w}) = \nu, \quad \text{and} \tag{2.17}$$
$$\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathrm{div})} \leq C\|\nu\|_{H_{0,I}^{-1/2}(\hat{F}_0)}.$$

Let $\widetilde{\boldsymbol{w}}$ denote the trivial extension of $\boldsymbol{w}$ from $\hat{K}$ to $\widetilde{K}_I$, i.e., $\widetilde{\boldsymbol{w}}$ vanishes on $\widetilde{K}_I \setminus K$ and equals $\boldsymbol{w}$ on K. It is easy to verify that $\widetilde{\boldsymbol{w}}$ is in $\boldsymbol{H}(\mathrm{div}, \widetilde{K}_I)$.

We decompose $\widetilde{\boldsymbol{w}}$ by a continuous Helmholtz-Hodge type decomposition [8] applied on the convex domain $\widetilde{K}_I$ to get

$$\widetilde{\boldsymbol{w}} = \mathbf{curl}\,\boldsymbol{\phi} + \boldsymbol{\theta} \tag{2.18}$$

where $\boldsymbol{\phi}$ is in $\boldsymbol{H}(\mathbf{curl}\,, \widetilde{K}_I)$ and $\boldsymbol{\theta}$ is in $\boldsymbol{H}^1(\widetilde{K}_I)$. Now, since $\widetilde{\boldsymbol{w}}$ vanishes on $\widetilde{K}_I \setminus \hat{K}$,

$$\mathbf{curl}\,\boldsymbol{\phi}\big|_{\widetilde{K}_I\setminus\hat{K}} = -\boldsymbol{\theta}\big|_{\widetilde{K}_I\setminus\hat{K}} \in \boldsymbol{H}^1(\widetilde{K}_I \setminus \hat{K}).$$

Hence $\boldsymbol{\phi}$ is in $\boldsymbol{H}^1(\mathbf{curl}\,, \widetilde{K}_I \setminus \hat{K})$. Applying the volumetric extension of Proposition 2.2 with $k = 1$, we obtain an extension $\boldsymbol{\phi}'$ of $\boldsymbol{\phi}$ from $\widetilde{K}_I \setminus \hat{K}$ to all $\widetilde{K}_I$ with the property that $\mathbf{curl}\,\boldsymbol{\phi}'$ is in $\boldsymbol{H}^1(\widetilde{K}_I)$. Thus

$$\widetilde{\boldsymbol{w}} = \mathbf{curl}\,\boldsymbol{\phi}'' + \boldsymbol{\theta}'' \tag{2.19}$$

where $\boldsymbol{\phi}'' = \boldsymbol{\phi} - \boldsymbol{\phi}'$ and $\boldsymbol{\theta}'' = \mathbf{curl}\,\boldsymbol{\phi}' + \boldsymbol{\theta}$. Clearly, $\boldsymbol{\phi}''$ is in $\boldsymbol{H}(\mathbf{curl}\,, \widetilde{K}_I)$ and $\boldsymbol{\theta}''$ is in $\boldsymbol{H}^1(\widetilde{K}_I)$. Moreover, both $\boldsymbol{\phi}''$ and $\boldsymbol{\theta}''$ vanish on $\widetilde{K}_I \setminus \hat{K}$.

Finally, applying the normal trace operator to (2.19) we obtain

$$\mathrm{trc}_n\,\widetilde{\boldsymbol{w}} = \mathrm{curl}_\tau\,\mathrm{trc}_\tau\,\boldsymbol{\phi}'' + \mathrm{trc}_n\,\boldsymbol{\theta}''.$$

Observe that since all components of $\boldsymbol{\theta}''$ vanish on $\widetilde{K}_I \setminus \hat{K}$,

$$(\mathrm{trc}_n\,\boldsymbol{\theta}'')\big|_{\widetilde{F}_{I,l}} \in H^{1/2}(\widetilde{F}_I) \quad \text{and} \quad \mathrm{supp}(\,(\mathrm{trc}_n\,\boldsymbol{\theta}'')\big|_{\widetilde{F}_I}) \subseteq \hat{F}_0.$$

Hence $\vartheta \equiv \mathrm{trc}_n\,\boldsymbol{\theta}''|_{\hat{F}_0}$ is in $H_{0,I}^{1/2}(\hat{F}_0)$. Similarly, since $\boldsymbol{\phi}''$ vanishes on $\widetilde{K}_I \setminus \hat{K}$, the restriction of the tangential trace $\mathrm{trc}_\tau\,\boldsymbol{\phi}''$ to $\hat{F}_0$, denoted by $\boldsymbol{\varphi}$, is in $\boldsymbol{X}_{0,I}^{-1/2}(\hat{F}_0)$. Hence

$$\nu = R_l\,\mathrm{trc}_n(\boldsymbol{w}) = \mathrm{curl}_\tau\,\boldsymbol{\varphi} + \vartheta$$

is the required decomposition. Its stability follows from the continuity of all the intermediate steps, including the stability of the decomposition (2.18), the continuity of the volumetric extension $\boldsymbol{\phi} \mapsto \boldsymbol{\phi}'$ (Proposition 2.2), the continuity of the lifting in (2.17), and the continuity of the trace maps. □

3. Primary extension operator

By "primary extensions" we mean, as in Parts I and II [6, 7], extensions of data specified on a planar surface. The construction of such extensions forms the first step in designing polynomial extensions of data from piecewise planar manifolds like the boundary of a tetrahedron. Define the primary extension for the $\boldsymbol{H}(\mathrm{div})$ case by

$$\mathcal{E}^{\mathrm{div}} w = 2\int_0^1\int_0^{1-t}\begin{pmatrix} s\\ t\\ -1\end{pmatrix} w(x+sz, y+tz)\ ds\,dt, \tag{3.1}$$

for all smooth functions $w(x,y)$.

We will now rewrite this expression using the affine coordinates of the tetrahedron. This will help generalize the expression to yield an extension from any face of a general tetrahedron. Let λ_i denote the affine (or barycentric coordinates) of a general tetrahedron K. As in Parts I and II [6, 7], for any permutation $\{i,j,k,l\}$ of $\{0,1,2,3\}$, we define the subtriangle

$$T_l(r_i,r_j,r_k) = \{\boldsymbol{x}\in F_l : \quad \lambda_\ell^{F_l}(\boldsymbol{x}) \ge r_\ell \text{ for } \ell = i,j, \text{ and } k\},$$

where $\lambda_m^{F_l} \equiv \lambda_m|_{F_l}$ (for $m = i,j,$ or k) are the barycentric coordinates of the face F_l. Now consider the expression in (3.1) as an extension into the "reference tetrahedron" $\hat{K}$ with vertices $\hat{\boldsymbol{a}}_0 = (0,0,0), \hat{\boldsymbol{a}}_1 = (1,0,0), \hat{\boldsymbol{a}}_2 = (0,1,0), \hat{\boldsymbol{a}}_3 = (0,0,1)$ from the face $\hat{F}_3$ opposite to $\hat{\boldsymbol{a}}_3$. Transforming (3.1) by the variable change $x' = x+sz, y' = y+tz$ and using barycentric coordinates, we have

$$\begin{aligned}\mathcal{E}^{\mathrm{div}} w &= \frac{2}{z^2}\int_x^{x+z}\int_y^{x+y+z-x'}\begin{pmatrix}(x'-x)/z\\ (y'-y)/z\\ -1\end{pmatrix} w(x',y')\ dx'\,dy'\\ &= \frac{1}{|\hat{F}_3|\lambda_3^2}\iint\limits_{T_3(\lambda_0,\lambda_1,\lambda_2)}\begin{pmatrix}\widetilde{\lambda}_1\\ \widetilde{\lambda}_2\\ -1\end{pmatrix} w(\boldsymbol{s})\ d\boldsymbol{s}.\end{aligned}$$

Here and elsewhere, while λ_ℓ denotes the barycentric coordinates of the tetrahedron under consideration, the symbol $\widetilde{\lambda}_\ell$ denotes a barycentric coordinate of the region of integration under consideration. The subtriangles that form our regions of integration are always considered as having their node enumeration inherited from the parent triangle. So, in the above formula, $\{\widetilde{\lambda}_0,\widetilde{\lambda}_1,\widetilde{\lambda}_2\}$ are the barycentric coordinates of $T_3(\lambda_0,\lambda_1,\lambda_2)$. Now observe that the vector part of the integrand can be rewritten as

$$\begin{aligned}\begin{pmatrix}\widetilde{\lambda}_1\\ \widetilde{\lambda}_2\\ -1\end{pmatrix} &= \widetilde{\lambda}_1(\mathbf{grad}\,\lambda_2\times\mathbf{grad}\,\lambda_3) + \widetilde{\lambda}_2(\mathbf{grad}\,\lambda_3\times\mathbf{grad}\,\lambda_1) - (\mathbf{grad}\,\lambda_1\times\mathbf{grad}\,\lambda_2)\\ &= \sum_{i=0}^{2} -\widetilde{\lambda}_i(\mathbf{grad}\,\lambda_{i\dotplus 1}\times\mathbf{grad}\,\lambda_{i\dotplus 2}),\end{aligned}$$

where $\dotplus$ denotes addition mod 3. The last identity was arrived at by expressing $\mathbf{grad}\,\lambda_3$ in terms of the gradients of $\lambda_0,\ \lambda_1$, and λ_2.

Motivated by the above rearrangement, we define the primary extension into a general tetrahedron K from one of its face, say F_l, using affine coordinates. Unlike the H^1 and $\boldsymbol{H}(\mathbf{curl}\,)$

cases, we now need to track orientation. Let $\boldsymbol{a}_i, \boldsymbol{a}_j, \boldsymbol{a}_k, \boldsymbol{a}_l$ be the vertices of K where, as usual, $\{i,j,k,l\}$ is a permutation of $\{0,1,2,3\}$. We say that "(i,j,k) is *positively oriented* with respect to l" if the vertices $\boldsymbol{a}_i$, $\boldsymbol{a}_j$, and $\boldsymbol{a}_k$, in that order, form a counterclockwise enumeration of the vertices of the face F_l, when looking from the vertex $\boldsymbol{a}_l$ (see Figure 1). Define

$$\boldsymbol{\mathcal{E}}_l^{\mathrm{div}} w = \frac{1}{|F_l|\,\lambda_l^2} \iint\limits_{T_l(\lambda_i,\lambda_j,\lambda_k)} -\sum_{\sigma\in S(l)} \widetilde{\lambda}_{\sigma_1}\,(\mathbf{grad}\,\lambda_{\sigma_2} \times \mathbf{grad}\,\lambda_{\sigma_3})\, w(\boldsymbol{s})\, d\boldsymbol{s} \tag{3.2}$$

Here, for any index l, we denote by $S(l)$ the set of (three) cyclic permutations of the remaining three indices i,j,k ordered so that they are positively oriented with respect to l. The above sum thus runs over all such cyclic permutations σ in $S(l)$. In the summand, the three components of σ are denoted by $\sigma_1, \sigma_2, \sigma_3$. The symmetries are clearly evident from (3.2): The region of integration $T(\lambda_i, \lambda_j, \lambda_k)$ is unchanged with respect to even or odd permutations of i,j,k, while the integrand is antisymmetric under odd permutations of i,j,k. Note that when $K = \hat{K}$, $l = 3$, and $(i,j,k) = (0,1,2)$, the expression in (3.2) coincides with that in (3.1). The properties of the operator $\boldsymbol{\mathcal{E}}_l^{\mathrm{div}}$ are collected in the next theorem.

Theorem 3.1 (Primary extension)**.** *The operator $\boldsymbol{\mathcal{E}}_l^{\mathrm{div}}$ has the following properties:*

(1) $\mathbf{curl}\,(\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}\boldsymbol{v}) = \boldsymbol{\mathcal{E}}_l^{\mathrm{div}}(\mathrm{curl}_\tau\,\boldsymbol{v})$ *for all $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}(F_l)$.*
(2) $\boldsymbol{\mathcal{E}}_l^{\mathrm{div}}$ *is a continuous map from $H^{1/2}(F_l)$ into $\boldsymbol{H}^1(K)$.*
(3) $\boldsymbol{\mathcal{E}}_l^{\mathrm{div}}$ *is a continuous map from $H^{-1/2}(F_l)$ into $\boldsymbol{H}(\mathrm{div})$.*
(4) *The tangential trace of $\boldsymbol{\mathcal{E}}_l^{\mathrm{div}}w$ on F_l equals w for all in $H^{-1/2}(F_l)$.*
(5) *If w is in $P_p(F_l)$, then $\boldsymbol{\mathcal{E}}_l^{\mathrm{div}}w$ is in $\boldsymbol{P}_p(K)$.*

Proof. Proof of (1): Let $\boldsymbol{v}(x,y) = (v_1, v_2)^t$ be a smooth function on the reference tetrahedron $\hat{K}$. Recalling the expression for $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ on $\hat{K}$ from [7], namely

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v} = 2\int_0^1\int_0^{1-t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s & t \end{pmatrix} \boldsymbol{v}(x+sz, y+tz)\, ds\, dt$$

and computing its curl, we have

$$\begin{aligned}
\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}) &= 2\int_0^1\int_0^{1-t} \mathbf{curl}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s & t \end{pmatrix} \boldsymbol{v}(x+sz, y+tz)\, ds\, dt \\
&= 2\int_0^1\int_0^{1-t} \begin{pmatrix} s(\partial_2 v_1 - \partial_1 v_2)(x+sz, y+tz) \\ t(\partial_2 v_1 - \partial_1 v_2)(x+sz, y+tz) \\ (\partial_1 v_2 - \partial_2 v_1)(x+sz, y+tz) \end{pmatrix} ds\, dt,
\end{aligned}$$

Since $\mathrm{curl}_\tau\,\boldsymbol{v} = \partial_2 v_1 - \partial_1 v_2$ on the $z=0$ face (see (1.2)), the above expression equals $\boldsymbol{\mathcal{E}}^{\mathrm{div}}(\mathrm{curl}_\tau\,\boldsymbol{v})$. Thus, by mapping, the commutativity property $\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}) = \boldsymbol{\mathcal{E}}^{\mathrm{div}}(\mathrm{curl}_\tau\,\boldsymbol{v})$ holds for all smooth functions on any tetrahedron K. To complete by a density argument, first note that since $\mathcal{D}(\bar{K})$ is dense in $\boldsymbol{H}(\mathbf{curl}, K)$, smooth functions are dense in the space of traces of $\boldsymbol{H}(\mathbf{curl})$. Thus, for any $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}(F_l)$, taking an approximating sequence of smooth functions $\boldsymbol{v}_n$,

$$\begin{aligned}
\|\boldsymbol{\mathcal{E}}^{\mathrm{div}}(\mathrm{curl}_\tau\,\boldsymbol{v}_n)\|_{\boldsymbol{H}(\mathrm{div})} &= \|\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}_n)\|_{\boldsymbol{H}(\mathrm{div})}, && \text{by commutativity for smooth } \boldsymbol{v}_n, \\
&= \|\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}_n)\|_{L^2(K)} \\
&\le C\|\boldsymbol{v}_n\|_{\boldsymbol{X}^{-1/2}(F_l)}, && \text{by [7, Theorem 3.1]},
\end{aligned}$$

we find that the operator $\boldsymbol{\mathcal{E}}^{\text{div}}$ extends continuously to all $\text{curl}_\tau\,\boldsymbol{X}^{-1/2}(F_l)$ and the commutativity property holds there.

Proof of (2): This is a direct consequence of [7, Lemma 3.1] applied to each component of the extension.

Proof of (3): This follows from item (2) and commutativity. Indeed, given any w in $H^{-1/2}(F_l)$, decomposing it by Theorem 2.1 as $w = \text{curl}_\tau\,\boldsymbol{v} + \theta$ with θ in $H^{1/2}(F_l)$ and $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}(F_l)$ we have

$$\begin{aligned}
\|\boldsymbol{\mathcal{E}}_l^{\text{div}} w\|_{\boldsymbol{H}(\text{div})} &= \|\boldsymbol{\mathcal{E}}_l^{\text{div}}(\text{curl}_\tau\,\boldsymbol{v} + \theta)\|_{\boldsymbol{H}(\text{div})} \\
&= \|\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\text{curl}}\boldsymbol{v}) + \boldsymbol{\mathcal{E}}_l^{\text{div}}\theta\|_{\boldsymbol{H}(\text{div})}, && \text{by item (1),} \\
&\le \|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}(F_l)} + \|\boldsymbol{\mathcal{E}}_l^{\text{div}}\theta\|_{\boldsymbol{H}(\text{div})}, && \text{by [7, Theorem 3.1],} \\
&\le \|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}(F_l)} + \|\theta\|_{H^{1/2}(F_l)}, && \text{by item (2),} \\
&\le C\|w\|_{H^{-1/2}(F_l)}, && \text{by Theorem 2.1.}
\end{aligned}$$

Proof of (4): On the reference element, setting $z = 0$ in (3.1), it is obvious that

$$\text{trc}_n(\boldsymbol{\mathcal{E}}^{\text{div}} w) = w$$

for all smooth functions w. The general statement follows by mapping and density of smooth functions [8] in $\boldsymbol{H}(\text{div})$.

Proof of (5): Since polynomial spaces are invariant under affine transformations, it suffices to prove the statement for the extension on the reference tetrahedron given in (3.1). If w is in $P_p(F_l)$, then $c_0 + c_1 s + c_2 t)w(s(x+z), t(x+z))$ is a polynomial of degree at most p in x, y and z. Clearly, after the integration in (3.1), which is over s and t, we continue to have a polynomial of degree at most p in x, y, z. □

Remark 3.1. We had considered the operator $\boldsymbol{\mathcal{E}}^{\text{div}}$ previously in [6, Appendix B] in order to present a technique of norm estimation using the Fourier transform (the expression there can be brought to (3.1) by a change of variable). In particular, item (3) of Theorem 3.1 can also be proved using such techniques.

4. Face corrections

Consider the normal trace of a function in $\boldsymbol{H}_{0,F_i}(\text{div})$ on the union of F_i and one other face, say $F_l \cup F_i$. The extension of such a trace must have vanishing normal component on F_i. Hence before we solve the full extension problem from ∂K, it is natural to consider the following *two-face problem*: Given w in $H_{0,i}^{-1/2}(F_l)$ on F_l (which by definition is the normal trace on F_l of some function in $\boldsymbol{H}_{0,F_i}(\text{div})$), construct a polynomial extension of w from F_l into K such that the normal trace of the extension vanishes on F_i.

Our approach to solve the two-face problem is by constructing a "face correction" operator. Suppose w is a smooth function defined on the x-y face (denoted by $\hat{F}_3$, or just $\hat{F}$) of the reference tetrahedron $\hat{K}$. We first extend it into $\hat{K}$ by the primary extension to obtain $\boldsymbol{\mathcal{E}}^{\text{div}} w$. We need an extension that has zero normal trace on the y-z face ($\hat{F}_1$). To this end, we develop a face correction operator $\boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\text{div}}$ that does not alter the normal trace on $\hat{F}$ but is such that $\boldsymbol{\mathcal{E}}^{\text{div}} w - \boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\text{div}} w$

has the zero normal trace on $\hat{F}_1$. Define

$$\boldsymbol{\mathcal{E}}^{\rm div}_{\hat{F}_1} w = \frac{1}{x+z}\int_0^1\int_0^{1-t} \begin{pmatrix} 2sz+(1-s)x \\ -t(x+z) \\ (1-3s)z \end{pmatrix} w(s(x+z), t(x+z)+y)\, ds\, dt. \tag{4.1}$$

We will comment on the derivation of this formula later (Remark 4.1).

Before we discuss the properties of this operator, it will be convenient to state its generalization to any tetrahedron K using affine coordinates. To arrive at the generalization, let us rewrite (4.1) using the change of variable $x' = s(x+z), y' = t(x+z)+y$ as

$$\boldsymbol{\mathcal{E}}^{\rm div}_{\hat{F}_1} w = \frac{1}{(x+z)^3}\int_0^{x+z}\int_y^{x+y+z-x'} \Theta\, w(x',y')\, dy'\, dx'.$$

where $\Theta \equiv \Theta(x',y',x,y,z)$ is the vector kernel in (4.1) rewritten in the new variables. Let $(\lambda_0,\lambda_1,\lambda_2,\lambda_3)$ be the affine coordinates of (x,y,z). Observe that the region of integration above can be expressed as $T_3(0,\lambda_0,\lambda_2)$. Let $\widetilde{\lambda}_0,\widetilde{\lambda}_1,\widetilde{\lambda}_2$ denote the affine coordinates of the integration region $T_3(0,\lambda_0,\lambda_2)$ considered with its node enumeration inherited from $\hat{F}_3$. Then, simplifying Θ,

$$\begin{aligned}
\Theta &= -2sz\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - t(x+z)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - (s-1)\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \\
&= 2\widetilde{\lambda}_1\lambda_3(\mathbf{grad}\,\lambda_0\times\mathbf{grad}\,\lambda_2) - \widetilde{\lambda}_2(\lambda_1+\lambda_3)(\mathbf{grad}\,\lambda_3\times\mathbf{grad}\,\lambda_1) \\
&\quad - (\widetilde{\lambda}_1-1)\big(\lambda_1(\mathbf{grad}\,\lambda_2\times\mathbf{grad}\,\lambda_3) - \lambda_3(\mathbf{grad}\,\lambda_1\times\mathbf{grad}\,\lambda_2)\big) \\
&= 2\widetilde{\lambda}_1\lambda_3(\mathbf{grad}\,\lambda_0\times\mathbf{grad}\,\lambda_2) - \widetilde{\lambda}_2(\lambda_1+\lambda_3)(\mathbf{grad}\,\lambda_3\times\mathbf{grad}\,\lambda_1) \\
&\quad + (\widetilde{\lambda}_0+\widetilde{\lambda}_2)\,\mathbf{grad}\,\lambda_2\times(\lambda_1\,\mathbf{grad}\,\lambda_3 - \lambda_3\,\mathbf{grad}\,\lambda_1) \\
&= 2\widetilde{\lambda}_1\lambda_3(\mathbf{grad}\,\lambda_0\times\mathbf{grad}\,\lambda_2) + (\widetilde{\lambda}_0\,\mathbf{grad}\,\lambda_2)\times(\lambda_1\,\mathbf{grad}\,\lambda_3 - \lambda_3\,\mathbf{grad}\,\lambda_1) \\
&\quad - \widetilde{\lambda}_2\big(-\lambda_1(\mathbf{grad}\,\lambda_3\times\mathbf{grad}\,\lambda_0) + \lambda_3(\mathbf{grad}\,\lambda_1\times\mathbf{grad}\,\lambda_0)\big) \\
&= 2\widetilde{\lambda}_1\lambda_3(\mathbf{grad}\,\lambda_0\times\mathbf{grad}\,\lambda_2) - (\widetilde{\lambda}_2\,\mathbf{grad}\,\lambda_0 - \widetilde{\lambda}_0\,\mathbf{grad}\,\lambda_2)\times(\lambda_1\,\mathbf{grad}\,\lambda_3 - \lambda_3\,\mathbf{grad}\,\lambda_1).
\end{aligned}$$

This motivates the definition below of the operator $\boldsymbol{\mathcal{E}}^{\rm div}_{F_i,l}$ generalizing $\boldsymbol{\mathcal{E}}^{\rm div}_{\hat{F}_1}$.

Returning to the two-face problem on $F_i\cup F_l$ of the general tetrahedron K mentioned in the beginning of this section, suppose w is a given smooth function on F_l. Define the *face correction* by

$$\begin{aligned}
\boldsymbol{\mathcal{E}}^{\rm div}_{F_i,l} w = {} & (\mathbf{grad}\,\lambda_k\times\mathbf{grad}\,\lambda_j)\ \frac{\lambda_l}{|F_l|\,(\lambda_i+\lambda_l)^3}\iint_{T_l(0,\lambda_j,\lambda_k)} \widetilde{\lambda}_i(\boldsymbol{s})\, w(\boldsymbol{s})\, d\boldsymbol{s} \\
& - \frac{(\lambda_l\,\mathbf{grad}\,\lambda_i - \lambda_i\,\mathbf{grad}\,\lambda_l)}{2|F_l|\,(\lambda_i+\lambda_l)^3}\times\iint_{T_l(0,\lambda_j,\lambda_k)} (\widetilde{\lambda}_j\,\mathbf{grad}\,\lambda_k - \widetilde{\lambda}_k\,\mathbf{grad}\,\lambda_j)\, w(\boldsymbol{s})\, d\boldsymbol{s}
\end{aligned} \tag{4.2}$$

for any indices (i,j,k) that are positively oriented with respect to l (in the sense defined in the previous section). Note that the above expression is antisymmetric under transpositions of any two of indices i,j,k. For negatively oriented (i,j,k), the face correction is defined to be the above expression with opposite sign. From the discussion in the previous paragraph, it is clear that the expression in (4.2) coincides with (4.1) when $K=\hat{K}$, $(i,l)=(1,3)$, and $(j,k)=(0,2)$.

The boundedness of this map is stated in the next lemma. The proof of all lemmas are in Appendix A.

Lemma 4.1. $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{F_i,l}$ *extends to a continuous linear operator from* $H^{1/2}_{0,i}(F_l)$ *into* $\boldsymbol{H}(\mathrm{div})$*.*

With the help of the face correction operator, we can now give an extension operator that solves the two-face problem. It is defined by

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{i,l} w = \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{l} w - \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{F_i,l} w, \tag{4.3}$$

Recall that a similar extension for the $\boldsymbol{H}(\mathbf{curl}\,)$ case, denoted by $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}$, was defined in [7]. There is a commutativity property involving $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}$ and $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{i,l}$. In order to prove it, we will borrow a homotopy operator from differential geometry [1] (typically used in proving the Poincaré lemma), defined by

$$\boldsymbol{K_a} v = (\boldsymbol{x} - \boldsymbol{a})^{\perp} \int_0^1 t\; v(t(\boldsymbol{x}-\boldsymbol{a}) + \boldsymbol{a})\; dt \tag{4.4}$$

where

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\perp} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

Here $v(x,y)$ is a smooth function defined on $\hat{F}$ and $\boldsymbol{a}$ is any point in $\hat{F}$ or $\partial\hat{F}$. The utility of such operators in the context of high order finite elements is already known [5, 9, 11]. In particular, it is well known that the identity

$$\mathrm{curl}_{\tau}(\boldsymbol{K_a} w) = w \tag{4.5}$$

holds for smooth functions w, and by density for a larger class of functions [9]. In other words, $\boldsymbol{K_a}$ is a *right inverse* of curl_τ.

Remark 4.1. In fact we used $\boldsymbol{K_a}$ in the very derivation of the expression for the face correction (4.1). In the $\boldsymbol{H}(\mathbf{curl}\,)$ case [7], we were able to derive the face correction $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1}$ motivated by the commutativity property

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1}(\mathbf{grad}_\tau\, u) = \mathbf{grad}(\mathcal{E}^{\mathrm{grad}}_{\hat{F}_1} u).$$

In fact, we could guess the form of $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1}$ by just computing the right hand side and expressing it terms of $\mathbf{grad}_\tau\, u$ alone. For the $\boldsymbol{H}(\mathrm{div})$ case, we are again motivated by the target commutativity property

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{\hat{F}_1}(\mathrm{curl}_\tau\, \boldsymbol{v}) = \mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{v}). \tag{4.6}$$

However, our attempts at a similar elementary approach in the $\boldsymbol{H}(\mathrm{div})$ case succumbed to the savagery of the calculations required, hence the entrance of $\boldsymbol{K_a}$. Observe that if (4.6) holds, then by (4.5), we must have

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{\hat{F}_1} w = \mathbf{curl}\,\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{K_a} w.$$

By simplifying the right hand side, we get an expression for $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{\hat{F}_1} w$. These simplifications are tedious and we do not display them here, but some of them reappear disguised in the proof of the commutativity in the next proposition.

The next proposition collects and proves all the properties of $\boldsymbol{\mathcal{E}}_{i,l}^{\mathrm{div}}$ that we shall need. We will use the following lemma in the proof of the proposition. The proof of the lemma is in Appendix A.

Lemma 4.2. *Suppose $f_{xyz}(s,t)$ is a smooth function of $s,t,x,y,$ and z that is homogeneous of degree minus one, i.e.,*

$$f_{xyz}(\frac{s}{r},\frac{t}{r}) = \frac{1}{r} f_{xyz}(s,t).$$

Then for all smooth functions $w(s,t)$, we have the following two identities:

$$\int_0^1 \iint_{\hat{F}} f_{xyz}(s,t)\, w(rs(x+z), rt(x+z)+y)\; r\; ds\, dt\, dr = \iint_{\hat{F}} \frac{1-s-t}{s+t} f_{xyz}(s,t)\; w(s(x+z), t(x+z)+y)\; ds\, dt, \tag{4.7}$$

$$\int_0^1 \iint_{\hat{F}} f_{xyz}(s,t)\, w(rs(x+y+z), rt(x+y+z))\; r\; ds\, dt\, dr = \iint_{\hat{F}} \frac{1-s-t}{s+t} f_{xyz}(s,t)\; w(s(x+y+z), t(x+y+z))\; ds\, dt. \tag{4.8}$$

Proposition 4.1. *The following statements hold for $\boldsymbol{\mathcal{E}}_{i,l}^{\mathrm{div}}$:*

(1) Commutativity: $\boldsymbol{\mathcal{E}}_{i,l}^{\mathrm{div}} \operatorname{curl}_\tau \boldsymbol{v} = \mathbf{curl}\,(\boldsymbol{\mathcal{E}}_{i,l}^{\mathrm{curl}} \boldsymbol{v})$ *for all* $\boldsymbol{v} \in \boldsymbol{X}_{0,i}^{-1/2}(F_l)$.

(2) Continuity: $\boldsymbol{\mathcal{E}}_{i,l}^{\mathrm{div}}$ *is a continuous operator from* $H_{0,i}^{-1/2}(F_l)$ *into* $\boldsymbol{H}(\mathrm{div})$.

(3) Extension property: *For all $\boldsymbol{v}$ in $\boldsymbol{X}_{0,i}^{-1/2}(F_l)$,*

$$R_i \operatorname{trc}_n(\boldsymbol{\mathcal{E}}_{i,l}^{\mathrm{div}} w) = 0 \quad \textit{and} \quad R_l \operatorname{trc}_n(\boldsymbol{\mathcal{E}}_{i,l}^{\mathrm{div}} w) = w,$$

where R_ℓ is the restriction to F_ℓ defined previously (2.13)–(2.15).

(4) Polynomial preservation: *If w is in $P_p(F_l)$, then $\boldsymbol{\mathcal{E}}_{i,l}^{\mathrm{div}} w$ is in $\boldsymbol{P}_p(K)$.*

Proof. Proof of (1): Because of the commutativity property for the primary extension established in Theorem 3.1, it suffices to prove that

$$\boldsymbol{\mathcal{E}}_{F_i,l}^{\mathrm{div}}(\operatorname{curl}_\tau \boldsymbol{v}) = \mathbf{curl}\,(\boldsymbol{\mathcal{E}}_{F_i,l}^{\mathrm{curl}} \boldsymbol{v}), \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{X}_{0,i}^{-1/2}(F_l), \tag{4.9}$$

where $\boldsymbol{\mathcal{E}}_{F_i,l}^{\mathrm{curl}}$ is the face correction defined in [7]. Let $\boldsymbol{v}$ be in $\boldsymbol{X}_{0,i}^{-1/2}(F_l)$. Using the stable trace decomposition of [7], there is a φ in $H_{0,i}^{1/2}(F_l)$ and $\boldsymbol{\psi}$ in $\boldsymbol{H}_{0,i}^{1/2}(F_l)$ such that

$$\boldsymbol{v} = \mathbf{grad}_\tau\, \varphi + \boldsymbol{\psi}. \tag{4.10}$$

Recall that we have proved in [7] that

$$\boldsymbol{\mathcal{E}}_{F_i,l}^{\mathrm{curl}}(\mathbf{grad}_\tau\, \varphi) = \mathbf{grad}(\mathcal{E}_{F_i,l}^{\mathrm{grad}} \varphi), \qquad \text{for all } \varphi \in H_{0,i}^{1/2}(F_l). \tag{4.11}$$

Hence, substituting (4.10) into (4.9), we find that the proof of (4.9) will be finished if we prove that

$$\boldsymbol{\mathcal{E}}_{F_i,l}^{\mathrm{div}}(\operatorname{curl}_\tau \boldsymbol{\psi}) = \mathbf{curl}\,(\boldsymbol{\mathcal{E}}_{F_i,l}^{\mathrm{curl}} \boldsymbol{\psi}), \qquad \text{for all } \boldsymbol{\psi} \in \boldsymbol{H}_{0,i}^{1/2}(F_l). \tag{4.12}$$

In fact, by density, we only need to prove the above identity for all infinitely differentiable $\boldsymbol{\psi}$ on F_l that vanish on a neighborhood of the edge E_{jk} connecting vertices $\boldsymbol{a}_j$ and $\boldsymbol{a}_k$.

Moving to the reference tetrahedron $\hat{K}$, let $\boldsymbol{\psi}$ be a smooth function on $\hat{F}$ that vanishes in a neighborhood of the y-axis. Then $w = \mathrm{curl}_\tau \boldsymbol{\psi}$ also vanishes in the same neighborhood. We need to prove that

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{\hat{F}_1} w = \mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{\psi}), \tag{4.13}$$

where $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1}$ is the face correction on $\hat{K}$ given in [7], namely

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}} \boldsymbol{v} = \int_0^1 \int_0^{1-t} \begin{pmatrix} (3s-1)z & 3zt \\ 0 & 2z \\ 2zs + x(1-s) & 2zt - xt \end{pmatrix} \frac{\boldsymbol{v}(s(x+z), y + t(x+z))}{x+z}\, ds\, dt. \tag{4.14}$$

We first make the observation that in order to prove (4.13) it is enough to prove that

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{\hat{F}_1} w = \mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{K}_{\boldsymbol{a}} w) \tag{4.15}$$

holds for all points $\boldsymbol{a} = (0, \beta, 0)$ on the y-axis with $0 \le \beta \le 1$. This is because

$$\mathrm{curl}_\tau(\boldsymbol{\psi} - \boldsymbol{K}_{\boldsymbol{a}} w) = \mathrm{curl}_\tau \boldsymbol{\psi} - \mathrm{curl}_\tau \boldsymbol{K}_{\boldsymbol{a}} w = 0$$

by (4.5), which implies that there is a ϕ such that $\mathbf{grad}_\tau \phi = \boldsymbol{\psi} - \boldsymbol{K}_{\boldsymbol{a}} w$. Moreover, since $\boldsymbol{a}$ is on the y-axis, from the definition of $\boldsymbol{K}_{\boldsymbol{a}}$ in (4.4), it is clear that the tangential component $\boldsymbol{\psi} - \boldsymbol{K}_{\boldsymbol{a}} w$ vanishes on the y-axis. Hence ϕ can be chosen such that it vanishes on the y-axis. Consequently, once we have proven (4.15), we have

$$\begin{aligned} \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{\hat{F}_1} \mathrm{curl}_\tau \boldsymbol{\psi} &= \mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{K}_{\boldsymbol{a}} w) \\ &= \mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} (\boldsymbol{\psi} - \mathbf{grad}_\tau \phi)) \\ &= \mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{\psi}) - \mathbf{curl}\,(\mathbf{grad}\, \mathcal{E}^{\mathrm{grad}}_{\hat{F}_1} \phi) \quad \text{by (4.11)} \\ &= \mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{\psi}), \end{aligned}$$

and (4.13) follows.

Therefore, let us now prove that (4.15) holds. Setting $\boldsymbol{a} = (\alpha, \beta)$ in the x-y plane and calculating using (4.14), we have

$$\begin{aligned} \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}} \boldsymbol{K}_{(\alpha,\beta)} w = \frac{1}{x+z} \iint_{\hat{F}} \begin{pmatrix} (3s-1)z & 3zt \\ 0 & 2z \\ 2zs + x(1-s) & 2zt - xt \end{pmatrix} \int_0^1 \begin{pmatrix} y + t(x+z) - \beta \\ -s(x+z) + \alpha \end{pmatrix} r \\ w(r(s(x+z) - \alpha) + \alpha, r(y + t(x+z) - \beta) + \beta)\, dr\, ds\, dt. \end{aligned}$$

Putting $\alpha = 0$ and simplifying, we have

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}} \boldsymbol{K}_{(0,\beta)} w = \frac{1}{x+z} \iint_{\hat{F}} \int_0^1 \big((y-\beta)\boldsymbol{p} + (x+z)\boldsymbol{q}\big)\, \widetilde{w}_\beta(x, y, z, r, s, t)\, r \;\, dr\, ds\, dt, \tag{4.16}$$

where $\widetilde{w}_\beta(x, y, z, r, s, t) = w(rs(x+z), r(y + t(x+z) - \beta) + \beta)$,

$$\boldsymbol{p} = \begin{pmatrix} (3s-1)z \\ 0 \\ x(1-s) + 2zs \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} -zt \\ -2sz \\ xt \end{pmatrix}.$$

Next, we must compute the curl of the right hand side of (4.16):

$$\mathbf{curl}\,\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat F}\boldsymbol{K}_{(0,\beta)}w = \iint\limits_{\hat F}\int_0^1 \Bigg((y-\beta)\Big(\frac{(\mathbf{curl}\,\boldsymbol{p})\widetilde{w}_\beta + \boldsymbol{p}\times\mathbf{grad}(\widetilde{w}_\beta)}{x+z} - \boldsymbol{p}\times\widetilde{w}_\beta\,\mathbf{grad}(\frac{1}{x+z})\Big)$$

$$+\frac{\widetilde{w}_\beta}{x+z}\begin{pmatrix} x(1+s)+4zs\\ -zt(x+z)\\ -z(3s-1)\end{pmatrix} - \boldsymbol{q}\times\mathbf{grad}(\widetilde{w}_\beta)\Bigg)\, r\,dr\,ds\,dt, \tag{4.17}$$

where the gradient is with respect to x, y, and z. This gradient can be expressed alternately (by chain rule) in terms of s and t derivatives as

$$\mathbf{grad}\,\widetilde{w}_\beta = \frac{1}{x+z}\begin{pmatrix} s & t\\ 0 & 1\\ s & t\end{pmatrix}\begin{pmatrix}\partial_s w(rs(x+z), r(y+t(x+z)-\beta)+\beta)\\ \partial_t w(rs(x+z), r(y+t(x+z)-\beta)+\beta)\end{pmatrix}. \tag{4.18}$$

We substitute (4.18) in (4.17). We do not display the resulting lengthy expression, but let us denote it by $\boldsymbol{f}(x,y,z,\beta)$, i.e.,

$$\boldsymbol{f}(x,y,z,\beta) \equiv [\mathbf{curl}\,\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat F}\boldsymbol{K}_{(0,\beta)}w](x,y,z).$$

Before proceeding with further simplifications of $\boldsymbol{f}$, let us make an observation. If $\beta' \neq \beta$ is another number in $[0,1]$, then

$$\boldsymbol{f}(x,y,z,\beta) = \boldsymbol{f}(x,y,z,\beta'). \tag{4.19}$$

This is because by (4.5), $\mathrm{curl}_\tau(\boldsymbol{K}_{(0,\beta)}w - \boldsymbol{K}_{(0,\beta')}w) = 0$, so we know that $\boldsymbol{K}_{(0,\beta)}w - \boldsymbol{K}_{(0,\beta')}w = \mathbf{grad}_\tau\,\phi'$ for some ϕ' that vanishes along the y-axis. Then, a consequence of (4.11) is that

$$\mathbf{curl}\left(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat F_1}(\boldsymbol{K}_{(0,\beta)}w - \boldsymbol{K}_{(0,\beta')}w)\right) = \mathbf{curl}\left(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat F_1}\mathbf{grad}_\tau\,\phi'\right) = \mathbf{curl}\,(\mathbf{grad}\,\mathcal{E}^{\mathrm{grad}}_{\hat F_1}\phi) = 0,$$

or in other words, (4.19) holds.

By virtue of (4.19), in order to prove that (4.15) holds for all $\boldsymbol{a} = (0,\beta)$, it suffices to show that (4.15) holds with some choice of β that makes simplifications convenient. We will select $\beta = y$ (carefully noting that we must substitute $\beta = y$ only after the derivatives in the definition of $\boldsymbol{f}$ have been computed). Then, a number of terms in (4.17) with $y-\beta$ as a factor vanish. After some simplifications, setting $\widetilde{w} \equiv (\widetilde{w}_\beta)|_{\beta=y} \equiv w(rs(x+z), rt(x+z)+y)$, we have

$$\boldsymbol{f} = \iint\limits_{\hat F}\int_0^1 \frac{r\widetilde{w}}{x+z}\begin{pmatrix} x(1+s)+4zs\\ -zt(x+z)\\ -z(3s-1)\end{pmatrix} - \begin{pmatrix} -2zs^2 & -(2zs+x)t\\ (x+z)st & (x+z)t^2\\ 2zs^2 & (2s-1)zt\end{pmatrix}\frac{r(\mathbf{grad}_{st}\,\widetilde{w})}{x+z}\,dr\,ds\,dt.$$

The last term above, when integrated by parts on $\hat F$, equals

$$\int_0^1\iint\limits_{\hat F}\frac{-r\widetilde{w}}{x+z}\begin{pmatrix}-6zs-x\\ 3t(x+z)\\ 6zs-z\end{pmatrix}dr\,ds\,dt + \int_0^1\int_{\partial\hat F}\frac{r\widetilde{w}}{x+z}\begin{pmatrix} -2zs^2 & -(2zs+x)t\\ (x+z)st & (x+z)t^2\\ 2zs^2 & (2s-1)zt\end{pmatrix}\boldsymbol{\nu}_{\hat F}\,d\mu\,dr,$$

where $\boldsymbol{\nu}_{\hat F}$ is the outward unit normal on the boundary of $\hat F$ in the x-y plane. Let us denote the last term involving the integral over the boundary of $\hat F$ by $\boldsymbol{g}$. This term can be split into three terms each involving an integral over one of the three edges of $\hat F$. But only the contribution from the hypotenuse $\hat{H}$ survives. This surviving integral can be transformed via a change of variable as follows:

$$\begin{aligned}
\boldsymbol{g} &= \int_0^1 \int_{\hat{H}} \frac{r\widetilde{w}}{x+z} \begin{pmatrix} -2zs^2 & -(2zs+x)t \\ (x+z)st & (x+z)t^2 \\ 2zs^2 & (2s-1)zt \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \, d\mu \, dr, \\
&= \int_0^1 \int_0^1 \begin{pmatrix} -2z\mu - x(1-\mu) \\ (x+z)(1-\mu) \\ 2z\mu - z(1-\mu) \end{pmatrix} \widetilde{w}\; r \, d\mu \, dr \\
&= \int_0^1 \int_0^r \begin{pmatrix} -2zs' - x(r-s') \\ (x+z)(r-s') \\ 2zs' - z(r-s') \end{pmatrix} \widetilde{w} \, \frac{ds'}{r} \, dr \qquad\qquad (s' = r\mu) \\
&= \int_0^1 \int_0^{1-t} \frac{1}{(s+t)(x+z)} \begin{pmatrix} -2sz - tx \\ t(x+z) \\ (2s-t)z \end{pmatrix} \widetilde{w} \, ds \, dt \qquad\qquad (s = s', t = r - s').
\end{aligned}$$

Substituting this back into the expression for $\boldsymbol{f}$, we notice that it now only remains to simplify the triple integrals to double integrals. This is achieved through Lemma 4.2.

Applying Lemma 4.2 to each component of the triple integrals, we find that the simplification of $\boldsymbol{f}(x, y, z, \beta)$ with $\beta = y$ now reads as

$$\begin{aligned}
\boldsymbol{f} = \frac{-1}{x+z} \int_0^1 \int_0^{1-t} \qquad & \frac{1-s-t}{s+t} \begin{pmatrix} s(x-2z) \\ t(x+z) \\ 3zs \end{pmatrix} w(s(x+z), t(x+z)+y) \\
+ \quad & \frac{1}{s+t} \begin{pmatrix} -2sz - tx \\ t(x+z) \\ (2s-t)z \end{pmatrix} w(s(x+z), t(x+z)+y) \, ds \, dt.
\end{aligned}$$

Combining the terms above, we obtain the expression for the face correction in (4.1), i.e., $\boldsymbol{f} = \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{\hat{F}_1} w$. Summarizing, we have thus proved (4.15), from which (4.13) follows, which in turn proves the required commutativity property.

Proof of (2): To prove the continuity, first decompose any w in $H^{-1/2}_{0,i}(F_l)$ using Theorem 2.1 as

$$w = \mathrm{curl}_\tau \, \boldsymbol{v} + \theta$$

for some $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}_{0,i}(F_l)$ and θ in $H^{1/2}_{0,i}(F_l)$. Then,

$$\begin{aligned}
\|\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{i,l} w\|_{\boldsymbol{H}(\mathrm{div})} &= \|\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l} \boldsymbol{v}) + \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{i,l} \theta\|_{\boldsymbol{H}(\mathrm{div})}, && \text{by commutativity,} \\
&\le C\Big(\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}_{0,i}(F_l)} + \|\theta\|_{H^{1/2}_{0,i}(F_l)} \Big), && \text{by [7, Prop. 4.1] and Lemma 4.1,} \\
&\le C \|w\|_{H^{-1/2}_{0,i}(F_l)}, && \text{by Theorem 2.1.}
\end{aligned}$$

Proof of (3): First, consider the expression for the primary extension in affine coordinates, namely (3.2). When calculating its normal component on F_i, since $\boldsymbol{n}_i$ is parallel to $\mathbf{grad}\, \lambda_i$, among the three summands involving cyclic permutations of (i,j,k), only one survives. Furthermore, since $\lambda_i = 0$ on face F_i we have

$$\operatorname{trc}_n(\boldsymbol{\mathcal{E}}_l^{\mathrm{div}})\big|_{F_i} = \frac{1}{|F_l|\,\lambda_l^2} \iint\limits_{T_l(0,\lambda_j,\lambda_k)} -\boldsymbol{n}_i \cdot \widetilde{\lambda}_i\,(\mathbf{grad}\,\lambda_j \times \mathbf{grad}\,\lambda_k)\,w(\boldsymbol{s})\,d\boldsymbol{s}. \tag{4.20}$$

Next, observe that in the expression for the face correction (4.2), setting $\lambda_i = 0$ we have

$$\operatorname{trc}_n(\boldsymbol{\mathcal{E}}_{F_i,l}^{\mathrm{div}} w)\big|_{F_i} = \boldsymbol{n}_i \cdot (\mathbf{grad}\,\lambda_k \times \mathbf{grad}\,\lambda_j)\ \frac{1}{|F_l|\,\lambda_l^2} \iint\limits_{T_l(0,\lambda_j,\lambda_k)} \widetilde{\lambda}_i(\boldsymbol{s})\,w(\boldsymbol{s})\,d\boldsymbol{s}. \tag{4.21}$$

Note that the term involving the Whitney form $\lambda_i\,\mathbf{grad}\,\lambda_l - \lambda_l\,\mathbf{grad}\,\lambda_i$ in (4.2) does not contribute above as its cross product with $\boldsymbol{n}_i$ is zero on F_i. Subtracting (4.21) from (4.20) we obtain

$$R_i\,\operatorname{trc}_n(\boldsymbol{\mathcal{E}}_{i,l}^{\mathrm{div}} w) = 0.$$

The second assertion on the normal trace on F_l is also easy to see using the affine coordinate expression (4.2). On the face F_l, since $\lambda_l = 0$, the first term in (4.2) drops off. The second also vanishes when calculating normal trace as $\boldsymbol{n}_l \times (\lambda_i\,\mathbf{grad}\,\lambda_l - \lambda_l\,\mathbf{grad}\,\lambda_i) = 0$ on F_l. In other words,

$$\operatorname{trc}_n(\boldsymbol{\mathcal{E}}_{F_i,l}^{\mathrm{div}})\big|_{F_l} = 0.$$

Hence, by Theorem 3.1(4), we have $R_l\,\operatorname{trc}_n(\boldsymbol{\mathcal{E}}_{i,l}^{\mathrm{div}} w) = w$.

Proof of (4): It is enough to prove the polynomial preservation property on the reference element. Hence let $w(x,y)$ be in $P_p(\hat{F})$. We already know that $\boldsymbol{\mathcal{E}}^{\mathrm{div}} w$ is in $P_p(Kr)$ by Theorem 3.1. Hence we only need to prove that $\boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\mathrm{div}} w$ is in $P_p(\hat{K})$. We proceed considering three cases:

Case of constants: If $w(x,y)$ is a constant κ, then by (4.1),

$$\boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\mathrm{div}} w = \frac{\kappa}{x+z} \int_0^1 \int_0^{1-t} \begin{pmatrix} 2sz + (1-s)x \\ -t(x+z) \\ (1-3s)z \end{pmatrix} ds\,dt = \frac{\kappa}{x+z} \begin{pmatrix} (x+z)/3 \\ -(x+z)/6 \\ 0 \end{pmatrix},$$

which is a constant vector.

Case of one variable dependence: Suppose $p > 0$ and $w(x,y) = q_p(y)$ for some polynomial q_p in one variable y. Writing

$$q_p(y) = \sum_{n=0}^{p} c_n y^n,$$

and using the binomial expansion, we find that there is a polynomial $r_{p-1}(y,z)$ of degree at most $p-1$ such that

$$\begin{aligned} q_p(y + t\eta) = \sum_{n=0}^{p} c_n (y+tz)^n &= \sum_{n=0}^{p} c_n \big(y^n + (n-1) y^{n-1} (t\eta) + \dots\big) \\ &= q_p(y) +\ (t\eta)\ r_{p-1}(y, t\eta), \end{aligned}$$

or, in other words,

$$w(s(x+z), y + t(x+z)) = q_p(y) +\ t(x+z)\ r_{p-1}(y, t(x+z)).$$

Observing that $q_p(y)$ is a quantity independent of the integration variables s and t, we find that

$$\boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\mathrm{div}} w = q_p(y)\boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\mathrm{div}}(1) \;+\; \frac{1}{x+z}\int_0^1\int_0^{1-t}\begin{pmatrix} 2sz+(1-s)x \\ -t(x+z) \\ (1-3s)z \end{pmatrix} t(x+z)\; r_{p-1}(y, t(x+z)).$$

Canceling the common factor $x+z$ from the last integral, we conclude that it gives a polynomial of degree at most p in x, y, and z. The first term on the right hand side is a constant by the previous case. Hence $\boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\mathrm{div}} w$ is in $P_p(\hat{K})$.

The general case: Any w in $P_p(\hat{F})$ can be rewritten as

$$w(x,y) = q_p(y) + x v_{p-1}(x,y)$$

for some polynomial $q_p(y)$ of degree at most p in y and some $v_{p-1} \in P_{p-1}(\hat{F})$. Then

$$\boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\mathrm{div}} w = \boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\mathrm{div}}(q_p(y)) + \boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\mathrm{div}}(x v_{p-1}(x,y))$$

Referring to (4.1), we find that the last term is an integral whose integrand has a factor $s(x+z)$ which cancels with the denominator $x+z$ in (4.1). Hence after integration with respect to s and t, it gives a polynomial of degree at most p in x, y and z. The first term, namely $\boldsymbol{\mathcal{E}}_{\hat{F}_1}^{\mathrm{div}} q_p$, is also in $P_p(\hat{K})$ because of the previous case. □

5. Edge corrections

In the previous section we saw how to solve the two-face problem. This section is devoted to constructing an extension operator that solves the *three-face problem*, which is the next intermediate step towards solving the total extension problem. To describe the three-face problem, consider a polynomial $\boldsymbol{r}$ on K whose normal trace $w \equiv \mathrm{trc}_n(\boldsymbol{r})$ is zero on two faces $F_i \cup F_j$. Given the values of w on a third face F_l, the three-face problem is to find a extension $(\boldsymbol{\mathcal{E}}_{ij,l}^{\mathrm{div}} w)$ of w into K which is a polynomial of degree not more than $\boldsymbol{r}$ and whose normal trace coincides with w on $F_i \cup F_j \cup F_l$. Of course, the extension operator must also extend continuously to the appropriate infinite dimensional Sobolev space.

We will solve the three-face problem using an *edge correction* operator. On the reference element $\hat{K}$, the edge correction operator is

$$\boldsymbol{\mathcal{E}}_{\hat{E}}^{\mathrm{div}} w = \int_0^1\int_0^{1-t}\begin{pmatrix} 2x(s-1)+3xt-ys-zs \\ -xt+2y(t-1)+3ys-zt \\ z(3(s+t)-2) \end{pmatrix}\frac{w(s(x+y+z), t(x+y+z))}{x+y+z}\,ds\,dt. \tag{5.1}$$

Here $\hat{E} \equiv \hat{E}_{03}$ denotes the edge of $\hat{K}$ along the z-axis. This expression is derived using $\boldsymbol{K_a}$ and the target commutativity property $\mathbf{curl}\,(\boldsymbol{\mathcal{E}}_{\hat{E}}^{\mathrm{curl}}\boldsymbol{v}) = \boldsymbol{\mathcal{E}}_{\hat{E}}^{\mathrm{div}}(\mathrm{curl}_\tau\,\boldsymbol{v})$ as motivation (where $\boldsymbol{\mathcal{E}}_{\hat{E}}^{\mathrm{curl}}$ is the $\boldsymbol{H}(\mathbf{curl}\,)$-edge correction operator defined in [7]).

As in previous sections, we now generalize this edge correction to an operator on any tetrahedron K using affine coordinates. To rewrite (5.1) using the barycentric coordinates λ_i of (x,y,z) with respect to $\hat{K}$, we first observe that the region of integration can be transformed into $T_l(0,0,\lambda_0)$ by the variable change $x' = s(x+y+z),\ y' = t(x+y+z)$. Furthermore, denoting the kernel by Φ, we can rewrite it using the barycentric coordinates $\widetilde{\lambda}_0, \widetilde{\lambda}_1, \widetilde{\lambda}_2$ of the subtriangle

$T_l(0,0,\lambda_i)$ as

$$\begin{aligned}\Phi &\equiv \begin{pmatrix} 2x(s-1)+3xt-ys-zs \\ -xt+2y(t-1)+3ys-zt \\ z(3(s+t)-2) \end{pmatrix} \\ &= \big(2\lambda_1(1-\widetilde{\lambda}_1)+(\lambda_2+\lambda_3)\widetilde{\lambda}_1-3\lambda_1\widetilde{\lambda}_2\big)\,\mathbf{grad}\,\lambda_3\times\mathbf{grad}\,\lambda_2 \\ &+ \big((\lambda_1+\lambda_3)\widetilde{\lambda}_2+(2\lambda_2(1-\widetilde{\lambda}_2)-3\lambda_2\widetilde{\lambda}_1)\big)\,\mathbf{grad}\,\lambda_1\times\mathbf{grad}\,\lambda_3 \\ &+ \lambda_3(2-3(\widetilde{\lambda}_1+\widetilde{\lambda}_2))\,\mathbf{grad}\,\lambda_2\times\mathbf{grad}\,\lambda_1.\end{aligned}$$

Either by manipulating the above expression in affine coordinates, or by direct verification, we can show that

$$\begin{aligned}\Phi &= (\lambda_2\,\mathbf{grad}\,\lambda_1-\lambda_1\,\mathbf{grad}\,\lambda_2)\times(\widetilde{\lambda}_0\,\mathbf{grad}\,\lambda_3) \\ &\quad-(\lambda_3\,\mathbf{grad}\,\lambda_1-\lambda_1\,\mathbf{grad}\,\lambda_3)\times(\widetilde{\lambda}_2\,\mathbf{grad}\,\lambda_0+\widetilde{\lambda}_0\,\mathbf{grad}\,\lambda_2) \\ &\quad+(\lambda_3\,\mathbf{grad}\,\lambda_2-\lambda_2\,\mathbf{grad}\,\lambda_3)\times(\widetilde{\lambda}_1\,\mathbf{grad}\,\lambda_0+\widetilde{\lambda}_0\,\mathbf{grad}\,\lambda_1).\end{aligned}$$

As a result, we obtain the following general *edge correction operator* for the edge E_{il} connecting vertices $\boldsymbol{a}_i$ and $\boldsymbol{a}_l$ of a general tetrahedron K when the indices (i,j,k) are, as before, positively oriented with respect to l:

$$\begin{aligned}\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{E_{il},l}w &= \frac{\lambda_k\,\mathbf{grad}\,\lambda_j-\lambda_j\,\mathbf{grad}\,\lambda_k}{2|F_l|(1-\lambda_i)^3}\times\iint_{T_l(0,0,\lambda_i)}(\widetilde{\lambda}_i\,\mathbf{grad}\,\lambda_l)\,w(\boldsymbol{s})\,d\boldsymbol{s} \\ &-\frac{\lambda_l\,\mathbf{grad}\,\lambda_j-\lambda_j\,\mathbf{grad}\,\lambda_l}{2|F_l|(1-\lambda_i)^3}\times\iint_{T_l(0,0,\lambda_i)}(\widetilde{\lambda}_k\,\mathbf{grad}\,\lambda_i+\widetilde{\lambda}_i\,\mathbf{grad}\,\lambda_k)\,w(\boldsymbol{s})\,d\boldsymbol{s} \\ &+\frac{\lambda_l\,\mathbf{grad}\,\lambda_k-\lambda_k\,\mathbf{grad}\,\lambda_l}{2|F_l|(1-\lambda_i)^3}\times\iint_{T_l(0,0,\lambda_i)}(\widetilde{\lambda}_j\,\mathbf{grad}\,\lambda_i+\widetilde{\lambda}_i\,\mathbf{grad}\,\lambda_j)\,w(\boldsymbol{s})\,d\boldsymbol{s}.\end{aligned}\tag{5.2}$$

For negatively oriented indices, the correction operator is defined with the sign reversed. With this correction operator, we can provide the solution for the three-face problem through the following extension operator:

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ij,l}=\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{l}-\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{F_i,l}-\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{F_j,l}+\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{E_{kl},l}.\tag{5.3}$$

As in the case of the face correction, to analyze the continuity of this operator, we must first establish a continuity property in a positive order Sobolev space, as stated in the next lemma (proved in Appendix A).

Lemma 5.1. $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ij,l}$ *is a continuous operator from* $\boldsymbol{H}^{1/2}_{0,ij}(F_l)$ *into* $\boldsymbol{H}(\mathrm{div})$.

That $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ij,l}$ indeed solves the three-face problem is the result of the next proposition.

Proposition 5.1. *The following holds for the above defined* $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ij,l}$*:*

(1) Commutativity: $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ij,l}(\mathrm{curl}_\tau\,\boldsymbol{v})=\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ij,l}\boldsymbol{v})$ *for all* $\boldsymbol{v}\in\boldsymbol{X}^{-1/2}_{0,ij}(F_l)$.

(2) Continuity: $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ij,l}$ *extends to a continuous operator from* $H^{-1/2}_{0,ij}(F_l)$ *into* $\boldsymbol{H}(\mathrm{div})$.

(3) Extension property: *For all* $w \in H^{-1/2}_{0,ij}(F_l)$,

$$R_i \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ij,l} w) = 0, \qquad R_j \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ij,l} w) = 0, \qquad R_l \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ij,l} w) = w.$$

(4) Polynomial preservation: *If w is in $P_p(F_l)$, then $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ji,l} w$ is in $\boldsymbol{P}_p(K)$.*

Proof. Proof of (1): As in the beginning of the proof of Proposition 4.1(1), we first use (*i*) the commutativity properties of the primary extension (Theorem 3.1(1)) and the face correction (Proposition 4.1(1)), (*ii*) the commutativity property

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_i,l}(\mathbf{grad}_\tau\, \varphi) = \mathbf{grad}(\mathcal{E}^{\mathrm{grad}}_{F_i,l}\varphi) \qquad \text{for all } \varphi \in H^{1/2}_{0,ij}(F_l),$$

proven in [7], and (*iii*) the stable decomposition

$$\boldsymbol{v} = \mathbf{grad}_\tau\, \varphi + \boldsymbol{\psi}, \qquad \text{with } \varphi \in H^{1/2}_{0,ij}(F_l), \quad \boldsymbol{\psi} \in \boldsymbol{H}^{1/2}_{0,ij}(F_l),$$

also proved in [7], to conclude that it is enough to prove that $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{F_i,l}$ satisfies

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{E_{kl},l}(\mathrm{curl}_\tau\, \boldsymbol{\psi}) = \mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{E_{kl},l}\boldsymbol{\psi}) \tag{5.4}$$

for all smooth $\boldsymbol{\psi}$ that vanishes in a neighborhood of the edges E_{ik} and E_{jk}.

We shall again use $\boldsymbol{K_a}$, now with $\boldsymbol{a}$ set to the origin. Moving to the reference tetrahedron, we make the second observation that it is enough to prove that

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{\hat{E}} w = \mathbf{curl}\, \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{E}} \boldsymbol{K}_{(0,0)} w \tag{5.5}$$

for all smooth w that vanishes in a neighborhood of the x and y-axes. That (5.5) implies (5.4) is proved by the same type of argument as in the proof of Proposition 4.1 (see (4.15)) so we do not repeat. Let us now prove (5.5).

Recall the expression for $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{E}}$ from [7]. It can be rewritten as

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{E}}\boldsymbol{v} = \iint_{\hat{F}} \begin{pmatrix} (3s-1)z & 3zt \\ 3zs & z(3t-1) \\ x(1-s) - ys + 2zs & -xt + y(1-t) + 2zt \end{pmatrix} \frac{\boldsymbol{v}(s(x+y+z), t(x+y+z))}{x+y+z}\, ds\, dt.$$

Setting $\boldsymbol{v} = \boldsymbol{K}_{(0,0)} w$ and simplifying, we have

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{E}} \boldsymbol{K}_{(0,0)} w = \iint_{\hat{F}} \int_0^1 \begin{pmatrix} -tz \\ sz \\ xt - ys \end{pmatrix} w(rs(x+y+z), rt(x+y+z))\; r\; dr\, ds\, dt.$$

We must now compute the curl of this expression. Let $\boldsymbol{p} = (-tz, sz, xt - ys)^t$ and $\widetilde{w}(s,t) \equiv w(rs(x+y+z), rt(x+y+z))$. Then using

$$\mathbf{curl}\,(\widetilde{w}\boldsymbol{p}) = \widetilde{w}\; \mathbf{curl}\, \boldsymbol{p} - \boldsymbol{p} \times \mathbf{grad}\, \widetilde{w}$$

and transforming the x, y, z-derivatives to s, t derivatives via

$$\mathbf{grad}\, w(rs(x+y+z), rt(x+y+z)) = \frac{1}{x+y+z} \begin{pmatrix} s & t \\ s & t \\ s & t \end{pmatrix} \mathbf{grad}_{st}\, \widetilde{w}(s,t),$$

(where $\mathbf{grad}_{st}\,\widetilde{w}$ is a column vector with the s and t derivatives of $\widetilde{w}$ as its two components) we find that

$$\begin{aligned}
\mathbf{curl}\,\boldsymbol{\mathcal{E}}_{\hat{E}}^{\mathrm{curl}}\boldsymbol{K}_{(0,0)}w &= \iint\limits_{\hat{F}}\int_0^1 \big(-\boldsymbol{p}\times\mathbf{grad}\,\widetilde{w}+\widetilde{w}\,\mathbf{curl}\,\boldsymbol{p}\big)\,r\;dr\,ds\,dt\\
&= \iint\limits_{\hat{F}}\int_0^1 \begin{pmatrix}(xt-ys-zs)\,s & (xt-ys-zs)\,t\\ (ys-xt-zt)\,s & (ys-xt-zt)\,t\\ z\,(s+t)\,s & z\,(s+t)\,t\end{pmatrix}\frac{r\,\mathbf{grad}_{st}\,\widetilde{w}}{x+y+z}\,dr\,ds\,dt\\
&\quad - \iint\limits_{\hat{F}}\int_0^1 \begin{pmatrix}2s\\2t\\0\end{pmatrix}\widetilde{w}\,r\;ds\,dt\,dr.
\end{aligned}$$

Integrating by parts the first term on the right hand side above,

$$\begin{aligned}
\mathbf{curl}\,\boldsymbol{\mathcal{E}}_{\hat{E}}^{\mathrm{curl}}\boldsymbol{K}_{(0,0)}w &= \int_0^1\int_{\partial\hat{F}} \begin{pmatrix}(xt-ys-zs)\,s & (xt-ys-zs)\,t\\ (ys-xt-zt)\,s & (ys-xt-zt)\,t\\ z\,(s+t)\,s & z\,(s+t)\,t\end{pmatrix}\boldsymbol{\nu}_{\hat{F}}(\mu)\;\frac{r\,\widetilde{w}}{x+y+z}\,d\mu\,dr\\
&\quad + \int_0^1\iint\limits_{\hat{F}} \begin{pmatrix}ys-xt+zs\\ xt-ys+zt\\ -z(s+t)\end{pmatrix}\frac{3r\,\widetilde{w}}{x+y+z}ds\,dt\,dr - \int_0^1\iint\limits_{\hat{F}}\begin{pmatrix}2s\\2t\\0\end{pmatrix}\widetilde{w}\,r\;ds\,dt\,dr.
\end{aligned}$$

In the integral over the boundary $\partial\hat{F}$, only the part along the hypotenuse survives. We then use a variable change and transform this surviving integral as in the proof of Proposition 4.1(1). We also combine the last two integrals into one. Thus,

$$\begin{aligned}
\mathbf{curl}\,\boldsymbol{\mathcal{E}}_{\hat{E}}^{\mathrm{curl}}\boldsymbol{K}_{(0,0)}w &= \int_0^1\iint\limits_{\hat{F}} \begin{pmatrix}-x(3t+2s)+ys+zs\\ xt-y(3s+2t)+zt\\ -3\,z(s+t)\end{pmatrix}\frac{r\,\widetilde{w}}{x+y+z}ds\,dt\,dr\\
&\quad + \iint\limits_{\hat{F}}\begin{pmatrix}xt-ys-zs\\ -xt+ys-zt\\ z(s+t)\end{pmatrix}\frac{w(s(x+y+z),t(x+y+z))}{(s+t)(x+y+z)}ds\,dt
\end{aligned}$$

Finally, we apply Lemma 4.2 to simplify the triple integral into a double integral by homogeneity. The result is

$$\begin{aligned}
&\mathbf{curl}\,\boldsymbol{\mathcal{E}}_{\hat{E}}^{\mathrm{curl}}\boldsymbol{K}_{(0,0)}w \;=\\
&= \iint\limits_{\hat{F}}\big(\frac{1-s-t}{s+t}\big)\begin{pmatrix}-x(3t+2s)+ys+zs\\ xt-y(3s+2t)+zt\\ -3\,z(s+t)\end{pmatrix}\frac{w(s(x+y+z),t(x+y+z))}{x+y+z}ds\,dt\\
&\quad + \iint\limits_{\hat{F}}\frac{1}{s+t}\begin{pmatrix}xt-ys-zs\\ -xt+ys-zt\\ z(s+t)\end{pmatrix}\frac{w(s(x+y+z),t(x+y+z))}{x+y+z}ds\,dt.\\
&= \iint\limits_{\hat{F}}\frac{-1}{s+t}\begin{pmatrix}(s+t)\big(2x(1-s)-3xt+ys+zs\big)\\ (s+t)\big(xt+2y(1-t)-3ys+zt\big)\\ z(s+t)(2-3(s+t))\end{pmatrix}\frac{w(s(x+y+z),t(x+y+z))}{x+y+z}ds\,dt,
\end{aligned}$$

which is the same as $\boldsymbol{\mathcal{E}}^{\text{div}}_{\hat{E}} w$. This proves (5.5) and consequently the required commutativity property.

Proof of (2): The idea is the same as that of the proof of the commutativity of the two-face extension. We decompose any w in $H^{-1/2}_{0,ij}(F_l)$ using Theorem 2.1 as $w = \operatorname{curl}_\tau \boldsymbol{v} + \theta$ for some $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}_{0,ji}(F_l)$ and θ in $H^{1/2}_{0,ij}(F_l)$. Then,

$$\begin{aligned}
\|\boldsymbol{\mathcal{E}}^{\text{div}}_{ij,l} w\|_{\boldsymbol{H}(\text{div})} &= \|\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\text{curl}}_{ij,l}\boldsymbol{v}) + \boldsymbol{\mathcal{E}}^{\text{div}}_{ij,l}\theta\|_{\boldsymbol{H}(\text{div})}, && \text{by commutativity,}\\
&\le C\left(\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}_{0,ij}(F_l)} + \|\theta\|_{H^{1/2}_{0,ij}(F_l)}\right), && \text{by [7, Prop. 5.1] and Lemma 5.1,}\\
&\le C\|w\|_{H^{-1/2}_{0,ij}(F_l)}, && \text{by Theorem 2.1.}
\end{aligned}$$

Proof of (3): To prove that

$$R_i \ \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\text{div}}_{ij,l} w) = 0, \tag{5.6}$$

we consider the expression for the edge correction $\boldsymbol{\mathcal{E}}^{\text{div}}_{E_{kl},l}$. This expression can be obtained from (5.2) either by a cyclic permutation of (i,j,k) (preserving the positive orientation) or by interchanging i and k in (5.2) and then switching signs:

$$\begin{aligned}
\boldsymbol{\mathcal{E}}^{\text{div}}_{E_{kl},l} w = &- \frac{\lambda_i\,\mathbf{grad}\,\lambda_j - \lambda_j\,\mathbf{grad}\,\lambda_i}{2|F_l|(1-\lambda_k)^3} \times \iint\limits_{T_l(0,0,\lambda_k)} (\widetilde{\lambda}_k\,\mathbf{grad}\,\lambda_l)\; w(\boldsymbol{s})\; d\boldsymbol{s}\\
&+ \frac{\lambda_l\,\mathbf{grad}\,\lambda_j - \lambda_j\,\mathbf{grad}\,\lambda_l}{2|F_l|(1-\lambda_k)^3} \times \iint\limits_{T_l(0,0,\lambda_k)} (\widetilde{\lambda}_i\,\mathbf{grad}\,\lambda_k + \widetilde{\lambda}_k\,\mathbf{grad}\,\lambda_i)\; w(\boldsymbol{s})\; d\boldsymbol{s}\\
&- \frac{\lambda_l\,\mathbf{grad}\,\lambda_i - \lambda_i\,\mathbf{grad}\,\lambda_l}{2|F_l|(1-\lambda_k)^3} \times \iint\limits_{T_l(0,0,\lambda_k)} (\widetilde{\lambda}_j\,\mathbf{grad}\,\lambda_k + \widetilde{\lambda}_k\,\mathbf{grad}\,\lambda_j)\; w(\boldsymbol{s})\; d\boldsymbol{s}
\end{aligned} \tag{5.7}$$

Also note that from (4.2), by transposition of i, j followed by a change of sign, we have

$$\begin{aligned}
\boldsymbol{\mathcal{E}}^{\text{div}}_{F_j,l} w = &- \ (\mathbf{grad}\,\lambda_k \times \mathbf{grad}\,\lambda_i)\ \frac{\lambda_l}{|F_l|(\lambda_i+\lambda_l)^3} \iint\limits_{T_l(0,\lambda_i,\lambda_k)} \widetilde{\lambda}_j(\boldsymbol{s})\, w(\boldsymbol{s})\; d\boldsymbol{s}\\
&+ \frac{(\lambda_l\,\mathbf{grad}\,\lambda_j - \lambda_j\,\mathbf{grad}\,\lambda_l)}{2|F_l|(\lambda_j+\lambda_l)^3} \times \iint\limits_{T_l(0,\lambda_i,\lambda_k)} (\widetilde{\lambda}_i\,\mathbf{grad}\,\lambda_k - \widetilde{\lambda}_k\,\mathbf{grad}\,\lambda_i)\, w(\boldsymbol{s})\; d\boldsymbol{s}.
\end{aligned} \tag{5.8}$$

If $\boldsymbol{n}_i$ is the outward unit normal on the face F_i, then since

$$\lambda_i = 0, \quad \lambda_i + \lambda_j = 1 - \lambda_k, \quad \text{and} \quad \boldsymbol{n}_i \times (\lambda_i\,\mathbf{grad}\,\lambda_m - \lambda_m\,\mathbf{grad}\,\lambda_i)\big|_{F_i} = 0,$$

we find from (5.7) and (5.8) that

$$\begin{aligned}
\operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\text{div}}_{E_{kl},l} w)\big|_{F_i} &= \boldsymbol{n}_i \cdot \frac{\lambda_l\,\mathbf{grad}\,\lambda_j - \lambda_j\,\mathbf{grad}\,\lambda_l}{2|F_l|(1-\lambda_k)^3} \times \iint\limits_{T_l(0,0,\lambda_k)} (\widetilde{\lambda}_i\,\mathbf{grad}\,\lambda_k + \widetilde{\lambda}_k\,\mathbf{grad}\,\lambda_i)\; w(\boldsymbol{s})\; d\boldsymbol{s}\\
&= \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\text{div}}_{F_j,l} w)\big|_{F_i}
\end{aligned}$$

This together with the extension properties of the two-face extension operator (Proposition 4.1(3)) proves (5.6).

To prove the next identity, namely $R_j \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\rm div}_{ij,l} w) = 0$, we use the same type of arguments to get

$$\operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\rm div}_{E_{kl},l} w)\big|_{F_j} = -\boldsymbol{n}_j \cdot \frac{\lambda_l \,\mathbf{grad}\, \lambda_i - \lambda_i \,\mathbf{grad}\, \lambda_l}{2|F_l|(1-\lambda_k)^3} \times \iint\limits_{T_l(0,0,\lambda_k)} (\widetilde{\lambda}_j \,\mathbf{grad}\, \lambda_k + \widetilde{\lambda}_k \,\mathbf{grad}\, \lambda_j)\; w(\boldsymbol{s})\; d\boldsymbol{s}$$

$$= \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\rm div}_{F_i,l})\big|_{F_j}, \qquad \text{by (4.2).}$$

The identity follows by using Proposition 4.1(3) again.

To prove the last identity $R_l \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\rm div}_{ij,l} w) = w$, we only need to observe that the face corrections, written out in (5.8) and (4.2), and the edge correction in (5.7) have vanishing normal components on F_l. The identity then follows from the extension property of the primary extension as given in Theorem 3.1(4).

Proof of (4): Any w in $P_p(\hat{F})$ can be expressed as $\kappa + xu_{p-1} + yv_{p-1}$ for some constant κ and some polynomials u_{p-1} and v_{p-1} of degree at most $p-1$,

$$\boldsymbol{\mathcal{E}}^{\rm div}_{\hat{E}} w = \boldsymbol{\mathcal{E}}^{\rm div}_{\hat{E}} \kappa + \boldsymbol{\mathcal{E}}^{\rm div}_{\hat{E}} (xu_{p-1}) + \boldsymbol{\mathcal{E}}^{\rm div}_{\hat{E}} (yv_{p-1}).$$

The last two terms give polynomials because the denominator $x+y+z$ in (5.1) is canceled off by a factor of either $s(x+y+z)$ or $t(x+y+z)$. For the remaining term $\boldsymbol{\mathcal{E}}^{\rm div}_{\hat{E}}\kappa$, we have

$$\begin{aligned} \boldsymbol{\mathcal{E}}^{\rm div}_{\hat{E}} \kappa &= \frac{\kappa}{x+y+z} \int_0^1 \!\! \int_0^{1-t} \begin{pmatrix} 2x(1-s) - 3xt + ys + zs \\ xt + 2y(1-t) - 3ys + zt \\ z(2-3(s+t)) \end{pmatrix} \, ds\, dt \\ &= \frac{\kappa}{x+y+z} \begin{pmatrix} (x+y+z)/6 \\ (x+y+z)/6 \\ 0 \end{pmatrix}, \end{aligned}$$

which is a constant vector. Hence the result follows. □

6. Vertex corrections

In this section, as the last intermediate step towards solving the tetrahedral $\boldsymbol{H}(\mathrm{div})$ polynomial extension problem, we consider the *four-face problem*: Given a polynomial w of zero mean on any face F_l, find an extending polynomial whose normal traces on all other faces are zero. As in the previous cases, the extension should not increase the degree and must be extendable continuously to the appropriate Sobolev space of traces, namely the trace space $H^{-1/2}_{0,ijk}(F_l)$. Roughly speaking, a function w in $H^{-1/2}_{0,ijk}(F_l)$ is the normal trace on F_l of a function in $\boldsymbol{H}(\mathrm{div})$ which has zero normal traces on the remaining faces. When solving the four-face problem, we are seeking an extension operator that extends such w to a function that continues to have vanishing normal trace on the remaining three faces.

To solve the four face problem, we need a *vertex correction* operator. Define the vertex correction for l-th vertex by

$$\boldsymbol{\mathcal{E}}^{\rm div}_{V_l} w \;=\; \sum_{m \in \{i,j,k\}} \sum_{\sigma \in S(m)} \frac{\lambda_{\sigma_1}(\mathbf{grad}\, \lambda_{\sigma_2} \times \mathbf{grad}\, \lambda_{\sigma_3})}{|F_l|} \iint\limits_{F_l} \widetilde{\lambda}_m(\boldsymbol{s})\, w(\boldsymbol{s})\; d\boldsymbol{s}. \tag{6.1}$$

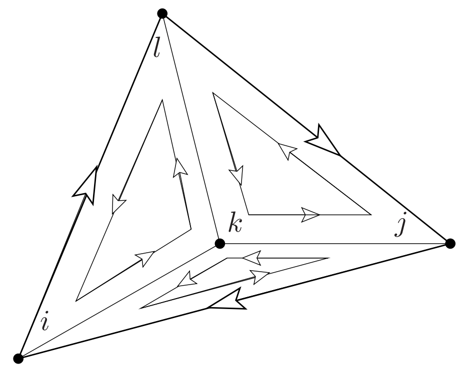

FIGURE 1. Positively oriented cycles of indices

The extension that solves the four-face problem can now be given by

$$\mathcal{E}^{\mathrm{div}}_{ijk,l}\boldsymbol{v} = \mathcal{E}^{\mathrm{div}}_{l}\boldsymbol{v} \;-\; \mathcal{E}^{\mathrm{div}}_{V_l}\boldsymbol{v} \;-\; \sum_{m\in\{i,j,k\}} \big(\mathcal{E}^{\mathrm{div}}_{F_m,l}\boldsymbol{v} - \mathcal{E}^{\mathrm{div}}_{E_{ml},l}\boldsymbol{v}\big), \tag{6.2}$$

where, as usual, we have assumed that (i,j,k) is positively oriented with respect to l.

Proposition 6.1. *The operator* $\mathcal{E}^{\mathrm{div}}_{ijk,l}$ *satisfies the following:*

(1) Commutativity: $\mathcal{E}^{\mathrm{div}}_{ijk,l}(\mathrm{curl}_\tau\, \boldsymbol{v}) = \mathbf{curl}\,(\mathcal{E}^{\mathrm{curl}}_{ijk,l}\boldsymbol{v})$ *for all* $\boldsymbol{v} \in \boldsymbol{X}^{-1/2}_{0,ijk}(F_l)$.

(2) Continuity: $\mathcal{E}^{\mathrm{div}}_{ijk,l}$ *is a continuous map from* $H^{-1/2}_{0,ijk}(F_l)$ *into* $\boldsymbol{H}(\mathrm{div})$.

(3) Extension property: *For all* $w \in H^{-1/2}_{0,ijk}(F_l)$, *we have*

$$R_l\ \mathrm{trc}_n(\mathcal{E}^{\mathrm{div}}_{ijk,l}w) = w,$$
$$R_i\ \mathrm{trc}_n(\mathcal{E}^{\mathrm{div}}_{ijk,l}w) = R_j\ \mathrm{trc}_n(\mathcal{E}^{\mathrm{div}}_{ijk,l}w) = R_k\ \mathrm{trc}_n(\mathcal{E}^{\mathrm{div}}_{ijk,l}w) = 0.$$

(4) Polynomial preservation: *Suppose* w *is in* $P_p(F_l)$, *then* $\mathcal{E}^{\mathrm{div}}_{ijk,l}w$ *is in* $P_p(K)$.

Proof. Proof of (1): First, we observe that it is enough to prove that

$$\mathcal{E}^{\mathrm{div}}_{V_l}(\mathrm{curl}_\tau\, \boldsymbol{\psi}) = \mathbf{curl}\,(\mathcal{E}^{\mathrm{curl}}_{V_l}\boldsymbol{\psi}) \tag{6.3}$$

for all smooth $\boldsymbol{\psi}$ that is compactly supported in F_l. This follows by the same type of arguments detailed in beginning of the proof of Proposition 4.1(1). The only differences now are that we should use the commutativity and stable decomposition properties appropriate for this case, namely, the commutativity property

$$\mathcal{E}^{\mathrm{curl}}_{V_l}(\mathbf{grad}_\tau\, \varphi) = \mathbf{grad}(\mathcal{E}^{\mathrm{grad}}_{V_l}\varphi) \qquad \text{for all } \varphi \in H^{1/2}_{0,ijk}(F_l),$$

proven in [7], and the stable decomposition

$$\boldsymbol{v} = \mathbf{grad}_\tau\, \varphi + \boldsymbol{\psi}, \qquad \text{with } \varphi \in H^{1/2}_{0,ijk}(F_l), \quad \boldsymbol{\psi} \in \boldsymbol{H}^{1/2}_{0,ijk}(F_l),$$

also proven in [7].

To prove (6.3), it will be convenient to go to the reference tetrahedron $\hat{K}$. Setting $l = 3$ and $(i,j,k) = (1,2,0)$ in (6.1), we find the expression for the vertex correction on $\hat{K}$ corresponding to the vertex $\hat{\boldsymbol{a}}_3 \equiv \hat{V}$. Simplifications are facilitated by the identity

$$\boldsymbol{x} - \hat{\boldsymbol{a}}_1 = \sum_{\sigma \in S(1)} -\lambda_{\sigma_1}(\mathbf{grad}\,\lambda_{\sigma_2} \times \mathbf{grad}\,\lambda_{\sigma_3}), \qquad \text{for all } \boldsymbol{x} \in \hat{K}. \tag{6.4}$$

Using also two other similar identities associated to vertices $\hat{\boldsymbol{a}}_0$ and $\hat{\boldsymbol{a}}_2$, we have

$$\begin{aligned}
\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{\hat{V}} w &= 2\int_0^1\int_0^{1-s} \left(-s \begin{pmatrix} x-1 \\ y \\ z \end{pmatrix} - t \begin{pmatrix} x \\ y-1 \\ z \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix} (1-s-t) \right) w(\boldsymbol{s})\, d\boldsymbol{s} \\
&= 2\int_0^1\int_0^{1-s} \begin{pmatrix} s-x \\ t-y \\ -z \end{pmatrix} w(s,t)\, ds\, dt. \qquad (6.5)
\end{aligned}$$

Recalling the expression for the $\boldsymbol{H}(\mathbf{curl}\,)$ vertex correction [7] and computing the curl,

$$\begin{aligned}
\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{V}} \boldsymbol{\psi}) &= \mathbf{curl} \iint_{\hat{F}} \begin{pmatrix} -z & 0 \\ 0 & -z \\ x-z & y-t \end{pmatrix} \boldsymbol{\psi}\, ds\, dt && \text{(by the expression in [7])} \\
&= 2 \iint_{\hat{F}} \begin{pmatrix} \psi_2 \\ -\psi_1 \\ 0 \end{pmatrix} ds\, dt && \text{(where } (\psi_1, \psi_2)^t \equiv \boldsymbol{\psi}) \\
&= 2 \iint_{\hat{F}} \begin{pmatrix} -s\, \partial_s \psi_2 \\ t\, \partial_t \psi_1 \\ 0 \end{pmatrix} ds\, dt && \text{(by integration by parts)} \\
&= 2 \iint_{\hat{F}} \begin{pmatrix} s\,(\partial_t \psi_1 - \partial_s \psi_2) \\ t\,(\partial_t \psi_1 - \partial_s \psi_2) \\ 0 \end{pmatrix} ds\, dt && \text{(because } \psi_1 \text{ and } \psi_2 \text{ are zero on } \partial\hat{F}).
\end{aligned}$$

Note that $\mathrm{curl}_\tau\, \boldsymbol{\psi}$ has zero mean on F_l. This is simply because $\mathrm{curl}_\tau\, \boldsymbol{\psi} = \boldsymbol{n} \cdot \mathbf{curl}\, \boldsymbol{w}$ for some smooth function $\boldsymbol{w}$ on K whose normal trace on F_{ijk} vanishes, and $\mathbf{curl}\, \boldsymbol{w}$ is divergence free. Hence,

$$\begin{aligned}
\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{V}} \boldsymbol{\psi}) &= 2 \iint_{\hat{F}} \begin{pmatrix} s-x \\ t-y \\ -z \end{pmatrix} \mathrm{curl}_\tau\, \boldsymbol{\psi}\, ds\, dt && \text{as the mean of } \mathrm{curl}_\tau\, \boldsymbol{\psi} \text{ is } 0 \\
&= \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{\hat{V}} (\mathrm{curl}_\tau\, \boldsymbol{\psi}) && \text{by (6.5).}
\end{aligned}$$

This proves the commutativity property.

Proof of (2): The continuity of the vertex correction operator from the positive order Sobolev space $H^{1/2}_{0,ijk}(F_l)$ into $\boldsymbol{H}(\mathrm{div})$ is obvious. Hence, decomposing any given w in $H^{-1/2}_{0,ijk}(F_l)$ by Theorem 2.1 as $w = \mathrm{curl}_\tau\, \boldsymbol{v} + \vartheta$ for some $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}_{0,ijk}(F_l)$ and ϑ in $H^{1/2}_{0,ijk}(F_l)$, we have

$$\begin{aligned}
\|\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ijk,l} w\|_{\boldsymbol{H}(\mathrm{div})} &= \|\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ijk,l} \boldsymbol{v}) + \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ijk,l} \vartheta\|_{\boldsymbol{H}(\mathrm{div})}, && \text{by commutativity} \\
&\le C\big(\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}_{0,ijk}(F_l)} + \|\vartheta\|_{H^{1/2}_{0,ijk}(F_l)}\big), && \text{by [7],} \\
&\le C\|w\|_{H^{-1/2}_{0,ijk}(F_l)}, && \text{by the decomposition's stability.}
\end{aligned}$$

Proof of (3): First, note that all the correction operators in (6.2) have zero normal trace on F_l. This follows from Propositions 4.1 and 5.1 for the face and edge correction. The vertex correction also has zero normal trace. This is most easily seen from (6.5), recalling the fact that all w in $H^{-1/2}_{0,ijk}(F_l)$ have zero mean. Hence the nonzero trace contribution comes from the primary extension, i.e.,

$$R_l \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ijk,l} w) = R_l \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{l} w) = w$$

by Theorem 3.1.

Next, consider the trace on face F_i. Rewriting (6.2) as

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ijk,l} w = \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{i,l} w - (\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{F_j,l} w - \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{E_{kl},l} w) - (\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{F_k,l} w - \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{E_{jl},l} w) + (\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{E_{il},l} w - \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{V_l} w), \tag{6.6}$$

observe that the first three terms on the right hand side has vanishing normal traces on F_i. Indeed,

$$\begin{aligned} \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{i,l} w)\big|_{F_l} &= 0, && \text{by Proposition 4.1(3), and} \\ \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{E_{kl},l} w - \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{F_j,l} w)|_{F_i} = \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{i,l} w - \boldsymbol{\mathcal{E}}^{\mathrm{div}}_{j,l} w)|_{F_i} &= 0, && \text{by Propositions 4.1(3) and 5.1(3),} \end{aligned}$$

and similarly for the third term. For the fourth term in (6.6), let us first calculate the normal trace of the edge correction on F_i by substituting $\lambda_i = 0$ in (5.2). We can omit terms orthogonal to $\mathbf{grad}\,\lambda_i$ as the outward normal $\boldsymbol{n}_i$ is parallel to $\mathbf{grad}\,\lambda_i$. Thus,

$$\begin{aligned} \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{E_{il},l} w)|_{F_i} = \quad & \boldsymbol{n}_i \cdot \big(\lambda_k \,\mathbf{grad}\,\lambda_j \times \mathbf{grad}\,\lambda_l - \lambda_j \,\mathbf{grad}\,\lambda_k \times \mathbf{grad}\,\lambda_l \\ & - \lambda_l \,\mathbf{grad}\,\lambda_j \times \mathbf{grad}\,\lambda_k + \lambda_j \,\mathbf{grad}\,\lambda_l \times \mathbf{grad}\,\lambda_k \\ & + \lambda_l \,\mathbf{grad}\,\lambda_k \times \mathbf{grad}\,\lambda_j - \lambda_k \,\mathbf{grad}\,\lambda_l \times \mathbf{grad}\,\lambda_j \big) \frac{1}{2|F_l|} \iint_{F_l} \widetilde{\lambda}_i \, w \, d\boldsymbol{s} \\ = \quad & 2\boldsymbol{n}_i \cdot \sum_{\sigma \in S(i)} \lambda_{\sigma_1} (\mathbf{grad}\,\lambda_{\sigma_2} \times \mathbf{grad}\,\lambda_{\sigma_3}) \frac{1}{2|F_l|} \iint_{F_l} \widetilde{\lambda}_i \, w \, d\boldsymbol{s} \end{aligned}$$

Now, consider the summands in (6.1). The normal component on F_i of the summands for $m = j, k$ vanish (this may be readily seen using an identity like (6.4) which also holds on any tetrahedron, with a minor modification). Hence the sum reduces to simply the $m = i$ summand, so

$$\operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{E_{il},l} w)|_{F_i} = \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{V_l} w)|_{F_i}.$$

This proves that

$$R_i \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ijk,l} w) = 0. \tag{6.7}$$

Since $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ijk,l} w$ is unchanged under a cyclic permutation of (i, j, k), we conclude that (6.7) implies that $R_j \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ijk,l} w)$ and $R_k \operatorname{trc}_n(\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ijk,l} w)$ also vanish.

Proof of (4): For the $p = 0$ case, note that the only constant polynomial in $H^{-1/2}_{0,ijk}(F_l)$ is zero. In this case, $\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{ijk,l} w$ is obviously zero. The result is obvious also for $p > 0$, because the vertex correction is always linear. □

7. The main result

Now we are in a position to construct an $\boldsymbol{H}(\mathrm{div})$-polynomial extension that extends traces given on the whole boundary of a general tetrahedron.

Let w be any function in the trace space of $\boldsymbol{H}(\mathrm{div})$ on ∂K, i.e., $w \in H^{-1/2}(\partial K)$. We construct the extension as in the H^1 and $\boldsymbol{H}(\mathbf{curl})$ cases [6, 7]. Define

$$\begin{aligned}
\boldsymbol{V}_i &= \boldsymbol{\mathcal{E}}_i^{\mathrm{div}} w, && \\
\boldsymbol{V}_j &= \boldsymbol{\mathcal{E}}_{i,j}^{\mathrm{div}} w_j, && \text{where } w_j = R_j(w - \mathrm{trc}_n\, \boldsymbol{V}_i), \\
\boldsymbol{V}_k &= \boldsymbol{\mathcal{E}}_{ij,k}^{\mathrm{div}} w_k, && \text{where } w_k = R_k(w - \mathrm{trc}_n\, \boldsymbol{V}_i - \mathrm{trc}_n\, \boldsymbol{V}_j), \\
\boldsymbol{V}_l &= \boldsymbol{\mathcal{E}}_{ijk,l}^{\mathrm{div}} w_l, && \text{where } w_l = R_l(w - \mathrm{trc}_n\, \boldsymbol{V}_i - \mathrm{trc}_n\, \boldsymbol{V}_j - \mathrm{trc}_n\, \boldsymbol{V}_k).
\end{aligned}$$

Here R_i is the restriction to face F_i defined earlier (see (2.13)), and the extensions $\boldsymbol{\mathcal{E}}_i^{\mathrm{div}}$, $\boldsymbol{\mathcal{E}}_{i,j}^{\mathrm{div}}$, $\boldsymbol{\mathcal{E}}_{ij,k}^{\mathrm{div}}$, and $\boldsymbol{\mathcal{E}}_{ijk,l}^{\mathrm{curl}}$ are as exhibited in (3.2), (4.3), (5.3), and (6.2), respectively. The total extension operator is then defined by

$$\boldsymbol{\mathcal{E}}_K^{\mathrm{div}} w = \boldsymbol{V}_i + \boldsymbol{V}_j + \boldsymbol{V}_k + \boldsymbol{V}_l. \tag{7.1}$$

With the help of the previously established results and the one additional lemma below, we can prove the required properties of this operator.

Lemma 7.1. *The functions w_j, w_k, and w_l defined above satisfy*

$$\begin{aligned}
\|w_j\|_{H_{0,i}^{-1/2}(F_j)} &\le C\|w\|_{H^{-1/2}(\partial K)}, \\
\|w_k\|_{H_{0,ij}^{-1/2}(F_k)} &\le C\|w\|_{H^{-1/2}(\partial K)}, \\
\|w_l\|_{H_{0,ijk}^{-1/2}(F_l)} &\le C\|w\|_{H^{-1/2}(\partial K)}.
\end{aligned}$$

Theorem 7.1. *The operator $\boldsymbol{\mathcal{E}}_K^{\mathrm{div}}$ in (7.1) has the following properties:*

1. Continuity: *$\boldsymbol{\mathcal{E}}_K^{\mathrm{div}}$ is a continuous operator from $H^{-1/2}(\partial K)$ into $\boldsymbol{H}(\mathrm{div})$.*
2. Commutativity: *$\mathbf{curl}\,(\boldsymbol{\mathcal{E}}_K^{\mathrm{curl}} \boldsymbol{v}) = \boldsymbol{\mathcal{E}}_K^{\mathrm{div}}(\mathrm{curl}_\tau\, \boldsymbol{v})$ for all tangential traces $\boldsymbol{v}$ of $\boldsymbol{H}(\mathbf{curl})$-functions, i.e., for all $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}(\partial K)$.*
3. Extension property: *The normal trace $\mathrm{trc}_n(\boldsymbol{\mathcal{E}}_K^{\mathrm{div}} w)$ coincides with w, for all w in $H^{-1/2}(\partial K)$.*
4. Polynomial preservation: *If w is a function on ∂K such that on each face $w|_{F_i}$ is in $P_p(F_i)$, then the extension $\boldsymbol{\mathcal{E}}_K^{\mathrm{div}} w$ is in $\boldsymbol{P}_p(K)$. In addition, if the mean of w on ∂K is zero, then the extension $\boldsymbol{\mathcal{E}}_K^{\mathrm{div}} w$ is a divergence free polynomial in $\boldsymbol{P}_p(K)$.*

Proof. The proof follows by combining the previous results.

Proof of (1): The proof of continuity follows by combining the continuity of $\boldsymbol{v} \mapsto w_m$ for $m = j, k, l$ (Lemma 7.1), the continuity of the primary extension (Theorem 3.1), and the continuity of the intermediate extension operators $\boldsymbol{\mathcal{E}}_{i,j}^{\mathrm{curl}}$ (Proposition 4.1), $\boldsymbol{\mathcal{E}}_{ij,k}^{\mathrm{curl}}$ (Proposition 5.1) and $\boldsymbol{\mathcal{E}}_{ijk,l}^{\mathrm{curl}}$ (Proposition 6.1).

Proof of (2): The proof of the commutativity property similarly follows because each of the intermediate operators satisfy commutativity properties, by Propositions 4.1, 5.1, and 6.1.

Proof of (3): This follows from the extension properties of the primary, the two-face, the three-face, and the four-face extensions.

Proof of (4): The polynomial preservation property is obvious from the previously established polynomial preservation properties of all the intermediate extensions. In addition, if w has zero mean, then $w = \mathrm{trc}_n(\boldsymbol{r})$ for some divergence free function $\boldsymbol{r}$. There is a vector potential $\boldsymbol{\psi}$ such that $\boldsymbol{r} = \mathbf{curl}\,\boldsymbol{\psi}$. Hence

$$\begin{aligned}\operatorname{div} \boldsymbol{\mathcal{E}}_K^{\mathrm{div}} w &= \operatorname{div} \boldsymbol{\mathcal{E}}_K^{\mathrm{div}}(\boldsymbol{n}\cdot\boldsymbol{r}) = \operatorname{div} \boldsymbol{\mathcal{E}}_K^{\mathrm{div}}(\boldsymbol{n}\cdot\mathbf{curl}\,\boldsymbol{\psi})\\ &= \operatorname{div} \boldsymbol{\mathcal{E}}_K^{\mathrm{div}}(\mathrm{curl}_\tau(\mathrm{trc}_\tau\,\boldsymbol{\psi}))\\ &= \operatorname{div} \mathbf{curl}\,(\boldsymbol{\mathcal{E}}_K^{\mathrm{curl}}\,\mathrm{trc}_\tau\,\boldsymbol{\psi}) = 0,\end{aligned}$$

hence the last assertion of the theorem. □

8. Conclusion

Combining the results of this paper with those of the previous two parts [6, 7], we conclude that we have constructed continuous *polynomial* extension operators $\mathcal{E}_K^{\mathrm{grad}}$, $\boldsymbol{\mathcal{E}}_K^{\mathrm{curl}}$, $\boldsymbol{\mathcal{E}}_K^{\mathrm{div}}$ on a tetrahedron K such that the following diagram commutes:

$$\begin{array}{ccccc} H^{1/2}(\partial K) & \xrightarrow{\mathbf{grad}_\tau} & \boldsymbol{X}^{-1/2}(\partial K) & \xrightarrow{\mathrm{curl}_\tau} & H^{-1/2}(\partial K)\\ \big\downarrow \mathcal{E}_K^{\mathrm{grad}} & & \big\downarrow \boldsymbol{\mathcal{E}}_K^{\mathrm{curl}} & & \big\downarrow \boldsymbol{\mathcal{E}}_K^{\mathrm{div}}\\ H^1(K) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div}). \end{array} \tag{8.1}$$

Appendix A. Proofs of the lemmas

We prove all the lemmas stated in the previous sections, in the order in which they appeared. We will need to use the continuity of certain operators discussed in [6]. Keeping the same notation as in [6], recall the definitions

$$A_3^\theta u\,(y,z) = 2\int_0^1\int_0^{1-s} \theta(s,t)\; u(sz, y+tz)\;\; dt\,ds, \tag{A.1}$$

$$B_2^\theta u\,(z) = 2\int_0^1\int_0^{1-s} \theta(s,t) u(sz,tz)\;\; dt\,ds. \tag{A.2}$$

$$J_\theta\phi\,(x,y,z) = \theta(x,y,z)\,\phi(y,x+z), \tag{A.3}$$

$$L_\theta\psi\,(x,y,z) = \theta(x,y,z)\,\psi(x+y+z), \tag{A.4}$$

Proof of Lemma 4.1. First, we investigate the continuity of the face correction, using its expression (4.1) for the reference tetrahedron. We find that it can be rewritten in terms of the above operators as follows:

$$\boldsymbol{\mathcal{E}}_{\hat F_1}^{\mathrm{div}} w = \begin{pmatrix} J_{\beta_2}\circ A_3^{\theta_{11}} w + J_{\beta_1}\circ A_3^{\theta_{12}} w\\ J_{\beta_0}\circ A_3^{\theta_{21}} w\\ J_{\beta_1}\circ A_3^{\theta_{31}} w\end{pmatrix} \tag{A.5}$$

where

$$\begin{aligned} &\theta_{11} = s, && \theta_{12} = \frac{1-s}{2}, && \beta_0 = 1, && \beta_1 = \frac{x}{x+z},\\ &\theta_{21} = -\frac{t}{2}, && \theta_{31} = \frac{1-3s}{2}, && && \beta_2 = \frac{x}{x+z}. \end{aligned}$$

Now we use the continuity properties of J_β and A_3^θ proved in [6]. Specifically, since all the β_i's are in $L^\infty(\hat{K})$, by [6, Lemma A.3],

$$J_{\beta_i} : \; L^2_z(\hat{F}_1) \longmapsto L^2(\hat{K})$$

is continuous. Additionally, computing the gradient of J_{β_i} (as already done in [7, Appendix A]), and applying [6, Lemma A.3] to each component, we find that

$$J_{\beta_i} : \; L^2_{1/z}(\hat{F}_1) \cap H^1_z(\hat{F}_1) \quad \longmapsto \quad H^1(\hat{K}) \tag{A.6}$$

is continuous. By [6, Lemma A.1],

$$A_3^{\theta_{mn}} : \; L^2_{1/x}(\hat{F}_3) \quad \longmapsto \quad L^2_{1/z}(\hat{F}_1) \cap H^1_z(\hat{F}_1) \tag{A.7}$$

is continuous. Combining the continuity of the maps in (A.6) and (A.7), we get that each of the composite operators in (A.5) of the form $J_{\beta_m} \circ A_3^{\theta_{ij}}$ is continuous from $L^2_{1/x}(\hat{F}_3)$ into $H^1(\hat{K})$. Transferring the result to a general tetrahedron, we find that the operator

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{F_i,l} : \; L^2_{1/\lambda_i}(F_l) \quad \longmapsto \quad \boldsymbol{H}^1(K) \tag{A.8}$$

is continuous. By Theorem 3.1(2) we know that

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}}_{l} : \; H^{1/2}(F_l) \quad \longmapsto \quad \boldsymbol{H}^1(K) \tag{A.9}$$

is continuous. Since $H^{1/2}_{0,i}(F_l) = H^{1/2}(F_l) \cap L_{1/\lambda_i}(F_l)$, we obtain the continuity stated in the lemma by combining (A.8) and (A.9). □

Proof of Lemma 4.2. We only prove the first identity (4.7) of the lemma, as the proof of the second identity is similar. We begin from the left hand side of (4.7), applying the variable change $s' = rs, t' = rt$. Let $\widetilde{w} = w(rs(x+z), rt(x+z)+y) = w(s'(x+z), t'(x+z)+y)$. Then

$$\begin{aligned}
&\int_0^1 \iint_{\hat{F}} f_{xyz}(s,t)\, \widetilde{w}\; r\; ds\, dt\, dr = \int_0^1\int_0^r\int_0^{r-s'} f_{xyz}\big(\frac{s'}{r}, \frac{t'}{r}\big)\; \widetilde{w} \;\; \frac{dt'\, ds'}{r}\, dr \\
&= \int_0^1\int_0^r\int_0^{r-s'} \frac{1}{r^2} f_{xyz}\big(s', t'\big)\; \widetilde{w} \;\; dt'\, ds'\, dr \qquad \text{(by homogeneity)} \\
&= \int_0^1 \frac{d}{dr}\big(-\frac{1}{r}\big) \int_0^r\int_0^{r-s'} f_{xyz}(s',t')\; \widetilde{w}\; dt'\, ds'\, dr \\
&= \int_0^1 \frac{1}{r}\frac{\partial}{\partial r} \int_0^r\int_0^{r-s'} f_{xyz}(s',t')\; \widetilde{w}\; dt'\, ds'\, dr \qquad \text{(by integration by parts)} \\
&\quad - \int_0^1\int_0^{1-s'} f_{xyz}(s',t')\; \widetilde{w}\; dt'\, ds' + \lim_{r\to 0} \frac{1}{r} \int_0^r\int_0^{r-s'} f_{xyz}(s',t')\; \widetilde{w}\; dt'\, ds'.
\end{aligned} \tag{A.10}$$

Since $f\widetilde{w}$ is a bounded (smooth) function, we immediately see that the last limit is zero. Carrying the r derivative inside the integral in the first term above and simplifying,

$$\int_0^1 \frac{1}{r}\frac{\partial}{\partial r}\int_0^r\int_0^{r-s'} f_{xyz}(s',t')\,\widetilde{w}\,dt'\,ds'\,dr =$$
$$= \int_0^1 \frac{1}{r}\int_0^r f_{xyz}(s',r-s')w(s'(x+z),(r-s')(x+z)+y)\,ds'\,dr$$
$$= \iint\limits_{\hat{F}} \frac{f_{xyz}(s,t)}{s+t}w(s(x+z),t(x+z)+y)\,ds\,dt,$$

where in the last step we have use the variable change $s=s', t=r-s'$. Using these in (A.10) we have

$$\int_0^1 \frac{1}{r}\frac{\partial}{\partial r}\int_0^r\int_0^{r-s'} f_{xyz}(s',t')\,\widetilde{w}\,dt'\,ds'\,dr =$$
$$= \iint\limits_{\hat{F}} \frac{f_{xyz}(s,t)}{s+t}w(s(x+z),t(x+z)+y)\,ds\,dt - \iint\limits_{\hat{F}} f_{xyz}(s,t)w(s(x+z),t(x+z)+y)\,ds\,dt,$$

thus finishing the proof of (4.7). □

Proof of Lemma 5.1. Let us rewrite the edge correction operator using the operators L_β and B_2^θ of (A.2) and (A.4):

$$\boldsymbol{\mathcal{E}}_{\hat{E}}^{\mathrm{div}} = \begin{pmatrix} L_{\beta_1}\circ B_2^{\theta_{11}} + L_{\beta_2}\circ B_2^{\theta_{12}} + L_{\beta_3}\circ B_2^{\theta_{13}} \\ L_{\beta_1}\circ B_2^{\theta_{21}} + L_{\beta_2}\circ B_2^{\theta_{22}} + L_{\beta_3}\circ B_2^{\theta_{23}} \\ L_{\beta_3}\circ B_{\theta_{31}} \end{pmatrix}. \tag{A.11}$$

where

$$\theta_{11} = s-1+\frac{3t}{2}, \qquad \theta_{12} = -\frac{s}{2} = \theta_{13}, \qquad \beta_1 = \frac{x}{x+y+z}, \qquad \beta_2 = \frac{y}{x+y+z},$$
$$\theta_{21} = -\frac{t}{2} = \theta_{23}, \qquad \theta_{31} = \frac{3}{2}(s+t)-1, \qquad \beta_3 = \frac{z}{x+y+z}.$$

By [6, Lemma A.4] applied to $L_{\beta_m}(\cdot)$ and the components of $\mathbf{grad}\,L_{\beta_m}(\cdot)$ (see [7, Appendix A] where this is explicitly done), we find that

$$L_{\beta_m} \;:\; L^2(\hat{E})\cap H^1_{z^2}(\hat{E}) \;\longmapsto\; H^1(\hat{K}) \tag{A.12}$$

is continuous. For the operators $B_2^{\theta_{ij}}$, we have by [6, Lemma A.2],

$$B_2^{\theta_{mn}} \;:\; L^2_{1/x}(\hat{F}_3)\cap L^2_{1/y}(\hat{F}_3) \longmapsto\; L^2(\hat{E}_{03})\cap H^1_{z^2}(\hat{E}_{03})$$

is continuous. Therefore, referring to (A.11), we conclude that $\boldsymbol{\mathcal{E}}_{\hat{E}}^{\mathrm{div}} : L^2_{1/x}(\hat{F})\cap L^2_{1/y}(\hat{F}) \mapsto \boldsymbol{H}^1(\hat{K})$ is continuous. In other words, on a general tetrahedron,

$$\boldsymbol{\mathcal{E}}_{E_{kl},l}^{\mathrm{div}} : L^2_{1/\lambda_i}(F_l)\cap L^2_{1/\lambda_j}(F_l) \longmapsto \boldsymbol{H}^1(K) \tag{A.13}$$

is continuous.

To complete the proof, the stated continuity of $\boldsymbol{\mathcal{E}}_{ij,l}^{\mathrm{curl}} = \boldsymbol{\mathcal{E}}_{l}^{\mathrm{curl}} - \boldsymbol{\mathcal{E}}_{F_i,l}^{\mathrm{curl}} - \boldsymbol{\mathcal{E}}_{F_j,l}^{\mathrm{curl}} + \boldsymbol{\mathcal{E}}_{E_{kl},l}^{\mathrm{curl}}$ now follows from (A.13), the continuity of the face corrections (Lemma 4.1) and the continuity of the primary extension (Theorem 3.1(2)). □

Proof of Lemma 7.1. There is a function $\boldsymbol{v}_w$ in $\boldsymbol{H}(\mathrm{div})$ which satisfies

$$\operatorname{trc}_n(\boldsymbol{v}_w) = w \qquad \text{and} \qquad \|\boldsymbol{v}_w\|_{\boldsymbol{H}(\mathrm{div})} \le C\|w\|_{H^{-1/2}(\partial K)}. \tag{A.14}$$

Let us prove the first estimate of the lemma. By the definition of $H_{0,i}^{-1/2}(F_l)$ as in (2.15), the restriction operator R_j is continuous from $H_{0,i}^{-1/2}(\partial K)$ onto $H_{0,i}^{-1/2}(F_l)$. Hence

$$\begin{aligned}
\|w_j\|_{H_{0,i}^{-1/2}(F_l)} &= \|R_j(w - \operatorname{trc}_n \boldsymbol{V}_i)\|_{H_{0,i}^{-1/2}(F_l)} \\
&\le C\|w - \operatorname{trc}_n \boldsymbol{V}_i\|_{H_{0,i}^{-1/2}(\partial K)} \\
&= C \inf_{\operatorname{trc}_n(\boldsymbol{v}) = w - \operatorname{trc}_n(\boldsymbol{V}_i)} \|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{div})} && \text{by (2.14),}
\end{aligned}$$

where the infimum runs over all $\boldsymbol{v}$ in $\boldsymbol{H}_{0,F_i}(\mathrm{div}, K)$ such that $\operatorname{trc}_n(\boldsymbol{v}) = w - \operatorname{trc}_n(\boldsymbol{V}_i)$. Since $\boldsymbol{v}_w - \boldsymbol{V}_i$ is in $\boldsymbol{H}_{0,F_i}(\mathrm{div}, K)$,

$$\begin{aligned}
\|w_j\|_{H_{0,i}^{-1/2}(F_l)} &\le C\|\boldsymbol{v}_w - \boldsymbol{V}_i\|_{\boldsymbol{H}(\mathrm{div})} \\
&\le C\|w\|_{H^{-1/2}(\partial K)} + \|\boldsymbol{\mathcal{E}}_i^{\mathrm{div}} w\|_{\boldsymbol{H}(\mathrm{div})} && \text{by (A.14)} \\
&\le C\|w\|_{H^{-1/2}(\partial K)} && \text{by Theorem 3.1.}
\end{aligned} \tag{A.15}$$

To prove the next estimate, we use similar arguments as above. In this case, we get, instead of (A.15), that

$$\|w_k\|_{H_{0,ij}^{-1/2}(F_k)} \le C\|\boldsymbol{v}_w - \boldsymbol{V}_i - \boldsymbol{V}_j\|_{\boldsymbol{H}(\mathrm{div})}$$

because $\boldsymbol{v}_w - \boldsymbol{V}_i - \boldsymbol{V}_j$ is in $\boldsymbol{H}_{0,F_i \cup F_j}(\mathrm{div}, K)$. The estimate is then proved as in the previous case. The third estimate is also similarly proved. □

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