POLYNOMIAL EXTENSION OPERATORS. PART II

LESZEK DEMKOWICZ, JAYADEEP GOPALAKRISHNAN, AND JOACHIM SCHÖBERL

Abstract. Consider the tangential trace of a vector polynomial on the surface of a tetrahedron. We construct an extension operator that extends such a trace function into a polynomial on the tetrahedron. This operator can be continuously extended to the trace space of $H(\mathbf{curl})$. Furthermore, it satisfies a commutativity property with an extension operator we constructed in Part I of this series. Such extensions are a fundamental ingredient of high order finite element analysis.

1. Introduction

This is the second in the series of papers devoted to constructing polynomial preserving continuous extension operators for Sobolev spaces satisfying the commuting diagram

$$
\begin{array}{ccccc}
H^{1/2}(\partial K) & \xrightarrow{\mathbf{grad}_\tau} & \mathrm{trc}_\tau(\boldsymbol{H}(\mathbf{curl})) & \xrightarrow{\mathrm{curl}_\tau} & \mathrm{trc}_n(\boldsymbol{H}(\mathrm{div})) \\
\Big\downarrow \mathcal{E}_K^{\mathrm{grad}} & & \Big\downarrow \boldsymbol{\mathcal{E}}_K^{\mathrm{curl}} & & \Big\downarrow \boldsymbol{\mathcal{E}}_K^{\mathrm{div}} \\
H^1(K) & \xrightarrow{\mathbf{grad}} & \boldsymbol{H}(\mathbf{curl}) & \xrightarrow{\mathbf{curl}} & \boldsymbol{H}(\mathrm{div}),
\end{array}
\tag{1.1}
$$

where K is a tetrahedron, $H^1(K)$, $\boldsymbol{H}(\mathbf{curl})$ and $\boldsymbol{H}(\mathrm{div})$ are the standard Sobolev spaces on K, and the trace operators are

$$
\begin{aligned}
\mathrm{trc}_\tau\, \boldsymbol{\phi} &= \big(\boldsymbol{\phi} - (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}\big)\big|_{\partial K}, && \text{(tangential trace)},\\
\mathrm{trc}_n\, \boldsymbol{\phi} &= (\boldsymbol{\phi}\cdot\boldsymbol{n})\big|_{\partial K}, && \text{(normal trace)},
\end{aligned}
$$

with $\boldsymbol{n}$ denoting the outward unit normal on ∂K. The first polynomial extension operator in (1.1), namely $\mathcal{E}_K^{\mathrm{grad}}$, was constructed in Part I [7]. The current part is devoted to the construction of $\boldsymbol{\mathcal{E}}_K^{\mathrm{curl}}$. The differential operators $\mathbf{grad}_\tau\, u$ and curl_τ in (1.1) denote the surface gradient and surface curl, respectively (see, e.g. [4] for definitions of differential operators on non-smooth polyhedral surfaces).

There are many applications in the analysis of high order finite elements for such an extension operator. Perhaps the most important one is in proving an approximation estimate for *hp* finite element spaces. Indeed, an approximation theory for high order $\boldsymbol{H}(\mathbf{curl})$ finite element spaces has been developed in [6] under the conjecture that such an extension operator exists. To describe one of the results there, suppose $\mathcal{T}$ is a tetrahedral finite element mesh of a polyhedral domain Ω, and let $\boldsymbol{V}_{hp} = \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\Omega) :\ \boldsymbol{v}|_K$ is a polynomial of degree at most p_K for all mesh elements K in $\mathcal{T}\}$. For any tetrahedron K, let ρ_K denote the diameter of the largest ball contained in K and let h_K denote the length of the longest edge of K. In finite element analysis, it is typical to assume that meshes are "shape regular", i.e., assume that there is a fixed positive constant γ such that $\max_{K\in\mathcal{T}} h_K/\rho_K < \gamma$

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for all meshes under consideration. In this situation, [6, Corollary 2] implies that, if an $\boldsymbol{H}(\mathbf{curl}\,)$ polynomial extension exists, then there is a constant C depending only on γ such that

$$\inf_{\boldsymbol{v}_{hp}\in \boldsymbol{V}_{hp}} \|\boldsymbol{v}-\boldsymbol{v}_{hp}\|_{\boldsymbol{H}(\mathbf{curl}\,)} \le C \sum_{K\in\mathcal{T}} h_K^{r+1}\frac{\ln p_K}{p_K^r}\left(|\boldsymbol{v}|^2_{H^r(K)}+|\mathbf{curl}\,\boldsymbol{v}|^2_{H^r(K)}\right)^{1/2} \tag{1.2}$$

for any $r > 1/2$. Thus, as a consequence of our construction of $\boldsymbol{\mathcal{E}}_K^{\mathrm{curl}}$, the approximation estimate (1.2) and other similar estimates in [6] are finally proved. The extension operator is important also in the analysis of spectral mixed methods (see remarks at the end of [10] for the need for an $\boldsymbol{H}(\mathbf{curl}\,)$ extension). Polynomial extensions also play an important role in the construction of good shape functions and preconditioning [17].

We will keep the same notation as in Part I (summarized in [7, § 1.5]) and employ the same overall technique developed there (summarized in [7, § 1.4]) for constructing the $\boldsymbol{H}(\mathbf{curl}\,)$ extension operator. In particular, we start with a primary extension operator, and then design suitable face, edge, and vertex correction operators to arrive at the total extension operator. The construction of both the primary and correction operators will be motivated by the need to satisfy the commutativity property in (1.1). For example, to obtain an expression for the $\boldsymbol{H}(\mathbf{curl}\,)$ primary extension of $\boldsymbol{v}$, denoted by $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}$, we took the expression for $\mathcal{E}^{\mathrm{grad}}u$ from [7], differentiated it, expressed the result in terms of $\mathbf{grad}_\tau\, u$, and then substituted $\mathbf{grad}_\tau\, u$ by $\boldsymbol{v}$. Clearly, this will guarantee the commutativity property $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\,\mathbf{grad}_\tau\, u = \mathbf{grad}\,\mathcal{E}^{\mathrm{grad}}u$. Such computations motivated the expressions for face and edge corrections as well. *The final $\boldsymbol{H}(\mathbf{curl}\,)$ polynomial extension operator and its properties are given in Theorem 7.1.*

Although we apply the same overall technique as in the H^1 case considered in Part I [7] of this series, a major difference between the $\boldsymbol{H}(\mathbf{curl}\,)$ case and the H^1 case is that the trace space of the former is more complicated. Only recently has the trace space of $\boldsymbol{H}(\mathbf{curl}\,)$ on polyhedral domains been fully characterized in terms of certain Sobolev spaces of negative index [4, 5]. In order to circumvent estimating negative norms, we proceed by first developing a new technical tool, namely a decomposition of the trace space, which when combined with commutativity, reduces the problem of norm bounds for the extension to Sobolev norms of positive index only. This seems to simplify the analysis considerably. Another new technique we introduce in this paper is proving a norm estimate for primary extensions in fractional Sobolev norms directly using Peetre's K-functional and interpolation theory. Other new aspects in the $\boldsymbol{H}(\mathbf{curl}\,)$ arena not seen in the H^1 case include symmetrization of integrals defining the extensions to obtain expressions invariant under relevant covariant transformations.

We begin by describing the decomposition of trace space using regular functions (Section 2). Then we study the primary extension from a plane (Section 3). The primary extension will then be corrected using face and edge correction operators given in Sections 4 and 5. The complete solution to the $\boldsymbol{H}(\mathbf{curl}\,)$ polynomial extension on a tetrahedron is given in Section 7. Appendix A contains proofs of all technical lemmas.

2. A characterization of the trace space

For smooth vector functions $\boldsymbol{\phi}$, we denote their tangential and normal traces on ∂K by

$$\begin{aligned}
\mathrm{trc}_\tau\, \boldsymbol{\phi} &= \big(\boldsymbol{\phi} - (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}\big)\big|_{\partial K},\\
\mathrm{trc}_n\, \boldsymbol{\phi} &= (\boldsymbol{\phi}\cdot\boldsymbol{n})\big|_{\partial K},
\end{aligned}$$

where $\boldsymbol{n}$ denote the outward unit normal on ∂K. It is well known that the operators trc_τ and $\boldsymbol{n} \times \operatorname{trc}_\tau$ extend continuously to $\boldsymbol{H}(\mathbf{curl})$ and that their ranges are subspaces of $\boldsymbol{H}^{-1/2}(\partial K)$ [1, 4, 9]. Letting $\langle\cdot,\cdot\rangle$ denote the duality pairing between $\boldsymbol{H}^{-1/2}(\partial K)$ and $\boldsymbol{H}^{1/2}(\partial K)$, define $\boldsymbol{H}_{0,S}(\mathbf{curl})$ for any subset S of ∂K of positive measure by

$$\boldsymbol{H}_{0,S}(\mathbf{curl}) = \{\boldsymbol{\phi} \in \boldsymbol{H}(\mathbf{curl}) : \ \langle \boldsymbol{n} \times \operatorname{trc}_\tau \boldsymbol{\phi},\ \boldsymbol{\psi}\rangle = 0 \quad \text{for all } \boldsymbol{\psi} \in \boldsymbol{H}^1_{\partial K\setminus S}(K)\},$$

where $\boldsymbol{H}^1_{\partial K\setminus S}(K)$ denotes the subspace of functions in $\boldsymbol{H}^1(K)$ whose tangential traces vanish on $\partial K \setminus S$. In addition, we shorten $\boldsymbol{H}_{0,\partial K}(\mathbf{curl})$ to simply $\boldsymbol{H}_0(\mathbf{curl})$.

We shall need the trace spaces of $\boldsymbol{H}_{0,S}(\mathbf{curl})$ when S is composed of one or more faces of K. Let $F_{ij} = F_i \cup F_j$ and $F_{ijk} = F_i \cup F_j \cup F_k$. We define the spaces by the range of the trace map:

$$\begin{aligned} \boldsymbol{X}^{-1/2} &= \operatorname{trc}_\tau \boldsymbol{H}(\mathbf{curl}), & \boldsymbol{X}^{-1/2}_{0,i} &= \operatorname{trc}_\tau \boldsymbol{H}_{0,F_i}(\mathbf{curl}), \\ \boldsymbol{X}^{-1/2}_{0,ij} &= \operatorname{trc}_\tau \boldsymbol{H}_{0,F_{ij}}(\mathbf{curl}), & \boldsymbol{X}^{-1/2}_{0,ijk} &= \operatorname{trc}_\tau \boldsymbol{H}_{0,F_{ijk}}(\mathbf{curl}). \end{aligned} \tag{2.1}$$

The above spaces $\boldsymbol{X}^{-1/2}_{0,I}$, for all subscripts I in the set $\{i, ij, ijk\}$, are subspaces of $\boldsymbol{H}^{-1/2}(\partial K)$. The precise subspace topology of $\boldsymbol{X}^{-1/2}$ in $\boldsymbol{H}^{-1/2}(\partial K)$ is given in [4]. One could attempt to use their techniques to characterize the subspace topologies of all $\boldsymbol{X}^{-1/2}_{0,I}$, but for our purposes it seems better to proceed by endowing all the sets in (2.1) with a natural quotient topology defined by

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}} := \inf_{\operatorname{trc}_\tau(\boldsymbol{\phi})=\boldsymbol{v}} \|\boldsymbol{\phi}\|_{\boldsymbol{H}(\mathbf{curl})}, \tag{2.2}$$

where the infimum runs over all $\boldsymbol{\phi}$ in $\boldsymbol{H}(\mathbf{curl})$ satisfying $\operatorname{trc}_\tau(\boldsymbol{\phi}) = \boldsymbol{v}$. Standard arguments then prove the following facts: Under the quotient norm in (2.2), the space $\boldsymbol{X}^{-1/2}$ is complete and the subsets $\boldsymbol{X}^{-1/2}_{0,I}$ are closed. Furthermore, there is a linear continuous lifting operator $\boldsymbol{E} : \boldsymbol{X}^{-1/2} \mapsto \boldsymbol{H}(\mathbf{curl})$ satisfying

$$\boldsymbol{E}\boldsymbol{X}^{-1/2}_{0,I} \subseteq \boldsymbol{H}_{0,F_I}(\mathbf{curl}), \quad \operatorname{trc}_\tau(\boldsymbol{E}\boldsymbol{v}) = \boldsymbol{v}, \quad \|\boldsymbol{E}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} = \|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}}, \tag{2.3}$$

for all $\boldsymbol{v} \in \boldsymbol{X}^{-1/2}$. We need to find an extension operator like $\boldsymbol{E}$, but one that has the additional polynomial preservation property.

We shall now characterize the $\boldsymbol{H}(\mathbf{curl})$ trace spaces using Sobolev spaces of positive index, namely $H^{1/2}(\partial K)$, and $\boldsymbol{H}^{1/2}_\tau := \operatorname{trc}_\tau \boldsymbol{H}^1(K)$. The space $\boldsymbol{H}^{1/2}_\tau$ is characterized in terms of the $\boldsymbol{H}^{1/2}$-norm of faces in [4], but we will simply work with the natural norm $\|\boldsymbol{\vartheta}\|_{\boldsymbol{H}^{1/2}_\tau}$ defined to be the infimum of $\|\boldsymbol{\phi}\|_{\boldsymbol{H}(\mathbf{curl})}$ over all $\boldsymbol{\phi} \in \boldsymbol{H}^1(K)$ for which $\operatorname{trc}_\tau \boldsymbol{\phi} = \boldsymbol{\vartheta}$. The idea for our characterization of the trace space is best revealed for the first space in (2.1), as we see next.

Proposition 2.1. *The space $\boldsymbol{X}^{-1/2}$ admits the following stable decomposition:*

$$\boldsymbol{X}^{-1/2} = \ \mathbf{grad}_\tau H^{1/2}(\partial K) \ + \ \boldsymbol{H}^{1/2}_\tau.$$

Proof. Consider any function $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}$ and its lifting $\boldsymbol{E}\boldsymbol{v}$ defined in (2.3). Since K is convex, by the well known Helmholtz-Hodge decomposition for $\boldsymbol{H}(\mathbf{curl})$ (see e.g. [9, Corollary I.3.4] or [13]), there is a $\varphi \in H^1(K)$ and $\boldsymbol{\psi} \in \boldsymbol{H}^1(K)$ such that

$$\boldsymbol{E}\boldsymbol{v} = \mathbf{grad}\,\varphi + \boldsymbol{\psi}. \tag{2.4}$$

Applying the tangential trace operator to this decomposition, we obtain the required decomposition:

$$\boldsymbol{v} = \mathbf{grad}_\tau(\varphi|_{\partial K}) + \operatorname{trc}_\tau(\boldsymbol{\psi}).$$

Its stability follows from the continuity of the trace maps. Indeed, there are positive constants C_1 and C_2 such that

$$\begin{aligned}\|\varphi\|_{H^{1/2}(\partial K)} + \|\operatorname{trc}_\tau \boldsymbol{\psi}\|_{\boldsymbol{H}_\tau^{1/2}} &\le C_1 \left(\|\varphi\|_{H^1(K)} + \|\boldsymbol{\psi}\|_{\boldsymbol{H}^1(K)}\right)\\ &\le C_2 \|\boldsymbol{E}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} = C_2 \|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}},\end{aligned}$$

where we have also used the stability of the decomposition in (2.4). □

Although the trace spaces in (2.1) were defined on the whole boundary ∂K, by virtue of Proposition 2.1, we can now speak of its restrictions on faces. Indeed, it is well known that the restriction to a face F_l is a continuous operation from $H^{1/2}(\partial K)$ into $H^{1/2}(F_l)$. Moreover, letting $\boldsymbol{H}^{1/2}(F_l)$ denote the space of tangential vector functions on F_l whose two components are in $H^{1/2}(F_l)$, the restriction operator is also a continuous map from $\boldsymbol{H}_\tau^{1/2}$ into $\boldsymbol{H}^{1/2}(F_l)$ (this follows, e.g., from the characterization of $\boldsymbol{H}_\tau^{1/2}$ in terms of standard Sobolev spaces found in [4]). Therefore, given any $\boldsymbol{v} \in \boldsymbol{X}^{-1/2}$, decomposing it by Proposition 2.1 as $\boldsymbol{v} = \mathbf{grad}_\tau\, \varphi + \boldsymbol{\psi}$ we can define the restriction operator $\boldsymbol{R}_l$ by

$$\boldsymbol{R}_l \boldsymbol{v} = \mathbf{grad}_\tau (\varphi\big|_{F_l}) + (\boldsymbol{\psi}\big|_{F_l}). \tag{2.5}$$

Clearly, $\boldsymbol{R}_l$ coincides with the usual restriction operator when applied to smooth $\boldsymbol{v}$. Moreoever, by the stability of the decomposition, $\boldsymbol{R}_l$ is a continuous map from $\boldsymbol{X}^{-1/2}$ into $\mathbf{grad}_\tau\, H^{1/2}(F_l) + \boldsymbol{H}^{1/2}(F_l)$. We define the *trace spaces on one face* as the range of this restriction operator:

$$\boldsymbol{X}^{-1/2}(F_l) = \boldsymbol{R}_l\, \boldsymbol{X}^{-1/2}, \qquad \|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}(F_l)} := \inf_{\boldsymbol{R}_l \boldsymbol{w} = \boldsymbol{v}} \|\boldsymbol{w}\|_{\boldsymbol{X}^{-1/2}}, \tag{2.6}$$

where the infimum runs over all $\boldsymbol{w}$ in $\boldsymbol{X}^{-1/2}$ satisfying $\boldsymbol{R}_l \boldsymbol{w} = \boldsymbol{v}$. The space $\boldsymbol{X}^{-1/2}(F_l)$ is complete under the above norm and the subsets $\boldsymbol{X}_{0,I}^{-1/2}(F_l) = \boldsymbol{R}_l \boldsymbol{X}_{0,I}^{-1/2}$ are closed. It is easy to verify that $\boldsymbol{R}_l$ has a continuous right inverse $\boldsymbol{L}_l : \boldsymbol{X}^{-1/2}(F_l) \mapsto \boldsymbol{X}^{-1/2}$ satisfying

$$\boldsymbol{L}_l \boldsymbol{X}_{0,I}^{-1/2}(F_l) \subseteq \boldsymbol{X}_{0,I}^{-1/2}, \quad \|\boldsymbol{L}_l \boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}} = \|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}(F_l)}, \quad \boldsymbol{R}_l \boldsymbol{L}_l \boldsymbol{v} = \boldsymbol{v}, \tag{2.7}$$

for all $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}(F_l)$.

We will now show that these trace spaces on the face F_l can be characterized using subspaces of $H^{1/2}(F_l)$ with zero boundary conditions. Recall the definitions of $H_{0,I}^{1/2}(F_l)$ for $I \in \{i, ij, ijk\}$ from Part I [7]:

$$\begin{aligned}H_{0,i}^{1/2}(F_l) &= H^{1/2}(F_l) \cap L^2_{1/\lambda_i}(F_l)\\ H_{0,ij}^{1/2}(F_l) &= H^{1/2}(F_l) \;\cap\; L^2_{1/\lambda_i}(F_l) \;\cap\; L^2_{1/\lambda_j}(F_l)\\ H_{0,ijk}^{1/2}(F_l) &= H^{1/2}(F_l) \cap L^2_{1/\lambda_i}(F_l) \cap L^2_{1/\lambda_j}(F_l) \cap L^2_{1/\lambda_k}(F_l).\end{aligned}$$

Here $L^2_{1/\lambda_i}(F_l)$ is the Lebesgue space with weight $1/\lambda_i$, so clearly, the functions in $H_{0,I}^{1/2}(F_l)$ vanish weakly on certain parts of the boundary ∂F_l. Also, let $\boldsymbol{H}_{0,I}^{1/2}(F_l)$ denote the set of tangential vector functions on F_l whose two components are in $H_{0,I}^{1/2}(F_l)$. Then we have the following theorem (where, like everywhere else in this paper, the indices i, j, k, l are a permutation of $0, 1, 2, 3$).

Theorem 2.1. *The spaces $\boldsymbol{X}^{-1/2}(F_l)$ and $\boldsymbol{X}^{-1/2}_{0,I}(F_l)$ of traces on F_l for all I in $\{i, ij, ijk\}$ admit the stable decompositions*

$$\boldsymbol{X}^{-1/2}(F_l) = \mathbf{grad}_\tau\, H^{1/2}(F_l) + \boldsymbol{H}^{1/2}(F_l),$$
$$\boldsymbol{X}^{-1/2}_{0,I}(F_l) = \mathbf{grad}_\tau\, H^{1/2}_{0,I}(F_l) + \boldsymbol{H}^{1/2}_{0,I}(F_l).$$

Proof. The first decomposition follows immediately from Proposition 2.1 (by restricting to F_l), so let us prove the second. Let $\boldsymbol{v}$ be any function in $\boldsymbol{X}^{-1/2}_{0,I}(F_l)$ and

$$\boldsymbol{\phi} = \boldsymbol{E}(\boldsymbol{L}_l \boldsymbol{v}), \tag{2.8}$$

where $\boldsymbol{E}$ and $\boldsymbol{L}_l$ are as in (2.3) and (2.7), respectively. Then, by the above mentioned properties of these operators, $\boldsymbol{\phi}$ is in $\boldsymbol{H}_{0,F_I}(\mathbf{curl}\,, K)$.

We need now to expand the domain K. Let $\widetilde{\boldsymbol{a}}_i$ be the reflection of the vertex $\boldsymbol{a}_i$ within the plane containing F_l, about the line containing $\boldsymbol{a}_j$ and $\boldsymbol{a}_k$. Then, depending on I in $\{i, ij, ijk\}$, define $\widetilde{F}_{I,l}$ by

$$\widetilde{F}_{i,l} = \operatorname{conv}(F_l, \widetilde{\boldsymbol{a}}_i), \quad \widetilde{F}_{ij,l} = \operatorname{conv}(F_l, \widetilde{\boldsymbol{a}}_i, \widetilde{\boldsymbol{a}}_j), \quad \widetilde{F}_{ijk,l} = \operatorname{conv}(F_l, \widetilde{\boldsymbol{a}}_i, \widetilde{\boldsymbol{a}}_j, \widetilde{\boldsymbol{a}}_k),$$

where conv denotes convex hull. The expanded domain is defined by $\widetilde{K}_I = \operatorname{conv}(\widetilde{F}_{I,l}, \boldsymbol{a}_l)$

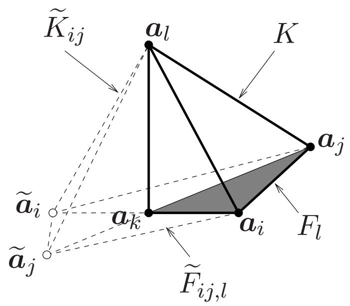

FIGURE 1. Notations in the proof of Theorem 2.1

(this domain, for the case $I = ij$ is illustrated in Fig. 1). It is easy to prove that the trivial extension of $\boldsymbol{\phi}$ defined by

$$\widetilde{\boldsymbol{\phi}} = \begin{cases} \boldsymbol{\phi} & \text{on } K, \\ \mathbf{0} & \text{on } \widetilde{K}_I \setminus K, \end{cases}$$

is in $\boldsymbol{H}_{0,F_I}(\mathbf{curl}\,, \widetilde{K}_I)$.

Next, we borrow a technique found in [3, Lemma 2.2] (see also [15, Proposition 5.1] and other related references mentioned there). We start by decomposing $\widetilde{\boldsymbol{\phi}}$ using the Helmholtz-Hodge decomposition on the convex domain $\widetilde{K}_I$ to get

$$\widetilde{\boldsymbol{\phi}} = \mathbf{grad}\,\varphi + \boldsymbol{\psi}, \qquad \text{with } \ \varphi \in H^1(\widetilde{K}_I),\ \boldsymbol{\psi} \in \boldsymbol{H}^1(\widetilde{K}_I). \tag{2.9}$$

Observe that since $\widetilde{\boldsymbol{\phi}}$ vanishes on $\widetilde{K}_I \setminus K$, the gradient of φ must coincide with $\boldsymbol{\psi}$ there. Hence

$$\varphi\big|_{\widetilde{K}_I \setminus K} \in H^2(\widetilde{K}_I \setminus K).$$

Therefore, there exists an H^2-extension (see, e.g. [18, Theorem VI.3.5, pp. 181], or our volume extension constructions in [8]) of φ to all $\widetilde{K}_I$, which we denote by φ'. Then

$$\widetilde{\boldsymbol{\phi}} = \mathbf{grad}\,\varphi'' + \boldsymbol{\psi}'', \qquad \text{with } \ \varphi'' = \varphi - \varphi',\ \ \boldsymbol{\psi}'' = \mathbf{grad}\,\varphi' + \boldsymbol{\psi}. \tag{2.10}$$

Clearly, φ'' is in $H^1(\widetilde{K}_I)$ and $\boldsymbol{\psi}''$ is in $\boldsymbol{H}^1(\widetilde{K}_I)$. Moreover both φ'' and $\boldsymbol{\psi}''$ vanish on $\widetilde{K}_I \setminus K$.

The required decomposition is now obtained by applying trc_τ to (2.10). Indeed, combining the definition of $\boldsymbol{\phi}$ in (2.8) with (2.10), we obtain

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{R}_l \operatorname{trc}_\tau(\boldsymbol{\phi}) = \boldsymbol{R}_l \operatorname{trc}_\tau(\widetilde{\boldsymbol{\phi}}\big|_K) \\ &= \mathbf{grad}_\tau(\varphi''\big|_{F_l}) + \boldsymbol{R}_l \operatorname{trc}_\tau \boldsymbol{\psi}''. \end{aligned} \tag{2.11}$$

Since $\boldsymbol{\psi}''$ is in $\boldsymbol{H}^1(\widetilde{K}_I)$, its trace $\boldsymbol{\psi}''\big|_{\widetilde{F}_{I,l}}$ is in $(H^{1/2}(\widetilde{F}_{I,l}))^3$ and all three components of this trace vanish on $\widetilde{F}_{I,l} \setminus F_l$. Moreover, since the tangential component of this trace on F_l coincides with $\boldsymbol{R}_l \operatorname{trc}_\tau \boldsymbol{\psi}''$, we conclude that the last term in (2.11) is in $\boldsymbol{H}^{1/2}_{0,I}(F_l)$. Moreover, since φ'' vanishes on $\widetilde{K}_I \backslash K$, its trace appearing in (2.11) is in $H^{1/2}_{0,I}(F_l)$. Thus the components in the decomposition (2.11) are in the required spaces.

The stability of the decomposition (2.11) follows from the stability of the decomposition (2.9), the H^2-continuity of the map $\varphi \mapsto \varphi'$, the continuity of various trace maps, and the continuity of the operators $\boldsymbol{E}$ and $\boldsymbol{L}_l$. □

Remark 2.1. The decomposition of Theorem 2.1 has a regular part, namely $\boldsymbol{H}^{1/2}_{0,I}(F_l)$, and a non-regular part, namely $\mathbf{grad}_\tau\, H^{1/2}_{0,I}(F_l)$ (which is generally only in $\boldsymbol{H}^{-1/2}(F_l)$). It is important to note that the theorem lets us choose the regular part to be a vector function with zero boundary conditions on *all* its components. Note also that the decomposition is not an orthogonal decomposition in $L^2(F_l)$.

Remark 2.2. The decomposition of Theorem 2.1 gives an equivalent norm on the trace space. E.g., from the results of [4, 5], it follows that the trace space $\boldsymbol{X}^{-1/2}(F_l)$ coincides with the space

$$\boldsymbol{H}^{-1/2}(\operatorname{curl}_\tau, F_l) := \{\boldsymbol{v} \in \boldsymbol{H}^{-1/2}(F_l) : \operatorname{curl}_\tau \boldsymbol{v} \in H^{-1/2}(F_l)\}$$

normed with $\|\boldsymbol{v}\|_{\boldsymbol{H}^{-1/2}(\operatorname{curl}_\tau, F_l)} := \left(\|\boldsymbol{v}\|^2_{\boldsymbol{H}^{-1/2}(F_l)} + \|\operatorname{curl}_\tau \boldsymbol{v}\|^2_{H^{-1/2}(F_l)}\right)^{1/2}$ where curl_τ denotes the scalar surface curl. Then our results imply that for any $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}(F_l)$, if $\boldsymbol{v} = \mathbf{grad}_\tau\, \varphi_{\boldsymbol{v}} + \boldsymbol{\psi}_{\boldsymbol{v}}$ denotes the decomposition given by Theorem 2.1, the norms

$$\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}(F_l)}, \quad \|\boldsymbol{v}\|_{\boldsymbol{H}^{-1/2}(\operatorname{curl}_\tau, F_l)}, \quad \text{and} \quad \|\varphi_{\boldsymbol{v}}\|_{H^{1/2}(F_l)} + \|\boldsymbol{\psi}_{\boldsymbol{v}}\|_{\boldsymbol{H}^{1/2}(F_l)},$$

are equivalent norms.

3. Primary extension operator

We first display the expression for the $\boldsymbol{H}(\mathbf{curl})$ primary extension when the data function $\boldsymbol{v}$ is smooth tangential vector function on the x-y plane (or the x-y face of the reference tetrahedron $\hat{K}$). The expression is

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}\,(x,y,z) = 2\int_0^1\int_0^{1-t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s & t \end{pmatrix} \boldsymbol{v}(x+sz, y+tz)\, ds\, dt \tag{3.1}$$

which by a change of variable can also be expressed as

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}\,(x,y,z) = \frac{2}{z^3}\int_x^{x+z}\int_y^{x+y+z-\widetilde{x}} \begin{pmatrix} z & 0 \\ 0 & z \\ \widetilde{x}-x & \widetilde{y}-y \end{pmatrix} \boldsymbol{v}(\widetilde{x},\widetilde{y})\, d\widetilde{y}\, d\widetilde{x}. \tag{3.2}$$

We derived this expression motivated by the commutativity property we need, namely $\mathbf{grad}\, \mathcal{E}^{\mathrm{grad}} u = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}\, \mathbf{grad}_\tau\, u$. Indeed, we took the expression for $\mathcal{E}^{\mathrm{grad}}$ from [7], differentiated it, expressed the result in terms of $\mathbf{grad}_\tau\, u$, and then substituted $\mathbf{grad}_\tau\, u$ by $\boldsymbol{v}$ to obtain (3.1). (This calculation is implicit in the proof of Theorem 3.1(1) to be given shortly, so we do not display it here.)

The above expression will give an extension operator on any other tetrahedron K once we use the right affine mapping that maps vector functions on $\hat{K}$ to K. As in the H^1 case, instead of exhibiting the mappings, we will simply give the general expressions in affine coordinates. In order to bring out the symmetry in the expressions, we shall write a smooth tangential vector function given on face F_l as

$$\boldsymbol{v} = \sum_{m\in\{i,j,k\}} v_m \ \mathbf{grad}_\tau \lambda_m. \tag{3.3}$$

with three smooth components v_m. Such a decomposition of $\boldsymbol{v}$ into component functions v_m is always possible, but is not unique. Indeed v_m for all m in $\{i,j,k\}$ coincides with one function $\bar{v}$ if and only if $\boldsymbol{v}$ is zero. With v_m as in (3.3), we can now rewrite the primary extension operator as follows:

$$\begin{aligned}
\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v} =& \frac{2}{z^3}\int_x^{x+z}\int_y^{x+y+z-\widetilde{x}} \begin{pmatrix} z & 0 \\ 0 & z \\ \widetilde{x}-x & \widetilde{y}-y \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix} v_0 + \begin{pmatrix} z \\ 0 \\ \widetilde{x}-x \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ z \\ \widetilde{y}-y \end{pmatrix} v_2 \, d\widetilde{y}\, d\widetilde{x} \\
=& \ \frac{2}{\lambda_3^2}\iint_{T_3(\lambda_0,\lambda_1,\lambda_2)} \Big(v_0(-\mathbf{grad}\,\lambda_1 - \mathbf{grad}\,\lambda_2 - (\widetilde{\lambda}_1+\widetilde{\lambda}_2)\,\mathbf{grad}\,\lambda_3) \\
&\qquad + v_1(\mathbf{grad}\,\lambda_1 + \widetilde{\lambda}_1\,\mathbf{grad}\,\lambda_3) + v_2(\mathbf{grad}\,\lambda_2 + \widetilde{\lambda}_2\,\mathbf{grad}\,\lambda_3)\Big)\, d\widetilde{y}\, d\widetilde{x}, \\
=& \ \frac{2}{\lambda_3^2}\iint_{T_3(\lambda_0,\lambda_1,\lambda_2)} \sum_{m=0}^{2}\Big(v_m - \sum_{\ell=0}^{2}\widetilde{\lambda}_\ell v_\ell\Big)\,\mathbf{grad}\,\lambda_m \, d\widetilde{y}\, d\widetilde{x},
\end{aligned} \tag{3.4}$$

where we have used the barycentric coordinates λ_i of the tetrahedron, and the barycentric coordinates $\widetilde{\lambda}_\ell(\boldsymbol{s})$ of the region of integration $T_3(\lambda_0,\lambda_1,\lambda_2)$. The symbol $\widetilde{\lambda}_\ell$ will generically denote the barycentric coordinates of whatever region of integration is under consideration, e.g., in the above, since the region is $T_3(\lambda_0,\lambda_1,\lambda_2)$, they are $\widetilde{\lambda}_1 = (\widetilde{x}-x)/z$, $\widetilde{\lambda}_1 = (\widetilde{y}-y)/z$, and $\widetilde{\lambda}_0 = 1-\widetilde{\lambda}_1-\widetilde{\lambda}_2$. Note also that in the above, we have continued to use the notations in [7], e.g., for any permutation $\{i,j,k,l\}$ of $\{0,1,2,3\}$, define $T_l(r_i,r_j,r_k) = \{\boldsymbol{x}\in F_l : \lambda_\ell^{F_l}(\boldsymbol{x}) \geq r_\ell \text{ for } \ell = i,j, \text{ and } k\}$, where $\lambda_m^{F_l} \equiv \lambda_m|_{F_l}$ (for $m = i,j,$ or k) are the barycentric coordinates of F_l.

Generalizing the above, we obtain the expression for the primary extension operator on a general tetrahedron K lifting from the face F_l:

$$\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}\boldsymbol{v}\ (\lambda_i,\lambda_j,\lambda_k,\lambda_l) = \frac{2}{\lambda_l^2}\iint_{T_l(\lambda_i,\lambda_j,\lambda_k)} \sum_{m\in\{i,j,k\}} D_m\boldsymbol{v}(\boldsymbol{s})\ \mathbf{grad}\,\lambda_m \ d\boldsymbol{s} \tag{3.5}$$

where

$$D_m\boldsymbol{v}(\boldsymbol{s}) = v_m(\boldsymbol{s}) - \sum_{\ell\in\{i,j,k\}} \widetilde{\lambda}_\ell(\boldsymbol{s})\, v_\ell(\boldsymbol{s}) \tag{3.6}$$

and $\widetilde{\lambda}_\ell(\boldsymbol{s})$, for ℓ in $\{i,j,k\}$, are the barycentric coordinate functions of the region of integration $T_l(\lambda_i,\lambda_j,\lambda_k)$, considered with its node enumeration inherited from K. Since the component representation in (3.3) is not unique, we must check that definitions like (3.6) are not affected, inasmuch as two different representations of the same function does not yield different results. That this is indeed the case, is readily checked by verifying that when we substitute $v_m = \bar{v}$ for all m in (3.6) and simplify, we do find $D_m \boldsymbol{v}$ to vanish.

We prove the properties of this primary extension operator in the next theorem. There are two new ingredients worth noting in the proof of continuity of $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}$. The first is the technique of proving continuity from $\boldsymbol{H}^{1/2}(F_l)$ into $\boldsymbol{H}^1(\hat{K})$ using Peetre's K-functional. (Note that this continuity only involves Sobolev norms of positive order.) The second is the technique of using continuity on positive order Sobolev spaces to obtain continuity on the trace space contained in the negative order Sobolev space $\boldsymbol{H}^{-1/2}(F_l)$. (In [7, Appendix B], we provided an alternate technique for proving the continuity using the Fourier transform.) We display the K-functional technique in Appendix A while proving the following lemma.

Lemma 3.1. *Let $\theta(x,y)$ be a smooth function on the unit triangle $\hat{F}$ (including the boundary $\partial\hat{F}$). Then the map $\mathcal{K}_\theta$ defined for smooth functions $u(x,y)$ on $\hat{F}$ by*

$$\mathcal{K}_\theta u\ (x,y,z) = \int_0^1 \int_0^{1-t} \theta(s,t)\ u(x+sz, y+tz)\ ds\, dt,$$

satisfies

$$\|\mathcal{K}_\theta u\|_{H^1(\hat{K})} \le C_\theta \|u\|_{H^{1/2}(\hat{F})}, \quad \textit{for all } u \in H^{1/2}(\hat{F}),$$

with some $C_\theta > 0$ that depends only on $\|\theta\|_{W_1^1(\hat{F})}$ and $\|\theta\|_{L^1(\partial\hat{F})}$.

The theorem on the primary extension will use this lemma. Before stating the theorem we need more notation for vector polynomial spaces: The space of vector functions on any domain D whose components are polynomials of degree at most p is denoted by $\boldsymbol{P}_p(D)$ and its subspace of homogeneous polynomials of degree p is denoted by $\bar{\boldsymbol{P}}_p(D)$. The Nédélec subspace (of the first kind) [14] of $\boldsymbol{P}_{p+1}$, denoted by $\boldsymbol{N}_p(D)$, is defined by

$$\boldsymbol{N}_p(D) = \{\boldsymbol{v}_p + \boldsymbol{r}_{p+1} : \quad \boldsymbol{v}_p \in \boldsymbol{P}_p(D), \text{ and } \boldsymbol{r}_{p+1} \in \bar{\boldsymbol{P}}_{p+1}(D) \text{ satisfies } \boldsymbol{r}_{p+1}\cdot\boldsymbol{x} = 0\}.$$

It is easy to see that

$$\boldsymbol{q} \in \boldsymbol{N}_p(D) \quad \text{if and only if} \quad \boldsymbol{q} \in \boldsymbol{P}_{p+1}(D) \text{ and } \boldsymbol{q}\cdot\boldsymbol{x} \in P_{p+1}(D). \tag{3.7}$$

In these characterizations of $\boldsymbol{N}_p(D)$, the vector $\boldsymbol{x}$ is the coordinate vector in the Euclidean space in which D lies, so it can have two or three components.

Theorem 3.1. *The primary extension operator $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}$ has the following properties:*

1. $\mathbf{grad}(\mathcal{E}_l^{\mathrm{grad}} u) = \boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}(\mathbf{grad}_\tau u)$ *for all u in $H^{1/2}(F_l)$.*
2. $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}$ *is a continuous map from $\boldsymbol{H}^{1/2}(F_l)$ into $\boldsymbol{H}^1(K)$.*
3. $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}$ *is a continuous map from $\boldsymbol{X}^{-1/2}(F_l)$ into $\boldsymbol{H}(\mathbf{curl}, K)$.*
4. *The tangential trace of $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}\boldsymbol{v}$ on F_l equals $\boldsymbol{v}$ for all $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}(F_l)$.*
5. *If $\boldsymbol{v}$ is in $\boldsymbol{P}_p(F_l)$, the extension $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}\boldsymbol{v}$ is in $\boldsymbol{P}_p(K)$. If $\boldsymbol{v}$ is in the Nédélec space $\boldsymbol{N}_p(F_l)$, its extension $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}\boldsymbol{v}$ is in $\boldsymbol{N}_p(K)$.*

Proof. Proof of (1): It suffices to prove the commutativity on the reference tetrahedron $\hat{K}$. Consider a smooth function $u(x,y)$ first. Recalling the expression for $\mathcal{E}^{\mathrm{grad}}$ on $\hat{K}$ from [7] and differentiating it, we have

$$
\begin{aligned}
\mathbf{grad}\, \mathcal{E}^{\mathrm{grad}} u &= 2\, \mathbf{grad} \int_0^1 \int_0^{1-t} u(x+sz, y+tz)\, ds\, dt \\
&= 2 \int_0^1 \int_0^{1-t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s & t \end{pmatrix} \mathbf{grad}_\tau\, u(x+sz, y+tz)\, ds\, dt \\
&= \boldsymbol{\mathcal{E}}^{\mathrm{curl}}\, \mathbf{grad}_\tau\, u.
\end{aligned}
$$

Here we have viewed gradients as column vectors, so the matrix above multiplies $\mathbf{grad}_\tau\, u = (\partial_x u, \partial_y u)^t$. Thus we have shown that the identity $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}(\mathbf{grad}_\tau\, u) = \mathbf{grad}(\mathcal{E}_l^{\mathrm{grad}} u)$ holds for all smooth functions u. Now, by [7, Theorem 2.1] (asserting the continuity of $\mathcal{E}_l^{\mathrm{grad}} u$ on $H^{1/2}(F_l)$), we have

$$
\|\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}\, \mathbf{grad}_\tau\, u\| = \|\, \mathbf{grad}\, \mathcal{E}_l^{\mathrm{grad}} u\| \le C \|u\|_{H^{1/2}(F_l)}.
$$

Hence the operator $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}$ extends continuously to the space $\mathbf{grad}\, H^{1/2}(F_l)$, so the commutativity property holds for all $u \in H^{1/2}(F_l)$.

Proof of (2): The continuity of $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ on $\boldsymbol{H}^{1/2}(\hat{F})$ follows by applying Lemma 3.1 to each of the components of $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}$ in (3.1). Since the Jacobian of the covariant transformation mapping functions on $\hat{K}$ to K is bounded, the result follows for $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}$ on any K.

Proof of (3): Given any $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}(F_l)$, decompose it using Theorem 2.1 to get

$$
\boldsymbol{v} = \mathbf{grad}_\tau\, \phi + \boldsymbol{\psi}, \qquad \text{with } \phi \in H^{1/2}(F_l),\ \boldsymbol{\psi} \in \boldsymbol{H}^{1/2}(F_l).
$$

Then

$$
\begin{aligned}
\|\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} &= \|\, \mathbf{grad}(\mathcal{E}_l^{\mathrm{grad}}\phi) + \boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}\boldsymbol{\psi}\|_{\boldsymbol{H}(\mathbf{curl})}, && \text{by item (1),} \\
&\le \|\mathcal{E}_l^{\mathrm{grad}}\phi\|_{H^1(K)} + \|\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}\boldsymbol{\psi}\|_{\boldsymbol{H}^1(K)}, \\
&\le C\left(\|\phi\|_{H^{1/2}(F_l)} + \|\boldsymbol{\psi}\|_{\boldsymbol{H}^{1/2}(F_l)} \right), && \text{by item (2) and [7, Theorem 2.1],} \\
&\le C \|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}(F_l)}, && \text{by stability (Theorem 2.1).}
\end{aligned}
$$

Proof of (4): Set $z = 0$ in (3.1). Then the result is obvious for smooth functions $\boldsymbol{v}$. Because of the continuity of $\boldsymbol{\mathcal{E}}_l^{\mathrm{curl}}$, the result follows for all functions in $\boldsymbol{X}^{-1/2}(F_l)$.

Proof of (5): It suffices to prove the polynomial preservation properties for the expression (3.1) on the reference tetrahedron $\hat{K}$ because the affine covariant mapping preserves both the polynomial spaces $\boldsymbol{P}_p(K)$ and $\boldsymbol{N}_p(K)$ [14].

So, consider a $\boldsymbol{v} \in \boldsymbol{P}_p(\hat{F})$. Then, each of the components of the integrand defining the extension $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}$ in (3.1) is a polynomial in x, y and z with coefficients depending on s and t. Hence, after integrating over s and t, we continue to have a polynomial in x, y, and z of degree at most p in x, y and z for each component.

Now suppose $\boldsymbol{v} \in \boldsymbol{N}_p$. Observe that

$$
\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v} &= 2 \int_0^1 \int_0^{1-t} \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s & t \end{pmatrix} \boldsymbol{v}(x+sz, y+tz)\, ds\, dt \\
&= 2 \int_0^1 \int_0^{1-t} \begin{pmatrix} x+sz \\ y+tz \end{pmatrix} \cdot \boldsymbol{v}(x+sz, y+tz)\, ds\, dt,
\end{aligned}
$$

By (3.7), $\boldsymbol{v} \cdot \boldsymbol{x}$ is a polynomial of degree at most $p+1$, hence the integrand in the last integral is a polynomial in $x+sz$ and $y+tz$ of degree at most p. Therefore, by repeating the argument of the previous paragraph, we find that $\boldsymbol{x} \cdot \boldsymbol{\mathcal{E}}^{\text{curl}}\boldsymbol{v}$ is a polynomial of degree at most $p+1$. Hence by (3.7), $\boldsymbol{\mathcal{E}}^{\text{curl}}\boldsymbol{v}$ is in $\boldsymbol{N}_p$. □

As in the H^1 case described in [7], the next step is to solve the two-face problem, for which we shall need a correction operator.

4. Face corrections

In general, the tangential traces of $\boldsymbol{\mathcal{E}}_l^{\text{curl}}\boldsymbol{v}$ are not zero on faces other than F_l even when $\boldsymbol{v}$ is a smooth function that vanishes on ∂F_l. Therefore, we must add a face correction. The face correction can be thought of as the solution to the $\boldsymbol{H}(\mathbf{curl})$ *two-face problem*: This problem concerns a polynomial $\boldsymbol{v}$ defined on F_l such that $\boldsymbol{v} \cdot \boldsymbol{t}|_{E_{jk}} = 0$, where $\boldsymbol{t}$ is the unit tangent vector along the edge E_{jk}. The problem is to find a polynomial extension with zero tangential trace on the face F_i.

We begin, as before, with the case of the reference tetrahedron $\hat{K}$. Suppose $\boldsymbol{v}$ is a polynomial defined on the x-y face $\hat{F}$ such that $\boldsymbol{v} \cdot \boldsymbol{t}|_{\hat{E}_{02}} = 0$ where $\boldsymbol{t}$ is the unit tangent vector along the edge. Then, we will first give an operator that maps $\boldsymbol{v}$ to a polynomial in $\hat{K}$ whose tangential trace on the x-y face vanishes, and whose tangential trace on the y-z face coincides with that of the primary extension of $\boldsymbol{v}$. Then subtracting this operator from the primary extension, we can solve the two-face problem. Define the face correction by

$$\begin{aligned}\boldsymbol{\mathcal{E}}^{\text{curl}}_{\hat{F}_1}\boldsymbol{v} =& \frac{2z}{x+z}\int_0^1\int_0^{1-t}\begin{pmatrix} s & t \\ 0 & 1 \\ s & t\end{pmatrix}\boldsymbol{v}(s(x+z), y+t(x+z))\,ds\,dt \\ &+ \frac{1}{x+z}\begin{pmatrix} -z \\ 0 \\ x\end{pmatrix}\int_0^1\int_0^{1-t}\begin{pmatrix} 1-s \\ -t\end{pmatrix}\cdot \boldsymbol{v}(s(x+z), y+t(x+z))\,ds\,dt.\end{aligned} \tag{4.1}$$

Before we give the properties of this correction operator, we briefly indicate how we derived the above expression. As in the case of the primary extension, we obtained the expression above by computing the gradient of the corresponding H^1 operator, namely the operator $\mathcal{E}^{\text{grad}}_{\hat{F}_1}$ given in [7] and observing what is needed for satisfying a commutativity property. Indeed, recalling the expression for $\mathcal{E}^{\text{grad}}_{\hat{F}_1}u$ and differentiating,

$$\begin{aligned}\mathbf{grad}\,\mathcal{E}^{\text{grad}}_{\hat{F}_1}u =& \frac{z}{x+z}\,\mathbf{grad}\,\mathcal{E}^{\text{grad}}u(0,y,x+z) + \mathcal{E}^{\text{grad}}u(0,y,x+z)\,\mathbf{grad}\left(\frac{z}{x+z}\right) \\ =& \frac{2z}{x+z}\int_0^1\int_0^{1-t}\begin{pmatrix} s & t \\ 0 & 1 \\ s & t\end{pmatrix}\mathbf{grad}_\tau\, u(s(x+z), y+t(x+z))\,ds\,dt \\ &+ \frac{2}{(x+z)^2}\begin{pmatrix} -z \\ 0 \\ x\end{pmatrix}\int_0^1\int_0^{1-t} u(s(x+z), y+t(x+z))\,ds\,dt.\end{aligned} \tag{4.2}$$

Therefore, in order to verify the commutativity property $\boldsymbol{\mathcal{E}}^{\text{curl}}_{\hat{F}_1}(\mathbf{grad}_\tau\, u) = \mathbf{grad}(\mathcal{E}^{\text{grad}}_{\hat{F}_1}u)$, we need to express the last term above in terms of $\mathbf{grad}_\tau\, u$ alone.

Since such a situation will recur often in this paper, we now describe our approach to handle this in some detail. To convert (4.2) into an expression depending on $\mathbf{grad}_\tau\, u$ alone, recall that for the two-face correction, we only need the commutativity for functions u that vanish along the edge on the y-axis. So we can apply the fundamental theorem of calculus

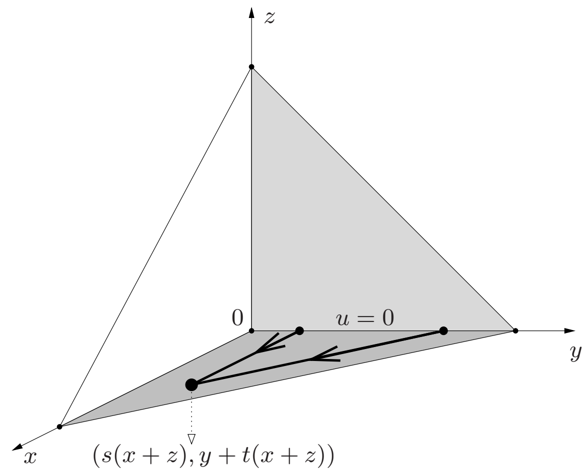

FIGURE 2. Integration paths symmetrizing the face correction $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1}\boldsymbol{v}$

and write

$$u(s(x+z), y+t(x+z)) = \int_0^{s(x+z)} \begin{pmatrix}1\\0\end{pmatrix} \cdot \mathbf{grad}_\tau\, u(r, y+t(x+z))\, dr. \tag{4.3}$$

Here we have chosen one of the many possible paths of integration. However, this choice is not invariant under affine automorphisms of $\hat{K}$ (that fix $\hat{\boldsymbol{a}}_1$ and $\hat{\boldsymbol{a}}_3$), because it can be mapped into the path in

$$u(s(x+z), y+t(x+z)) = \int_0^{s(x+z)} \begin{pmatrix}1\\-1\end{pmatrix} \cdot \mathbf{grad}_\tau\, u(r, y+(s+t)(x+z)-r)\, dr. \tag{4.4}$$

Hence, we must replace $u(s(x+z), y+t(x+z))$ in (4.2) by the average of the right hand sides of (4.3) and (4.4). (The paths in both the integrals are illustrated in Fig. 2, from which the symmetry with respect to the interchange of the two vertices on the y-axis is obvious.) After this replacement of u in (4.2), we have

$$\begin{aligned}
\mathbf{grad}\, \mathcal{E}^{\mathrm{grad}}_{\hat{F}_1} u = {} & \frac{2z}{x+z} \int_0^1 \int_0^{1-t} \begin{pmatrix} s & t\\ 0 & 1\\ s & t\end{pmatrix} \mathbf{grad}_\tau\, u(s(x+z), y+t(x+z))\, ds\, dt \\
& + \frac{2}{(x+z)^2} \begin{pmatrix}-z\\0\\x\end{pmatrix} \int_0^1 \int_0^{1-t} \frac{1}{2} \int_0^{s(x+z)} \begin{pmatrix}1\\0\end{pmatrix} \cdot \mathbf{grad}_\tau\, u(r, y+t(x+z))\, dr\, ds\, dt \\
& + \frac{2}{(x+z)^2} \begin{pmatrix}-z\\0\\x\end{pmatrix} \int_0^1 \int_0^{1-t} \frac{1}{2} \int_0^{s(x+z)} \begin{pmatrix}1\\-1\end{pmatrix} \cdot \mathbf{grad}_\tau\, u(r, y+(s+t)(x+z)-r)\, dr\, ds\, dt.
\end{aligned}$$

The last two terms above can be simplified so that the entire sum matches the expression for $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1}(\mathbf{grad}_\tau\, u)$ given by (4.1). The details are in the proof of the following lemma (in Appendix A), which gives several symmetry preserving ways to rewrite integrals of a scalar function in terms of its derivatives. This completes the discussion motivating the definition of the face correction operator in (4.1). A rigorous proof of the required commutativity property using the following lemma is in the proof of the succeeding proposition.

Lemma 4.1. *Let $u(s,t)$ be a smooth function on the unit triangle $\hat{F}$.*

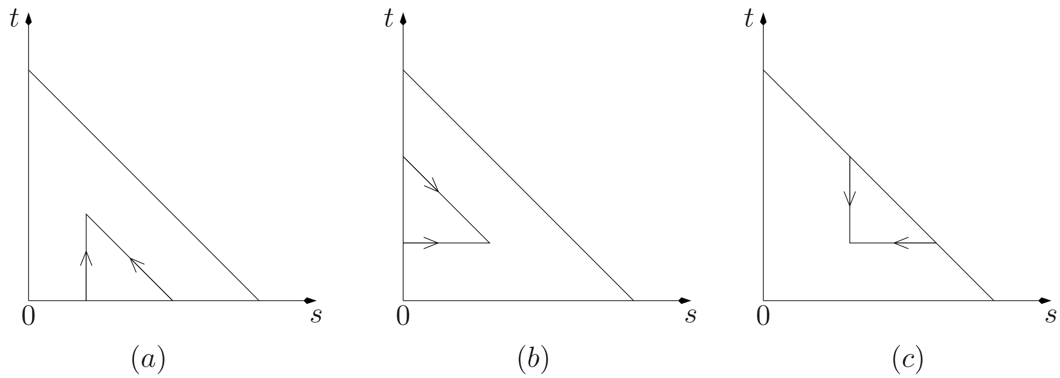

FIGURE 3. Integration paths for (the proof of) Lemma 4.1.

(1) *If* $u(0,t) = 0$ *then (integration along the two paths in* Fig. 3(a) *yields)*

$$\iint_{\hat{F}} u(s,t)\, ds\, dt = \frac{1}{2} \iint_{\hat{F}} \left((1-s)\frac{\partial u}{\partial s} + (-t)\frac{\partial u}{\partial t} \right)\, ds\, dt.$$

(2) *If* $u(s,0) = 0$ *then (integration along the two paths in* Fig. 3(b) *yields)*

$$\iint_{\hat{F}} u(s,t)\, ds\, dt = \frac{1}{2} \iint_{\hat{F}} \left((-s)\frac{\partial u}{\partial s} + (1-t)\frac{\partial u}{\partial t} \right)\, ds\, dt.$$

(3) *If* $u(s,1-s) = 0$ *then (integration along the two paths in* Fig. 3(c) *yields)*

$$\iint_{\hat{F}} u(s,t)\, ds\, dt = \frac{1}{2} \iint_{\hat{F}} \left((-s)\frac{\partial u}{\partial s} + (-t)\frac{\partial u}{\partial t} \right)\, ds\, dt.$$

Before we give the proposition detailing the properties involving our face correction, it will be useful to generalize the expression (4.1) to a general tetrahedron K in affine coordinates. To do this, we first split the given smooth tangential vector function $\boldsymbol{v}$ into components v_m as in (3.3). Then substituting

$$\boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} v_0 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_2$$

into the integrands in (4.1) and simplifying, we have

$$\frac{2z}{x+z} \begin{pmatrix} s & t \\ 0 & 1 \\ s & t \end{pmatrix} \boldsymbol{v} = \frac{2\lambda_3}{(\lambda_1+\lambda_3)} \big(D_j\boldsymbol{v}\ \ \mathbf{grad}\,\lambda_j + D_k\boldsymbol{v}\ \ \mathbf{grad}\,\lambda_k \big),$$

$$\frac{1}{x+z} \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} \begin{pmatrix} 1-s \\ -t \end{pmatrix} \cdot \boldsymbol{v} = \frac{\lambda_1\,\mathbf{grad}\,\lambda_3 - \lambda_3\,\mathbf{grad}\,\lambda_1}{(\lambda_1+\lambda_3)^3} D_1\boldsymbol{v},$$

where $D_\ell \boldsymbol{v}$ is as in (3.6) but now with $\widetilde{\lambda}_j(\boldsymbol{s})$ in (3.6) denoting the barycentric coordinates of the current region of integration, namely that of $T_3(0, \lambda_0, \lambda_2)$. Thus (4.1) becomes

$$
\begin{aligned}
\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{v} \; &= \frac{\lambda_1 \,\mathbf{grad}\, \lambda_3 - \lambda_3 \,\mathbf{grad}\, \lambda_1}{(\lambda_1 + \lambda_3)^3} \iint\limits_{T_3(0,\lambda_2,\lambda_0)} D_1 \boldsymbol{v} \\
&+ \frac{2\lambda_3}{(\lambda_1 + \lambda_3)^3} \iint\limits_{T_3(0,\lambda_2,\lambda_0)} (D_0 \boldsymbol{v} \,\mathbf{grad}\, \lambda_0 + D_2 \boldsymbol{v} \,\mathbf{grad}\, \lambda_2).
\end{aligned} \tag{4.5}
$$

In generalizing this operator as an extension into a general tetrahedron K from face F_l, we note that the region of integration becomes $T_l(0, \lambda_j, \lambda_k)$ (and $\widetilde{\lambda}_\ell$ becomes the barycentric coordinates of this region). Thus we have the following expression

$$
\begin{aligned}
\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_i,l} \boldsymbol{v} \; &= \frac{\lambda_i \,\mathbf{grad}\, \lambda_l - \lambda_l \,\mathbf{grad}\, \lambda_i}{(\lambda_i + \lambda_l)^3} \iint\limits_{T_l(0,\lambda_j,\lambda_k)} D_i \boldsymbol{v} \;\; d\boldsymbol{s} \\
&+ \frac{2\lambda_l}{(\lambda_i + \lambda_l)^3} \sum_{m \in \{j,k\}} \mathbf{grad}\, \lambda_m \iint\limits_{T_l(0,\lambda_j,\lambda_k)} D_m \boldsymbol{v} \;\; d\boldsymbol{s},
\end{aligned} \tag{4.6}
$$

which coincides with the expression in (4.5) when $(i, j, k) = (1, 2, 0)$. Clearly, if all the components of $\boldsymbol{v}$ coincide with a single function (so that $\boldsymbol{v}$ vanishes), the result of this extension is zero, so it is well defined. Note that this expression is symmetric with respect to indices j and k.

Now we can solve the $\boldsymbol{H}(\mathbf{curl}\,)$ two-face problem mentioned in the beginning of this section by subtracting the above operator from the primary extension. The operator that solves the two-face problem is

$$
\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l} \boldsymbol{v} = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{l} \boldsymbol{v} - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_i,l} \boldsymbol{v}. \tag{4.7}
$$

The following continuity from a positive order Sobolev space is established in Appendix A:

Lemma 4.2. $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}$ *is a continuous map from* $\boldsymbol{H}^{1/2}_{0,i}(F_l)$ *into* $\boldsymbol{H}(\mathbf{curl}\,)$.

Nonetheless, we need its continuity of $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}$ from an $\boldsymbol{H}(\mathbf{curl}\,)$ trace space. This is proved in the next proposition, where we also prove its other properties.

Proposition 4.1. *The two face extension* $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}$ *satisfies the following:*

(1) Commutativity: $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l} \,\mathbf{grad}_\tau\, u = \mathbf{grad}(\mathcal{E}^{\mathrm{grad}}_{i,l} u)$ *for all* $u \in H^{1/2}_{0,i}(F_l)$.

(2) Continuity: $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}$ *extends to a continuous operator from* $\boldsymbol{X}^{-1/2}_{0,i}(F_l)$ *into* $\boldsymbol{H}(\mathbf{curl}\,)$.

(3) Extension property: *For all* $\boldsymbol{v} \in \boldsymbol{X}^{-1/2}_{0,i}(F_l)$,

$$
\mathrm{trc}_\tau(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l} \boldsymbol{v})\big|_{F_i} = \mathbf{0}, \qquad \mathrm{trc}_\tau(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l} \boldsymbol{v})\big|_{F_l} = \boldsymbol{v}.
$$

(4) Polynomial preservation: *Suppose* $\boldsymbol{v} \in \boldsymbol{P}_p(F_l)$ *is such that* $\boldsymbol{v} \cdot \boldsymbol{t} = 0$ *on the edge* E_{jk}*. Then the extension* $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l} \boldsymbol{v}$ *is in* $\boldsymbol{P}_p(K)$*. If in addition* $\boldsymbol{v}$ *is in the Nédélec space* $\boldsymbol{N}_p(F_l)$*, then its extension* $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l} \boldsymbol{v}$ *is in* $\boldsymbol{N}_p(K)$.

Proof. Proof of (1): It suffices to prove this identity for smooth functions u on F_l vanishing on the edge where λ_i is zero. Indeed, once the identity is established for such functions, the continuity of $\mathcal{E}^{\mathrm{grad}}_{i,l}$ established in [7] implies that the operator $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}$ extends continuously to $\mathbf{grad}\, H^{1/2}_{0,i}(F_l)$ wherein the commutativity property holds (by a minor modification of the argument in the proof of Theorem 3.1(1)). Furthermore, because of Theorem 3.1(1), we only need to prove that $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_i,l}\,\mathbf{grad}_\tau\, u = \mathbf{grad}(\mathcal{E}^{\mathrm{grad}}_{F_i,l}u)$. To prove this identity, we obviously only need to prove its analogue on the reference tetrahedron $\hat{K}$, namely

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1}(\mathbf{grad}_\tau\, u) = \mathbf{grad}(\mathcal{E}^{\mathrm{grad}}_{\hat{F}_1}u). \tag{4.8}$$

Here $\mathcal{E}^{\mathrm{grad}}_{\hat{F}_1}$ is the corresponding operator given in [7] and $u(x,y)$ is a smooth function vanishing on the y-axis.

To prove (4.8), we start by computing the gradient on the right hand side of (4.8), which we have already done in (4.2). To convert (4.2) into an expression depending on $\mathbf{grad}_\tau\, u$ alone, we use Lemma 4.1. Applying Lemma 4.1(2) to the last term in (4.2) we get

$$\begin{aligned}
&\frac{2}{(x+z)^2}\begin{pmatrix}-z\\0\\x\end{pmatrix}\int_0^1\!\!\int_0^{1-t} u(s(x+z),y+t(x+z))\,ds\,dt\\
&\qquad = \frac{2}{(x+z)^2}\begin{pmatrix}-z\\0\\x\end{pmatrix}\int_0^1\!\!\int_0^{1-t}\frac{1}{2}\Big((1-s)\frac{\partial}{\partial s}-t\frac{\partial}{\partial t}\Big)u(s(x+z),y+t(x+z))\,ds\,dt,
\end{aligned}$$

hence

$$\begin{aligned}
\mathbf{grad}\,\mathcal{E}^{\mathrm{grad}}_{\hat{F}_1}u &= \frac{2z}{x+z}\int_0^1\!\!\int_0^{1-t}\begin{pmatrix}s & t\\0 & 1\\s & t\end{pmatrix}\mathbf{grad}_\tau\, u(s(x+z),y+t(x+z))\,ds\,dt\\
&\quad + \frac{1}{x+z}\begin{pmatrix}-z\\0\\x\end{pmatrix}\int_0^1\!\!\int_0^{1-t}\begin{pmatrix}1-s\\-t\end{pmatrix}\cdot\mathbf{grad}_\tau\, u(s(x+z),y+t(x+z))\,ds\,dt,
\end{aligned}$$

which is the same as $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1}(\mathbf{grad}_\tau\, u)$.

Proof of (2): To prove the continuity estimate, apply Theorem 2.1 and decompose $\boldsymbol{v}$ as

$$\boldsymbol{v} = \mathbf{grad}_\tau\,\phi + \boldsymbol{\psi}, \qquad \text{with } \phi\in H^{1/2}_{0,i}(F_l), \text{ and } \boldsymbol{\psi}\in \boldsymbol{H}^{1/2}_{0,i}(F_l).$$

Then,

$$\begin{aligned}
\|\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} &= \|\,\mathbf{grad}(\mathcal{E}^{\mathrm{grad}}_{i,l}\phi)+\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}\boldsymbol{\psi}\|_{\boldsymbol{H}(\mathbf{curl})}, && \text{by commutativity,}\\
&\le C\Big(\|\phi\|_{H^{1/2}_{0,i}(F_l)}+\|\boldsymbol{\psi}\|_{\boldsymbol{H}^{1/2}_{0,i}(F_l)}\Big), && \text{by [7, Prop. 3.1] and Lemma 4.2,}\\
&\le C\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}_{0,i}(F_l)}, && \text{by Theorem 2.1.}
\end{aligned}$$

Proof of (3): Since $\lambda_i = 0$ on F_i,

$$\begin{aligned}
\mathrm{trc}_\tau(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{l}\boldsymbol{v})|_{F_i} &= \frac{2}{\lambda_l^2}\iint\limits_{T_l(0,\lambda_j,\lambda_k)}\sum_{m\in\{i,j,k\}}D_m\boldsymbol{v}(\boldsymbol{s})\;\mathbf{grad}_\tau\,\lambda_m\;\;d\boldsymbol{s}, && \text{by (3.5),}\\
\mathrm{trc}_\tau(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_i,l}\boldsymbol{v})|_{F_i} &= \frac{2\lambda_l}{(\lambda_i+\lambda_l)^3}\iint\limits_{T_l(0,\lambda_j,\lambda_k)}\sum_{m\in\{j,k\}}D_m\boldsymbol{v}\;\mathbf{grad}_\tau\,\lambda_m\;\;d\boldsymbol{s}, && \text{by (4.6),}
\end{aligned}$$

as $\mathrm{trc}_\tau(\lambda_i\,\mathbf{grad}\,\lambda_l-\lambda_l\,\mathbf{grad}\,\lambda_i)|_{F_i}=\mathbf{0}$. Therefore,

$$\mathrm{trc}_\tau(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}\boldsymbol{v})|_{F_i} = \mathrm{trc}_\tau(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{l}\boldsymbol{v}-\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_i,l}\boldsymbol{v})|_{F_i} = \mathbf{0}.$$

Proof of (4): As in the proof of Theorem 3.1(5), it suffices to prove the polynomial preservation properties for the expression (4.1) on $\hat{K}$.

Any polynomial $\boldsymbol{v}(x,y)$ in $\boldsymbol{P}_p(\hat{F})$ whose tangential component along the y-axis vanishes, can be written as

$$\boldsymbol{v}(x,y) = \begin{pmatrix} v_1(x,y) \\ x v_2(x,y) \end{pmatrix} \tag{4.9}$$

for some $v_1 \in P_p(\hat{F})$ and $v_2 \in P_{p-1}(\hat{F})$. This implies

$$\begin{aligned} \boldsymbol{v}(x,y) &= \boldsymbol{v} - \mathbf{grad}_\tau(x v_1) + \mathbf{grad}_\tau(x v_1) \\ &= \begin{pmatrix} v_1 - v_1 - x\partial_x v_1 \\ x v_2 - x \partial_y v_1 \end{pmatrix} + \mathbf{grad}_\tau(x v_1) \\ &= x\, \widetilde{\boldsymbol{v}} + \mathbf{grad}_\tau(x v_1), \qquad \text{with } \widetilde{\boldsymbol{v}} = \begin{pmatrix} -\partial_x v_1 \\ v_2 - \partial_y v_1 \end{pmatrix} \in \boldsymbol{P}_{p-1}(\hat{F}). \end{aligned}$$

With this decomposition,

$$\begin{aligned} \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{v} &= \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1}(x\, \widetilde{\boldsymbol{v}} + \mathbf{grad}_\tau(x v_1)), \\ &= \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1}(x\, \widetilde{\boldsymbol{v}}) + \mathbf{grad}\, \mathcal{E}^{\mathrm{grad}}_{\hat{F}_1}(x v_1), \qquad \text{by commutativity.} \end{aligned}$$

By the polynomial preservation properties of $\mathcal{E}^{\mathrm{grad}}_{\hat{F}_1}$ established in [7], the last term is clearly in $\boldsymbol{P}_p(\hat{K})$. For the remaining term, referring to (4.1), we find that

$$\begin{aligned} \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{v} = & \frac{2z}{x+z} \int_0^1 \int_0^{1-t} \begin{pmatrix} s & t \\ 0 & 1 \\ s & t \end{pmatrix} s(x+z) \widetilde{\boldsymbol{v}}(s(x+z), y+t(x+z))\, ds\, dt \quad + \\ & \frac{1}{x+z} \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} \int_0^1 \int_0^{1-t} \begin{pmatrix} 1-s \\ -t \end{pmatrix} \cdot s(x+z) \widetilde{\boldsymbol{v}}(s(x+z), y+t(x+z))\, ds\, dt, \end{aligned}$$

so the $x+z$ term in the denominator cancels out. Since $\widetilde{\boldsymbol{v}} \in \boldsymbol{P}_{p-1}(\hat{F})$, by arguments similar to the proof of Theorem 3.1(5), we find that $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{v}$ is in $\boldsymbol{P}_p(\hat{K})$.

To prove that the Nédélec space is preserved, observe that (4.1) implies

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{v} = \frac{2z}{x+z} \int_0^1 \int_0^{1-t} \begin{pmatrix} s(x+z) \\ y + t(x+z) \end{pmatrix} \cdot \boldsymbol{v}(s(x+z), y+t(x+z))\, ds\, dt.$$

If $\boldsymbol{v}$ is in $\boldsymbol{N}_p(\hat{F})$, then by (3.7), the integrand is a polynomial of degree at most $p+1$. Furthermore, since $\boldsymbol{v}$ has the form in (4.9), the integrand has $s(x+z)$ as a scalar factor. Hence the $x+z$ term in the denominator cancels out. Usual arguments then yield that $\boldsymbol{x} \cdot \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{F}_1} \boldsymbol{v}$ is in $P_{p+1}(\hat{K})$, so the proof is finished by appealing to (3.7) again. □

5. Edge corrections

As in the H^1 case, edge corrections are necessary now, because successive applications of different face corrections alter the previously zeroed tangential traces. Consider the *three-face problem* of finding a polynomial extension of $\boldsymbol{v}$ given on face F_l that has zero tangential trace on F_i and F_j whenever $\boldsymbol{v}$ is a smooth function whose tangential component vanishes on edges E_{jk} and E_{ik}. To solve this intermediate problem, we define the next operator.

Beginning with the case of the reference tetrahedron $\hat{K}$, let $\boldsymbol{v}$ be a smooth function on the x-y face $\hat{F}$ whose tangential components along the edges on x and y axes vanish. Define the edge correction for the edge along the z-axis by

$$
\begin{aligned}
\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{E}_{03}}\boldsymbol{v}\,(x,y,z) = {}& \frac{1}{x+y+z}\begin{pmatrix}-z\\0\\x\end{pmatrix}\int_0^1\int_0^{1-t}\begin{pmatrix}1-s\\-t\end{pmatrix}\cdot\boldsymbol{v}(s(x+y+z),t(x+y+z))\,ds\,dt\\
&+\frac{1}{x+y+z}\begin{pmatrix}0\\-z\\y\end{pmatrix}\int_0^1\int_0^{1-t}\begin{pmatrix}-s\\1-t\end{pmatrix}\cdot\boldsymbol{v}(s(x+y+z),t(x+y+z))\,ds\,dt\\
&+\frac{2z}{x+y+z}\int_0^1\int_0^{1-t}\begin{pmatrix}s&t\\s&t\\s&t\end{pmatrix}\boldsymbol{v}(s(x+y+z),t(x+y+z))\,ds\,dt.
\end{aligned}
\tag{5.1}
$$

As in the previous sections, we next generalize this expression to the case of the edge E_{il} of the general tetrahedron K. Split $\boldsymbol{v}$ into components as in (3.3), then substitute this form of $\boldsymbol{v}$ into (5.1). A few simplifications then transform the above expression to

$$
\begin{aligned}
\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{E}_{03}}\boldsymbol{v}\,(x,y,z) = {}&\frac{\lambda_2\,\mathbf{grad}\,\lambda_3-\lambda_3\,\mathbf{grad}\,\lambda_2}{(1-\lambda_0)^3}\iint\limits_{T_3(0,0,\lambda_0)} D_2\boldsymbol{v}\;d\boldsymbol{s}\\
&+\frac{\lambda_0\,\mathbf{grad}\,\lambda_3-\lambda_3\,\mathbf{grad}\,\lambda_0}{(1-\lambda_0)^3}\iint\limits_{T_3(0,0,\lambda_0)} D_0\boldsymbol{v}\;d\boldsymbol{s}\\
&+\frac{2\lambda_3\,\mathbf{grad}\,\lambda_0}{(1-\lambda_0)^3}\iint\limits_{T_3(0,0,\lambda_0)} D_0\boldsymbol{v}\;d\boldsymbol{s}.
\end{aligned}
$$

Thus we obtain the general formula on any tetrahedron K:

$$
\begin{aligned}
\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{E_{il},l}\boldsymbol{v} = {}&\sum_{m\in\{j,k\}}\frac{\lambda_m\,\mathbf{grad}\,\lambda_l-\lambda_l\,\mathbf{grad}\,\lambda_m}{(1-\lambda_i)^3}\iint\limits_{T_l(0,0,\lambda_i)} D_m\boldsymbol{v}\;d\boldsymbol{s}\\
&+\frac{2\lambda_l\,\mathbf{grad}\,\lambda_i}{(1-\lambda_i)^3}\iint\limits_{T_l(0,0,\lambda_i)} D_i\boldsymbol{v}\;d\boldsymbol{s},
\end{aligned}
\tag{5.2}
$$

where $D_m\boldsymbol{v}$ is as defined in (3.6) but now with $\widetilde{\lambda}_j(\boldsymbol{s})$ denoting the barycentric coordinates of the current region of integration $T(0,0,\lambda_i)$. It is easy to check that if all $v_i=\bar{v}$, then the expression above vanishes, so it is independent of the non-uniqueness in the splitting in (3.3).

Let us now solve the three-face problem. The required extension operator is

$$
\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ij,l} = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{l} - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_i,l} - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_j,l} + \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{E_{kl},l},
\tag{5.3}
$$

whose properties appear in the next proposition. As in the case of the face correction, to analyze this operator, we first establish a continuity property in a positive order Sobolev space, as seen in the next lemma (proved in Appendix A).

Lemma 5.1. $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ij,l}$ *is a continuous operator from* $\boldsymbol{H}^{1/2}_{0,ij}(F_l)$ *into* $\boldsymbol{H}(\mathbf{curl}\,)$.

We use this together with the trace decomposition to prove the required continuity from the trace space. All the properties of this extension we shall need are in the next proposition.

Proposition 5.1. *The three face extension* $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ij,l}$ *satisfies the following:*

(1) Commutativity: $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ij,l}\,\mathbf{grad}_\tau\, u = \mathbf{grad}(\mathcal{E}^{\mathrm{grad}}_{ij,l}u)$ *for all* $u\in H^{1/2}_{0,ij}(F_l)$.

(2) Continuity: $\boldsymbol{\mathcal{E}}^{\text{curl}}_{ij,l}$ *extends to a continuous operator from* $\boldsymbol{X}^{-1/2}_{0,ij}(F_l)$ *into* $\boldsymbol{H}(\mathbf{curl}\,)$.

(3) Extension property: *For all* $\boldsymbol{v} \in \boldsymbol{X}^{-1/2}_{0,ij}(F_l)$,

$$\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}^{\text{curl}}_{ij,l}\boldsymbol{v})\big|_{F_i} = \mathbf{0}, \qquad \operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}^{\text{curl}}_{ij,l}\boldsymbol{v})\big|_{F_j} = \mathbf{0}, \qquad \operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}^{\text{curl}}_{ij,l}\boldsymbol{v})\big|_{F_l} = \boldsymbol{v}.$$

(4) Polynomial preservation: *Suppose* $\boldsymbol{v} \in \boldsymbol{P}_p(F_l)$ *is such that* $\boldsymbol{v}\cdot\boldsymbol{t} = 0$ *on the edges* E_{jk} *and* E_{ik}. *Then the extension* $\boldsymbol{\mathcal{E}}^{\text{curl}}_{ij,l}\boldsymbol{v}$ *is in* $\boldsymbol{P}_p(K)$. *If in addition* $\boldsymbol{v}$ *is in the Nédélec space* $\boldsymbol{N}_p(F_l)$, *then* $\boldsymbol{\mathcal{E}}^{\text{curl}}_{ij,l}\boldsymbol{v}$ *is in* $\boldsymbol{N}_p(K)$.

Proof. Proof of (1): We will prove that

$$\boldsymbol{\mathcal{E}}^{\text{curl}}_{\hat{E}_{03}}(\mathbf{grad}_\tau\, u) = \mathbf{grad}(\mathcal{E}^{\text{grad}}_{\hat{E}_{03}} u) \tag{5.4}$$

for a smooth function $u(x,y)$ that vanishes along the x and y edges. The required commutativity property stated in item (1) then follows by arguments similar to those detailed in the proof of Proposition 3(1), which we shall not repeat here. To prove (5.4), we start by computing the gradient of the expression for $\mathcal{E}^{\text{grad}}_{\hat{E}_{03}} u$ given in [7]:

$$\begin{aligned}\mathbf{grad}(\mathcal{E}^{\text{grad}}_{\hat{E}_{03}} u) &= \frac{2z}{x+y+z}\int_0^1\!\!\int_0^{1-s} \begin{pmatrix} s & t\\ s & t\\ s & t\end{pmatrix} \mathbf{grad}_\tau\, u(s(x+y+z), t(x+y+z))\,dt\,ds\\ &\quad + \frac{2}{(x+y+z)^2}\begin{pmatrix} -z\\ -z\\ x+y\end{pmatrix}\int_0^1\!\!\int_0^{1-s} u(s(x+y+z), t(x+y+z)\,dt\,ds.\end{aligned} \tag{5.5}$$

We must now express the last integral in terms of surface gradients alone. Since u vanishes along the x and y-axis, we can apply parts (1) and (2) of Lemma 4.1 to the last term in (5.5). (While applying this lemma, as is clear from its proof, we are integrating along the path shown in Fig. 4, obtained by combining the paths in Fig. 3(a) and 3(b). Hence the symmetries with respect to the z-edge are not lost.)

$$\begin{aligned}\mathbf{grad}(\mathcal{E}^{\text{grad}}_{\hat{E}_{03}} u) &= \frac{2z}{x+y+z}\int_0^1\!\!\int_0^{1-t} \begin{pmatrix} s & t\\ s & t\\ s & t\end{pmatrix} \mathbf{grad}_\tau\, u(s(x+y+z), t(x+y+z))\,ds\,dt\\ &\quad + \frac{1}{x+y+z}\begin{pmatrix} -z\\ 0\\ x\end{pmatrix}\int_0^1\!\!\int_0^{1-t}\begin{pmatrix} 1-s\\ -t\end{pmatrix}\cdot \mathbf{grad}_\tau\, u(s(x+y+z), t(x+y+z))\,ds\,dt\\ &\quad + \frac{1}{x+y+z}\begin{pmatrix} 0\\ -z\\ y\end{pmatrix}\int_0^1\!\!\int_0^{1-t}\begin{pmatrix} -s\\ 1-t\end{pmatrix}\cdot \mathbf{grad}_\tau\, u(s(x+y+z), t(x+y+z))\,ds\,dt.\end{aligned}$$

This expression is the same as (5.1) with $\mathbf{grad}_\tau\, u$ in place of $\boldsymbol{v}$. Thus we have proved (5.4).

Proof of (2): We use the regular decomposition again: By Theorem 2.1,

$$\boldsymbol{v} = \mathbf{grad}_\tau\,\phi + \boldsymbol{\psi}, \qquad \text{with } \phi \in H^{1/2}_{0,ij}(F_l), \text{ and } \boldsymbol{\psi} \in \boldsymbol{H}^{1/2}_{0,ij}(F_l).$$

Applying the three face extension to this decomposition,

$$\begin{aligned}\|\boldsymbol{\mathcal{E}}^{\text{curl}}_{ij,l}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl}\,)} &= \|\,\mathbf{grad}(\mathcal{E}^{\text{grad}}_{ij,l}\phi) + \boldsymbol{\mathcal{E}}^{\text{curl}}_{ij,l}\boldsymbol{\psi}\|_{\boldsymbol{H}(\mathbf{curl}\,)}, && \text{by commutativity (item (1))},\\ &\le C\Big(\|\phi\|_{H^{1/2}_{0,ij}(F_l)} + \|\boldsymbol{\psi}\|_{\boldsymbol{H}^{1/2}_{0,ij}(F_l)}\Big), && \text{by [7, Prop. 4.1] and Lemma 5.1},\\ &\le C\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}_{0,ij}(F_l)}, && \text{by Theorem 2.1.}\end{aligned}$$

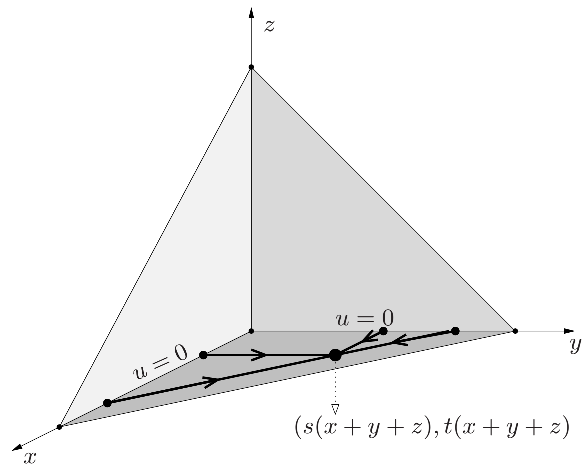

FIGURE 4. Integration paths symmetrizing the edge correction

Proof of (3): To show that $\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{ij,l}^{\text{curl}}\boldsymbol{v})\big|_{F_i} = 0$,

$$\begin{aligned}\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{ij,l}^{\text{curl}}\boldsymbol{v})\big|_{F_i} &= \operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{i,l}^{\text{curl}}\boldsymbol{v}) - \operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{F_j,l}^{\text{curl}}\boldsymbol{v}) + \operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{E_{kl},l}^{\text{curl}}\boldsymbol{v})\Big|_{F_i}, && \text{by (4.7)}\\ &= -\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{F_j,l}^{\text{curl}}\boldsymbol{v})|_{F_i} + \operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{E_{kl},l}^{\text{curl}}\boldsymbol{v})|_{F_i}, && \text{by Prop. 4.1(3).}\end{aligned}$$

Now, by (4.6) and (5.2),

$$\begin{aligned}\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{F_j,l}^{\text{curl}}\boldsymbol{v}) =& \frac{2\lambda_l}{(\lambda_j+\lambda_l)^3} \iint\limits_{T_l(0,\lambda_i,\lambda_k)} \sum_{m\in\{i,k\}} D_m\boldsymbol{v}\ \mathbf{grad}_\tau\,\lambda_m\ \ d\boldsymbol{s}\\ &+ \frac{\lambda_j\,\mathbf{grad}_\tau\,\lambda_l - \lambda_l\,\mathbf{grad}_\tau\,\lambda_j}{(\lambda_j+\lambda_l)^3}, \iint\limits_{T_l(0,\lambda_i,\lambda_k)} D_j\boldsymbol{v}\ d\boldsymbol{s}, \qquad \text{and}\end{aligned}$$

$$\begin{aligned}\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{E_{kl},l}^{\text{curl}}\boldsymbol{v}) =& \sum_{m\in\{i,j\}} \frac{\lambda_m\,\mathbf{grad}_\tau\,\lambda_l - \lambda_l\,\mathbf{grad}_\tau\,\lambda_m}{(1-\lambda_k)^3} \iint\limits_{T_l(0,0,\lambda_k)} D_m\boldsymbol{v}\ d\boldsymbol{s}\\ &+ \frac{2\lambda_l\,\mathbf{grad}_\tau\,\lambda_k}{(1-\lambda_k)^3} \iint\limits_{T_l(0,0,\lambda_k)} D_k\boldsymbol{v}\ d\boldsymbol{s}.\end{aligned}$$

These two expressions coincide on F_i because on F_i we have $\lambda_i = 0$, $\mathbf{grad}_\tau\,\lambda_i = \mathbf{0}$, $\lambda_j+\lambda_l = 1-\lambda_k$, and $T_l(0,\lambda_i,\lambda_k) = T_l(0,0,\lambda_k)$. Hence

$$\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{F_j,l}^{\text{curl}}\boldsymbol{v} - \boldsymbol{\mathcal{E}}_{E_{kl},l}^{\text{curl}}\boldsymbol{v})|_{F_i} = \mathbf{0}, \tag{5.6}$$

and so $\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{ij,l}^{\text{curl}}\boldsymbol{v})|_{F_i} = 0$. That $\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{ij,l}^{\text{curl}}\boldsymbol{v})|_{F_j} = 0$ now immediately follows because the expression for the three face extension $\boldsymbol{\mathcal{E}}_{ij,l}^{\text{curl}}$ is symmetric with respect to i and j. The third identity $\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}_{ij,l}^{\text{curl}}\boldsymbol{v})|_{F_l} = \boldsymbol{v}$ holds because all the correction operators have vanishing tangential traces on F_l.

Proof of (4): To show that the expression in (5.1) is in $\boldsymbol{P}_p(\hat{K})$ is easy. Indeed, since $\boldsymbol{v}$ has vanishing tangential components along both the x and y-axes, it has the form $\boldsymbol{v}(x,y) = (xv_1(x,y), yv_2(x,y))^t$. Hence the denominator term $x+y+z$ in (5.1) cancels out showing that $\boldsymbol{\mathcal{E}}_{\hat{E}_{03}}^{\text{curl}}\boldsymbol{v}$ is in $\boldsymbol{P}_p(\hat{K})$.

If $\boldsymbol{v}$ is in $\boldsymbol{N}_p(\hat{F})$, then since (5.1) implies

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \boldsymbol{\mathcal{E}}^{\text{curl}}_{\hat{E}_{03}} \boldsymbol{v} = \frac{2z}{x+y+z} \int_0^1 \int_0^{1-t} \begin{pmatrix} s(x+y+z) \\ t(x+y+z) \end{pmatrix} \cdot \boldsymbol{v}(s(x+y+z), t(x+y+z))\, ds\, dt,$$

and (3.7) implies $\boldsymbol{x} \cdot \boldsymbol{v}$ is in $P_{p+1}(\hat{F})$, we have $\boldsymbol{x} \cdot \boldsymbol{\mathcal{E}}^{\text{curl}}_{\hat{E}_{03}} \boldsymbol{v}$ is in $P_{p+1}(\hat{K})$. This proves the last statement of the proposition. □

6. Extension of a tangential face bubble

Now consider a tangential vector function on the face F_l of a general tetrahedron K, whose tangential components along all the three edges of F_l vanish. The *four-face problem* is the problem of finding an extension of $\boldsymbol{v}$ into K whose tangential traces are zero on all the other three faces of K.

We have all the main ingredients to solve the four-face problem right away. The required extension operator is

$$\boldsymbol{\mathcal{E}}^{\text{curl}}_{ijk,l} \boldsymbol{v} = \boldsymbol{\mathcal{E}}^{\text{curl}}_{l} \boldsymbol{v} \;-\; \boldsymbol{\mathcal{E}}^{\text{curl}}_{V_l} \boldsymbol{v} \;-\; \sum_{m \in \{i,j,k\}} \big(\boldsymbol{\mathcal{E}}^{\text{curl}}_{F_m,l} \boldsymbol{v} - \boldsymbol{\mathcal{E}}^{\text{curl}}_{E_{ml},l} \boldsymbol{v} \big), \tag{6.1}$$

where $\boldsymbol{\mathcal{E}}^{\text{curl}}_{l}$ is the primary extension operator defined in (3.5), $\boldsymbol{\mathcal{E}}^{\text{curl}}_{F_i,l}$ is the face correction operator defined in (4.6), $\boldsymbol{\mathcal{E}}^{\text{curl}}_{E_{il},l}$ is the edge correction operator defined in (5.2), and $\boldsymbol{\mathcal{E}}^{\text{curl}}_{V_l}$ is a *vertex correction operator* defined by

$$\boldsymbol{\mathcal{E}}^{\text{curl}}_{V_l} \boldsymbol{v} = \sum_{m \in \{i,j,k\}} (\lambda_m \,\mathbf{grad}\, \lambda_l - \lambda_l \,\mathbf{grad}\, \lambda_m) \iint_{F_l} D_m \boldsymbol{v}\; d\boldsymbol{s} \tag{6.2}$$

where $D_m \boldsymbol{v}$ is as defined before in (3.6) but now with $\widetilde{\lambda}_j(\boldsymbol{s})$ in (3.6) denoting the barycentric coordinates of F_l, i.e., now $\widetilde{\lambda}_j = \lambda_j|_{F_l}$.

Proposition 6.1. *The four-face extension* $\boldsymbol{\mathcal{E}}^{\text{curl}}_{ijk,l}$ *satisfies the following:*

1. Commutativity: $\boldsymbol{\mathcal{E}}^{\text{curl}}_{ijk,l} \,\mathbf{grad}_\tau\, u = \mathbf{grad}(\mathcal{E}^{\text{grad}}_{ijk,l} u)$ *for all* $u \in H^{1/2}_{0,ijk}(F_l)$.
2. Continuity: $\boldsymbol{\mathcal{E}}^{\text{curl}}_{ijk,l}$ *is a continuous map from* $\boldsymbol{X}^{-1/2}_{0,ijk}(F_l)$ *into* $\boldsymbol{H}(\mathbf{curl})$.
3. Extension property: *For all* $\boldsymbol{v} \in \boldsymbol{X}^{-1/2}_{0,ijk}(F_l)$, *the tangential traces of* $\boldsymbol{\mathcal{E}}^{\text{curl}}_{ijk,l} \boldsymbol{v}$ *on all faces of the tetrahedron are zero except for the face* F_l, *where it equals* $\boldsymbol{v}$.
4. Polynomial preservation: *Suppose* $\boldsymbol{v} \in \boldsymbol{P}_p(F_l)$ *is such that* $\boldsymbol{v} \cdot \boldsymbol{t} = 0$ *on* ∂F_l. *Then the extension* $\boldsymbol{\mathcal{E}}^{\text{curl}}_{ijk,l} \boldsymbol{v}$ *is in* $\boldsymbol{P}_p(K)$. *Furthermore, if* $\boldsymbol{v}$ *is in the Nédélec space* $\boldsymbol{N}_p(F_l)$, *then its extension* $\boldsymbol{\mathcal{E}}^{\text{curl}}_{ijk,l} \boldsymbol{v}$ *is in* $\boldsymbol{N}_p(K)$.

Proof. Proof of (1): We have already proven the commutativity properties of all the operators in (6.1) except $\boldsymbol{\mathcal{E}}^{\text{curl}}_{V_l}$. Therefore, it is enough to prove that

$$\boldsymbol{\mathcal{E}}^{\text{curl}}_{V_l} \,\mathbf{grad}_\tau\, u = \mathbf{grad}(\mathcal{E}^{\text{grad}}_{V_l} u), \quad \text{for all } u \in H^{1/2}_{0,ijk}(F_l), \tag{6.3}$$

for the operator $\mathcal{E}^{\text{grad}}_{V_l}$ defined in [7]. Furthermore, by mapping, it is enough to prove (6.3) for the specific case of the reference tetrahedron with $l = 3$. In this case, the right hand side of (6.3) simplifies to

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{V_3}(\mathbf{grad}_\tau\, u)\,(x,y,z) = \begin{pmatrix}0\\0\\1\end{pmatrix}\int_0^1\!\!\int_0^{1-s}\begin{pmatrix}-s\\-t\end{pmatrix}\cdot\mathbf{grad}_\tau\, u(s,t)\;ds\,dt\;\;+$$

$$\begin{pmatrix}-z\\0\\x\end{pmatrix}\int_0^1\!\!\int_0^{1-s}\begin{pmatrix}1\\0\end{pmatrix}\cdot\mathbf{grad}_\tau\, u\;ds\,dt + \begin{pmatrix}0\\-z\\y\end{pmatrix}\int_0^1\!\!\int_0^{1-s}\begin{pmatrix}0\\1\end{pmatrix}\cdot\mathbf{grad}_\tau\, u\;ds\,dt.$$

Because u vanishes on the boundary, the last two terms on the right hand side are zero, and the remaining term can be rewritten using Lemma 4.1(3):

$$\begin{aligned}\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{V_3}(\mathbf{grad}_\tau\, u) &= \begin{pmatrix}0\\0\\1\end{pmatrix}\int_0^1\!\!\int_0^{1-s}\begin{pmatrix}-s\\-t\end{pmatrix}\cdot\mathbf{grad}_\tau\, u(s,t)\;ds\,dt\\ &= \begin{pmatrix}0\\0\\1\end{pmatrix}\iint\limits_{\hat F} 2\,u(s,t)\;ds\,dt\\ &= \mathbf{grad}(\mathcal{E}^{\mathrm{grad}}_{V_3}u).\end{aligned}$$

Proof of (2): First observe that the continuity of the vertex correction $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{V_l}$ from $\boldsymbol{H}_{0,ijk}(F_l)$ into $\boldsymbol{H}^1(K)$ is obvious. To obtain the continuity stated in the proposition, we use Theorem 2.1: Split

$$\boldsymbol{v} = \mathbf{grad}_\tau\,\phi + \boldsymbol{\psi}, \qquad \text{with } \phi\in H^{1/2}_{0,ijk}(F_l), \text{ and } \boldsymbol{\psi}\in \boldsymbol{H}^{1/2}_{0,ijk}(F_l).$$

Then by the commutativity property already proved, $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ijk,l}\boldsymbol{v} = \mathbf{grad}(\mathcal{E}^{\mathrm{grad}}_{ijk,l}\phi) + \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ijk,l}\boldsymbol{\psi}$. Hence, using the obvious continuity of $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{V_l} : \boldsymbol{H}_{0,ijk}(F_l)\mapsto \boldsymbol{H}^1(K)$, we have

$$\begin{aligned}\|\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ijk,l}\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} &\le C\big(\|\phi\|_{H^{1/2}_{0,ijk}(F_l)} + \|\boldsymbol{\psi}\|_{\boldsymbol{H}^{1/2}_{0,ijk}(F_l)}\big), && \text{by [7, Prop. 5.1]},\\ &\le C\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}_{0,ijk}(F_l)}, && \text{by Theorem 2.1.}\end{aligned}$$

Proof of (3): To prove the extension property, we first rewrite the terms in (6.1) as

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ijk,l}\boldsymbol{v} = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l}\boldsymbol{v} - (\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_j,l}\boldsymbol{v} - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{E_{kl},l}\boldsymbol{v}) - (\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_k,l}\boldsymbol{v} - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{E_{jl},l}\boldsymbol{v}) + (\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{E_{il},l}\boldsymbol{v} - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{V_l}\boldsymbol{v}). \tag{6.4}$$

Note that in the course of the proof of Proposition 5.1(3), we have shown that $\mathrm{trc}_\tau(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_j,l}\boldsymbol{v} - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{E_{kl},l}\boldsymbol{v})$ vanishes on F_i – see (5.6). Hence the middle two terms in (6.4) have vanishing tangential traces on F_i. The first term also has vanishing tangential trace on F_i by Proposition 4.1(3). Hence,

$$\begin{aligned}\mathrm{trc}_\tau(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ijk,l}\boldsymbol{v})|_{F_i} &= \mathrm{trc}_\tau(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{E_{il},l}\boldsymbol{v} - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{V_l}\boldsymbol{v})|_{F_i}\\ &= \sum_{m\in\{j,k\}}\frac{(\lambda_m\,\mathbf{grad}_\tau\,\lambda_l - \lambda_l\,\mathbf{grad}_\tau\,\lambda_m)|_{F_i}}{(1-0)^3}\iint\limits_{T_l(0,0,0)} D_m\boldsymbol{v}\;d\boldsymbol{s}\\ &\qquad - \sum_{m\in\{i,j,k\}}(\lambda_m\,\mathbf{grad}_\tau\,\lambda_l - \lambda_l\,\mathbf{grad}_\tau\,\lambda_m)|_{F_i}\iint\limits_{F_l} D_m\boldsymbol{v}\;d\boldsymbol{s}\\ &= \mathbf{0},\end{aligned}$$

because, on the face F_i, we have $\lambda_i = 0$, $\mathbf{grad}_\tau\,\lambda_i = \mathbf{0}$, and $T_l(0,0,0) = F_l$. Since $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ijk,l}$ is symmetric with respect to i, j, and k, the above implies that the tangential trace vanishes on

$F_i \cup F_j \cup F_k$. That $\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}^{\rm curl}_{ijk,l}\boldsymbol{v})$ coincides with $\boldsymbol{v}$ on F_l follows because all correction operators in (6.1) have vanishing tangential traces on F_l, while the primary extension reproduces $\boldsymbol{v}$ as its tangential trace on F_l.

Proof of (4): From the expression (6.2), it is clear that the vertex correction always results in a lowest order function in the Nédélec space (a Whitney form). Hence, the polynomial preservation property follows from the already established results in Proposition 4.1(4) and Proposition 5.1(4). □

7. Extension from the whole boundary of the tetrahedron

Consider any function $\boldsymbol{v}$ in the trace space of $\boldsymbol{H}(\mathbf{curl})$ on ∂K, i.e., $\boldsymbol{v} \in \boldsymbol{X}^{-1/2}$. Let us now solve the problem of extending this function from ∂K into K in a polynomial preserving way. The construction, at this stage, is completely analogous to the H^1 case: Define

$$\begin{aligned}
\boldsymbol{U}_i &= \boldsymbol{\mathcal{E}}^{\rm curl}_i \boldsymbol{v}, \\
\boldsymbol{U}_j &= \boldsymbol{\mathcal{E}}^{\rm curl}_{i,j} \boldsymbol{w}_j, && \text{where } \boldsymbol{w}_j = \boldsymbol{R}_j(\boldsymbol{v} - \operatorname{trc}_\tau \boldsymbol{U}_i), \\
\boldsymbol{U}_k &= \boldsymbol{\mathcal{E}}^{\rm curl}_{ij,k} \boldsymbol{w}_k, && \text{where } \boldsymbol{w}_k = \boldsymbol{R}_k(\boldsymbol{v} - \operatorname{trc}_\tau \boldsymbol{U}_i - \operatorname{trc}_\tau \boldsymbol{U}_j), \\
\boldsymbol{U}_l &= \boldsymbol{\mathcal{E}}^{\rm curl}_{ijk,l} \boldsymbol{w}_l, && \text{where } \boldsymbol{w}_l = \boldsymbol{R}_l(\boldsymbol{v} - \operatorname{trc}_\tau \boldsymbol{U}_i - \operatorname{trc}_\tau \boldsymbol{U}_j - \operatorname{trc}_\tau \boldsymbol{U}_k),
\end{aligned}$$

where $\boldsymbol{R}_i$ is the restriction to face F_i defined in (2.5), and the extensions $\boldsymbol{\mathcal{E}}^{\rm curl}_i$, $\boldsymbol{\mathcal{E}}^{\rm curl}_{i,j}$, $\boldsymbol{\mathcal{E}}^{\rm curl}_{ij,k}$, and $\boldsymbol{\mathcal{E}}^{\rm curl}_{ijk,l}$ are as defined in (3.5), (4.7), (5.3), and (6.1), respectively. The total extension operator is then defined by

$$\boldsymbol{\mathcal{E}}^{\rm curl}_K \boldsymbol{v} = \boldsymbol{U}_i + \boldsymbol{U}_j + \boldsymbol{U}_k + \boldsymbol{U}_l. \tag{7.1}$$

Lemma 7.1. *The functions $\boldsymbol{w}_j$, $\boldsymbol{w}_k$, and $\boldsymbol{w}_l$ defined above satisfy*

$$\begin{aligned}
\|\boldsymbol{w}_j\|_{\boldsymbol{X}^{-1/2}_{0,i}(F_j)} &\le C\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}}, \\
\|\boldsymbol{w}_k\|_{\boldsymbol{X}^{-1/2}_{0,ij}(F_k)} &\le C\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}}, \\
\|\boldsymbol{w}_l\|_{\boldsymbol{X}^{-1/2}_{0,ijk}(F_l)} &\le C\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}}.
\end{aligned}$$

Theorem 7.1. *The operator $\boldsymbol{\mathcal{E}}^{\rm curl}_K$ in (7.1) has the following properties:*

1. Continuity: *$\boldsymbol{\mathcal{E}}^{\rm curl}_K$ is a continuous operator from $\boldsymbol{X}^{-1/2}$ into $\boldsymbol{H}(\mathbf{curl})$.*
2. Commutativity: *$\mathbf{grad}(\mathcal{E}^{\rm grad}_K u) = \boldsymbol{\mathcal{E}}^{\rm curl}_K(\mathbf{grad}_\tau u)$ for all u in $H^{1/2}(\partial K)$.*
3. Extension property: *The tangential trace $\operatorname{trc}_\tau(\boldsymbol{\mathcal{E}}^{\rm curl}_K\boldsymbol{v})$ coincides with $\boldsymbol{v}$ for all $\boldsymbol{v}$ in $\boldsymbol{X}^{-1/2}$.*
4. Full polynomial preservation: *If $\boldsymbol{v}$ is the tangential trace of a function in $\boldsymbol{P}_p(K)$, then $\boldsymbol{\mathcal{E}}^{\rm curl}_K\boldsymbol{v}$ is in $\boldsymbol{P}_p(K)$.*
5. Nédélec polynomial preservation: *If $\boldsymbol{v}$ is the tangential trace of a function in $\boldsymbol{N}_p(K)$, then $\boldsymbol{\mathcal{E}}^{\rm curl}_K\boldsymbol{v}$ is in $\boldsymbol{N}_p(K)$.*

Proof. The proof follows by combining the previous results. E.g., the proof of continuity follows by combining the continuity of $\boldsymbol{v} \mapsto \boldsymbol{w}_m$ for $m = j, k, l$ (Lemma 7.1), the continuity of the primary extension (Theorem 3.1), and the continuity of the intermediate extension operators $\boldsymbol{\mathcal{E}}^{\rm curl}_{i,j}$ (Proposition 4.1), $\boldsymbol{\mathcal{E}}^{\rm curl}_{ij,k}$ (Proposition 5.1) and $\boldsymbol{\mathcal{E}}^{\rm curl}_{ijk,l}$ (Proposition 6.1). The proof of the commutativity property similarly follows because each of the intermediate operators satisfy commutativity properties. The remaining properties are also proved similarly. □

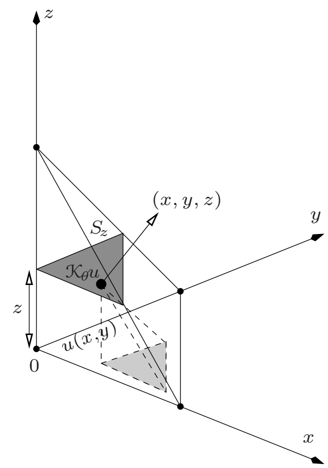

FIGURE 5. The value of $\mathcal{K}_\theta u$ at a point (x, y, z) in the slice S_z is determined by integrating u over the triangle in the x-y plane shown above. Even if $u(x, y)$ is not differentiable, $\mathcal{K}_\theta u$ can be differentiable. But the derivatives of $\mathcal{K}_\theta u$ degenerate as $z \to 0$, unless u is differentiable (see Lemma A.2).

APPENDIX A. PROOFS OF THE LEMMAS

We now prove all the lemmas in the order in which they appeared in the previous sections. For these proofs, we will use the lemmas established in [7], as well as a few new auxiliary results. We begin with the following auxiliary lemma:

Lemma A.1. *Let $S_z = \{(x', y', z') \in \hat{K} : \ z' = z\}$, $\theta(x, y)$ be a smooth function on $\hat{F}$, and*

$$\begin{aligned} G_0 u\,(x, y, z) &= \int_0^1 \theta(s, 1-s)\; u(x + sz, y + (1-s)z)\; ds, \\ G_1 u\,(x, y, z) &= \int_0^1 \theta(0, t)\; u(x, y + tz)\; dt, \\ G_2 u\,(x, y, z) &= \int_0^1 \theta(s, 0)\; u(x + sz, y)\; ds. \end{aligned}$$

Then, for any $0 < z < 1$,

$$\begin{aligned} \sqrt{2}\, \|G_0 u\|_{L^2(S_z)} &\le \|\theta\|_{L^1(\hat{E}_{12})} \|u\|_{L^2(\hat{F})} \\ \|G_1 u\|_{L^2(S_z)} &\le \|\theta\|_{L^1(\hat{E}_{20})} \|u\|_{L^2(\hat{F})}, \\ \|G_2 u\|_{L^2(S_z)} &\le \|\theta\|_{L^1(\hat{E}_{01})} \|u\|_{L^2(\hat{F})}. \end{aligned}$$

Proof. The three estimates have very similar proofs, so we will only prove the last one:

$$\begin{aligned} \|G_2 u\|^2_{L^2(S_z)} &= \iint\limits_{S_z} \left| \int_0^1 \theta(s, 0)\; u(x + sz, y)\; ds \right|^2 dx\, dy \\ &= \iint\limits_{S_z} \left(\int_0^1 \theta(s_1, 0)\; u(x + s_1 z, y)\; ds_1 \right) \left(\int_0^1 \theta(s_2, 0)\; u(x + s_2 z, y)\; ds_2 \right) dx\, dy \\ &= \int_0^1 \int_0^1 \theta(s_1, 0)\; \theta(s_2, 0) \left(\iint\limits_{S_z} u(x + s_1 z, y)\; u(x + s_2 z, y)\;\; dx\, dy \right) ds_1\, ds_2 \end{aligned}$$

by Fubini's theorem. Now applying Cauchy-Schwarz inequality to the integral over S_z in the parentheses above, and increasing the integration domain to all (x, y) in $\hat{F}$, we obtain

$$\begin{aligned}
\|G_2 u\|^2_{L^2(S_z)} &\le \int_0^1 \int_0^1 |\theta(s_1, 0)\, \theta(s_2, 0)|\, \|u\|_{L^2(\hat{F})} \|u\|_{L^2(\hat{F})}\, ds_1\, ds_2 \\
&= \left(\int_0^1 |\theta(s, 0)|\, ds \right)^2 \|u\|^2_{L^2(\hat{F})},
\end{aligned}$$

from which the last estimate of the lemma follows. □

Next, we present a result for the integral operator

$$\mathcal{K}_\theta u\ (x, y, z) = \int_0^1 \int_0^{1-t} \theta(s, t)\ u(x + sz, y + tz)\ ds\, dt,$$

with a smooth kernel θ. This is a smoothing integral, but the smoothness of the resulting function degenerates as $z \to 0$. The following lemma quantifies this by examining norms of derivatives on slices S_z (see Fig. 5) parallel to and approaching the x-y plane.

Lemma A.2. *Let $\theta(x, y)$ be a smooth function on $\hat{F}$. Then the map $\mathcal{K}_\theta$ defined above for smooth functions $u(x, y)$ on $\hat{F}$, extends to a continuous operator from $L^2(\hat{F})$ into $L^2(\hat{K})$. Moreover, letting $S_z = \{(x', y', z') \in \hat{K} :\ z' = z\}$, the following inequalities hold for any $0 < z < 1$:*

$$\|\mathcal{K}_\theta u\|_{L^2(S_z)} \le \kappa_1 \|u\|_{L^2(\hat{F})}, \tag{A.1}$$

$$\|\,\mathbf{grad}(\mathcal{K}_\theta u)\|_{L^2(S_z)} \le \kappa_2\, z^{-1} \|u\|_{L^2(\hat{F})}, \tag{A.2}$$

$$\|\,\mathbf{grad}(\mathcal{K}_\theta u)\|_{L^2(S_z)} \le \kappa_3 \|\,\mathbf{grad}_\tau\, u\|_{L^2(\hat{F})}, \tag{A.3}$$

where $\kappa_1 = \|\theta\|_{L^1(\hat{F})}$, $\kappa_2 = 2\sqrt{3} \left(\|\theta\|^2_{W^1_1(\hat{F})} + \|\theta\|^2_{L^1(\partial \hat{F})} \right)^{1/2}$, *and* $\kappa_3 = \sqrt{3}\, \|\theta\|_{L^1(\hat{F})}$.

Proof. The proof of the first estimate (A.1) is similar to the proof of Lemma A.1, so we omit it. To prove the second estimate (A.2), we rewrite the expression for $\mathcal{K}_\theta u$ as

$$\begin{aligned}
\mathcal{K}_\theta u\ (x, y, z) &= \int_0^1 \int_0^{1-t} \theta(s, t)\ u(x + sz, y + tz)\ ds\, dt \\
&= \frac{1}{z^2} \int_x^{x+z} \int_y^{x+y+z-x'} \theta\Big(\frac{x' - x}{z}, \frac{y' - y}{z}\Big)\ u(x', y')\ dy' dx',
\end{aligned} \tag{A.4}$$

and differentiate it (so that no derivatives fall on u). Then we obtain the following identity:

$$\mathbf{grad}(\mathcal{K}_\theta u) = \frac{1}{z} \begin{pmatrix} -\mathcal{K}_{\partial_s \theta} u + G_0 u - G_1 u \\ -\mathcal{K}_{\partial_t \theta} u + G_0 u - G_2 u \\ -2\mathcal{K}_\theta u - \mathcal{K}_{(s\partial_s \theta + t \partial_t \theta)} u + G_0 u \end{pmatrix}, \tag{A.5}$$

where $\mathcal{K}_\alpha$ (appearing above with $\alpha = \partial_s \theta, \partial_t \theta$, and $s\partial_s \theta + t\partial_t \theta$) denotes the same expression as on the right hand side of (A.4), but with $\theta(s, t)$ replaced by $\alpha(s, t)$. By applying Lemma A.1 and (A.1) to estimate the terms on the right hand side of (A.5), we obtain (A.2).

To prove the last estimate of the lemma, we express $\mathbf{grad}(\mathcal{K}_\theta u)$ differently from (A.5), this time letting all the derivatives fall on u:

$$\mathbf{grad}(\mathcal{K}_\theta u) = \int_0^1 \int_0^{1-t} \theta(s,t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s & t \end{pmatrix} \mathbf{grad}_\tau\, u(x+sz, y+tz)\, ds\, dt$$

$$= \begin{pmatrix} \mathcal{K}_\theta(\partial_x u) \\ \mathcal{K}_\theta(\partial_y u) \\ \mathcal{K}_{s\theta}(\partial_x u) + \mathcal{K}_{t\theta}(\partial_y u) \end{pmatrix}.$$

Thus, (A.3) follows by applying (A.1) to each term on the right hand side above. □

Proof of Lemma 3.1 *(the K-functional technique).* We use the real method of interpolation of spaces [2] and Peetre's K-functional [16]. It is well known [11, 12] that an equivalent norm on space $H^{1/2}(\hat{F})$ is

$$\|u\|_{H^{1/2}(\hat{F})} = \left(\int_0^\infty t^{-2} |K(t,u)|^2 \, dt \right)^{1/2},$$

where the K-functional is defined by

$$K(t,u)^2 = \inf_{u=u_0+u_1} \|u_0\|^2_{L^2(\hat{F})} + t^2 \|u_1\|^2_{H^1(\hat{F})}.$$

The infimum is taken over all decompositions $u = u_0 + u_1$ of u in $H^{1/2}(F_l)$ with u_0 in $L^2(\hat{F})$ and u_1 in $H^1(\hat{F})$. For such a decomposition, (A.2) and (A.3) of Lemma A.2 gives

$$\left\| \mathbf{grad}\, \mathcal{K}_\theta u_0 \right\|^2_{L^2(S_z)} \le C z^{-2} \|u_0\|^2_{L^2(\hat{F})},$$
$$\left\| \mathbf{grad}\, \mathcal{K}_\theta u_1 \right\|^2_{L^2(S_z)} \le C \|u_1\|^2_{H^1(\hat{F})},$$

where S_z is the slice defined previously (see Fig. 5). Using these to estimate the $H^1(\hat{K})$-norm, we have

$$\|\mathcal{K}_\theta u\|^2_{H^1(\hat{K})} = \int_0^1 \left(\|\mathcal{K}_\theta u\|^2_{L^2(S_z)} + \left\| \mathbf{grad}\left(\mathcal{K}_\theta(u_0+u_1) \right) \right\|^2_{L^2(S_z)} \right) dz$$
$$\le C \int_0^1 \|u\|^2_{L^2(\hat{F})} + z^{-2} \left(\|u_0\|^2_{L^2(\hat{F})} + z^2 \|u_1\|^2_{H^1(\hat{F})} \right) dz,$$

where we have also used (A.1) of Lemma A.2. Taking the infimum over all the decompositions,

$$\left\| \mathcal{K}_\theta u \right\|^2_{H^1(\hat{K})} \le C \int_0^1 z^{-2} K(z,u)^2 \, dz \le \; C \, \|u\|^2_{H^{1/2}(\hat{F})}.$$

□

Proof of Lemma 4.1. The proofs of the first, second, and third identities rely on an application of the fundamental theorem of calculus along the integration paths shown in Fig. 3(a), 3(b), and 3(c), respectively. Since the three proofs are very similar, we will only prove the first identity.

First, integrating $\partial u/\partial s$ along the vertical path in Fig. 3(a), we have

$$\begin{aligned}
&\int_0^1 \int_0^{1-t} u(s,t)\, ds\, dt \\
&\quad = \int_0^1 \int_0^{1-t} \int_0^s \frac{\partial u}{\partial s}(s',t)\, ds' ds\, dt && \text{(Fundamental theorem of calculus)}\\
&\quad = \int_0^1 \int_0^{1-t} \int_{s'}^{1-t} ds \;\; \frac{\partial u}{\partial s}(s',t) \;\; ds' dt && \text{(Fubini's theorem)}\\
&\quad = \int_0^1 \int_0^{1-t} (1-t-s)\, \frac{\partial u}{\partial s}(s,t) \;\; ds\, dt && \text{(variable change: } s' \to s\text{).}
\end{aligned}$$

Next, we integrate along the slanted line in Fig. 3(a) to get

$$\begin{aligned}
\int_0^1 \int_0^{1-t} u(s,t)\, ds\, dt &= \int_0^1 \int_0^{\beta} u(\alpha, \beta-\alpha)\, d\alpha d\beta && \text{(variable change: } \alpha = s,\ \beta = s+t\text{)}\\
&= \int_0^1 \int_0^{\beta} \int_0^{\alpha} \frac{d}{d\alpha} u(\alpha', \beta-\alpha')\, d\alpha' d\alpha d\beta \\
&= \int_0^1 \int_0^{\beta} \int_0^{\alpha} \left(\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} \right) (\alpha', \beta-\alpha')\, d\alpha' d\alpha d\beta \\
&= \int_0^1 \int_0^{\beta} \int_{\alpha'}^{\beta} d\alpha \, \left(\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} \right) (\alpha', \beta-\alpha')\, d\alpha' d\beta \\
&= \int_0^1 \int_0^{\beta} (\beta-\alpha) \left(\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} \right) (\alpha, \beta-\alpha)\, d\alpha d\beta \\
&= \int_0^1 \int_0^{1-t} t \left(\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} \right) (s,t)\, ds\, dt && \text{(variable change).}
\end{aligned}$$

Taking the average of the two identities we get the first identity of the lemma. □

Next, let us prove the continuity of the face and edge correction operators. Recall the averaging operators A_3^θ, B_2^θ and the interpolatory operators J_θ, L_θ analyzed in [7, Appendix A]:

$$A_3^\theta u\,(y,z) = 2 \int_0^1 \int_0^{1-s} \theta(s,t)\; u(sz, y+tz) \;\; dt\, ds, \tag{A.6}$$

$$B_2^\theta u\,(z) = 2 \int_0^1 \int_0^{1-s} \theta(s,t) u(sz,tz) \;\; dt\, ds. \tag{A.7}$$

$$J_\theta \phi\,(x,y,z) = \theta(x,y,z)\, \phi(y, x+z), \tag{A.8}$$

$$L_\theta \psi\,(x,y,z) = \theta(x,y,z)\, \psi(x+y+z), \tag{A.9}$$

which we used in the analysis of the H^1 face and edge correction operators. We will use them here in the $\boldsymbol{H}(\mathbf{curl})$ case as well.

Proof of Lemma 4.2. Combining the two terms in the definition of the face correction (4.1), write

$$\boldsymbol{\mathcal{E}}_{\hat F}^{\mathrm{curl}} \boldsymbol{v} = \int_0^1 \int_0^{1-t} \begin{pmatrix} (3s-1)z & 3zt \\ 0 & 2z \\ 2zs + x(1-s) & 2zt - xt \end{pmatrix} \frac{\boldsymbol{v}(s(x+z), y+t(x+z))}{x+z}\, ds\, dt.$$

In terms of the operators in (A.6) and (A.8), this expression becomes

$$\boldsymbol{\mathcal{E}}_{\hat F}^{\mathrm{curl}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} J_{\beta_1} \circ A_3^{\theta_1} v_1 + J_{\beta_1} \circ A_3^{\theta_2} v_2 \\ J_{\beta_1} \circ A_3^{\theta_3} v_2 \\ J_{\beta_1} \circ A_3^{\theta_4} v_1 + J_{\beta_2} \circ A_3^{\theta_5} v_1 + J_{\beta_1} \circ A_3^{\theta_6} v_2 - J_{\beta_2} \circ A_3^{\theta_6/2} v_2 \end{pmatrix} \tag{A.10}$$

with

$$\theta_1 = \frac{3s-1}{2} \quad \theta_2 = \frac{3t}{2}, \quad \theta_3 = 1, \quad \beta_1 = \frac{z}{x+z}$$
$$\theta_4 = s, \quad \theta_5 = \frac{1-s}{2} \quad \theta_6 = t, \quad \beta_2 = \frac{x}{x+z}. \tag{A.11}$$

Since $|\beta_i|$ are bounded, we can apply [7, Lemma A.3] to conclude that the map J_{β_i} : $L^2_z(\hat{F}_1) \longmapsto L^2(\hat{K})$ is continuous. In addition, for the specific β_1 and β_2 in (A.11) above, we have

$$\mathbf{grad}(J_{\beta_1}\phi) = \begin{pmatrix} J_{\beta_1}(\partial_z\phi) - J_{\beta_1}(\phi/z) \\ J_{\beta_1}(\partial_y\phi) \\ J_{\beta_1}(\partial_z\phi) + J_{\beta_2}(\phi/z) \end{pmatrix}, \quad \mathbf{grad}(J_{\beta_2}\phi) = \begin{pmatrix} J_{\beta_2}(\partial_z\phi) + J_{\beta_1}(\phi/z) \\ J_{\beta_2}(\partial_y\phi) \\ J_{\beta_2}(\partial_z\phi) - J_{\beta_2}(\phi/z) \end{pmatrix}.$$

Applying [7, Lemma A.3] again to these gradients, we conclude that the map

$$J_{\beta_i} :\ L^2_{1/z}(\hat{F}_1) \cap H^1_z(\hat{F}_1) \quad \longmapsto \quad H^1(\hat{K}) \tag{A.12}$$

is continuous. Furthermore, since the θ_i in (A.11) are smooth, applying [7, Lemma A.1], we find that

$$A_3^{\theta_i} :\ L^2_{1/x}(\hat{F}_3) \quad \longmapsto \quad L^2_{1/z}(\hat{F}_1) \cap H^1_z(\hat{F}_1) \tag{A.13}$$

is continuous. Combining the continuity of the maps in (A.12) and (A.13), we get that each of the operators in (A.10) of the form $J_{\beta_i} \circ A_3^{\theta_j}$ is continuous from $L^2_{1/x}(\hat{F}_3)$ into $H^1(\hat{K})$.

Since $H^{1/2}_{0,i}(F_l) = H^{1/2}(F_l) \cap L^2_{1/\lambda_i}(F_l)$, the continuity of the two-face extension $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{i,l} = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_l - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_i,l}$, now follows from the continuity of $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_l$ proved in Theorem 3.1 and the continuity of the face correction established above. □

Proof of Lemma 5.1. Let us first consider the expression (5.1) for the edge correction, summing its the three terms, namely

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{E}}\boldsymbol{v} = \iint_{\hat{F}} \begin{pmatrix} (3s-1)z & 3zt \\ 3zs & z(3t-1) \\ x(1-s) - ys + 2zs & -xt + y(1-t) + 2zt \end{pmatrix} \frac{\boldsymbol{v}(s(x+y+z), t(x+y+z))}{x+y+z}\, ds\, dt. \tag{A.14}$$

Using the B_2^θ in (A.7) and the L_θ in (A.9), we can rewrite this expression as

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{E}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} L_{\beta_3} \circ B_2^{\theta_1} v_1 & L_{\beta_3} \circ B_2^{\theta_2} v_2 \\ L_{\beta_3} \circ B_2^{3\theta_3} v_1 & L_{\beta_3} \circ B_2^{\theta_4} v_2 \\ L_{\beta_3} \circ (B_2^{\theta_5} + B_2^{2\theta_3}) v_1 - L_{\beta_2} \circ B_2^{\theta_3} v_1 & L_{\beta_2} \circ (B_2^{\theta_6} + B_2^{2\theta_2}) v_2 - L_{\beta_1} \circ B_2^{2\theta_2} v_2 \end{pmatrix}, \tag{A.15}$$

with

$$\theta_1 = \frac{3s-1}{2} \quad \theta_2 = \frac{t}{2}, \quad \theta_3 = \frac{s}{2}, \quad \beta_1 = \frac{x}{x+y+z}, \quad \beta_2 = \frac{y}{x+y+z}$$
$$\theta_4 = \frac{3t-1}{2}, \quad \theta_5 = \frac{1-s}{2} \quad \theta_6 = \frac{1-t}{2}, \quad \beta_3 = \frac{z}{x+y+z}. \tag{A.16}$$

Note that the above β_i take values in the bounded interval $[0,1]$. Hence [7, Lemma A.4] implies that $L_{\beta_i} : L^2_{z^2}(\hat{E}_{03}) \mapsto L^2(\hat{K})$ is continuous. However, since

$$\mathbf{grad}(L_{\beta_1}\psi(z)) = \frac{1}{(x+y+z)^2}\begin{pmatrix} y+z \\ -x \\ -x \end{pmatrix}\psi(x+y+z) + \frac{x}{x+y+z}\begin{pmatrix} \psi'(x+y+z) \\ \psi'(x+y+z) \\ \psi'(x+y+z) \end{pmatrix},$$
$$= \begin{pmatrix} L_{\beta_2}(\psi/z) + L_{\beta_3}(\psi/z) \\ -L_{\beta_1}(\psi/z) \\ -L_{\beta_1}(\psi/z) \end{pmatrix} + \begin{pmatrix} L_{\beta_1}(\psi') \\ L_{\beta_1}(\psi') \\ L_{\beta_1}(\psi') \end{pmatrix},$$

and since similar identities hold for the gradients of $L_{\beta_2}\psi$ and $L_{\beta_3}\psi$, applying [7, Lemma A.4] to the components of these gradients, we find a stronger continuity property, namely

$$L_{\beta_i} \ : \ L^2(\hat{E}_{03}) \cap H^1_{z^2}(\hat{E}_{03}) \ \longmapsto \ H^1(\hat{K}) \tag{A.17}$$

is continuous. Next, since θ_i in (A.11) are smooth, applying [7, Lemma A.2], we also have

$$B_2^{\theta_j} \ : \ L^2_{1/x}(\hat{F}_3) \cap L^2_{1/y}(\hat{F}_3) \longmapsto \ L^2(\hat{E}_{03}) \cap H^1_{z^2}(\hat{E}_{03}). \tag{A.18}$$

Since all the operators in (A.15) are of the form $L_{\beta_i} \circ B_2^{\theta_j}$, combining the continuity properties of (A.17) and (A.18), we find that the edge correction

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{\hat{E}} : [L^2_{1/x}(\hat{F}) \cap L^2_{1/y}(\hat{F})]^2 \mapsto \boldsymbol{H}^1(\hat{K}) \tag{A.19}$$

is continuous.

The required continuity of the three-face extension $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{ij,l} = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{l} - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_i,l} - \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{F_j,l} + \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_{E_{kl},l}$ now follows from the continuity of (A.19), the continuity of the face corrections (established in the proof of Lemma 4.2) and the continuity of the primary extension (Theorem 3.1). □

Proof of Lemma 7.1. By the definition of the space $\boldsymbol{X}^{-1/2}_{0,I}(F_l)$, its norm is

$$\|\boldsymbol{w}_j\|_{\boldsymbol{X}^{-1/2}_{0,I}(F_j)} = \inf_{\boldsymbol{R}_l\boldsymbol{u}=\boldsymbol{w}_j, \boldsymbol{u}\in\boldsymbol{X}^{-1/2}_{0,I}} \|\boldsymbol{u}\|_{\boldsymbol{X}^{-1/2}}.$$

Hence

$$\begin{aligned}
\|\boldsymbol{w}_j\|_{\boldsymbol{X}^{-1/2}_{0,i}(F_j)} &\le \|\boldsymbol{v} - \mathrm{trc}_\tau \boldsymbol{U}_i\|_{\boldsymbol{X}^{-1/2}} \\
&\le \|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}} + \|\,\mathrm{trc}_\tau \boldsymbol{\mathcal{E}}^{\mathrm{curl}}_i \boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}} \\
&\le \|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}} + C\|\boldsymbol{\mathcal{E}}^{\mathrm{curl}}_i \boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl})} && \text{by trace theorem} \\
&\le \|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}} + C\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}(F_l)} && \text{by Theorem 3.1} \\
&\le C\|\boldsymbol{v}\|_{\boldsymbol{X}^{-1/2}} && \text{by (2.6).}
\end{aligned}$$

The remaining estimates are proved similarly. □

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