POLYNOMIAL EXTENSION OPERATORS. PART I

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ABSTRACT. In this series of papers, we construct operators that extend certain given functions on the boundary of a tetrahedron into the interior of the tetrahedron, with continuity properties in appropriate Sobolev norms. These extensions are novel in that they have certain polynomial preservation properties important in the analysis of high order finite elements. This part of the series is devoted to introducing our new technique for constructing the extensions, and its application to the case of polynomial extensions from $H^{1/2}(\partial K)$ into $H^1(K)$, for any tetrahedron K.

1. INTRODUCTION

This paper is the first in a series of papers that construct extension operators with certain polynomial preservation properties for the three basic first order Sobolev spaces

$$(1.1) \qquad H^1(D) = \{u \in L^2(D) : \ \mathbf{grad}\, u \in \boldsymbol{L}^2(D)\}$$

$$(1.2) \qquad \boldsymbol{H}(\mathbf{curl}, D) = \{\boldsymbol{u} \in \boldsymbol{L}^2(D) : \ \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(D)\}$$

$$(1.3) \qquad \boldsymbol{H}(\mathrm{div}, D) = \{\boldsymbol{u} \in \boldsymbol{L}^2(D) : \ \mathrm{div}\, \boldsymbol{u} \in L^2(D)\}.$$

Here the derivatives are understood in the distributional sense, $L^2(D)$ denotes the set of square integrable functions (with respect to the Lebesgue measure) on an open subset D of the three dimensional Euclidean space, and $\boldsymbol{L}^2(D)$ denotes the set of vector functions whose components are in $L^2(D)$. The domain into which our extensions are performed is a tetrahedron K.

Extension operators are right inverses of trace maps. To describe the traces of the spaces in (1.1)–(1.3), let ϕ be a smooth scalar function and $\boldsymbol{\phi}$ is a smooth vector function on K. Then three standard trace operators are

$$\begin{aligned} \mathrm{trc}\,\phi &= \phi|_{\partial K}, & \text{(scalar trace)}, \\ \mathrm{trc}_\tau\,\boldsymbol{\phi} &= \big(\boldsymbol{\phi} - (\boldsymbol{\phi} \cdot \boldsymbol{n})\boldsymbol{n}\big)\big|_{\partial K}, & \text{(tangential trace)}, \\ \mathrm{trc}_n\,\boldsymbol{\phi} &= (\boldsymbol{\phi} \cdot \boldsymbol{n})\big|_{\partial K}, & \text{(normal trace)}. \end{aligned}$$

where $\boldsymbol{n}$ denotes the outward unit normal on ∂K. It is well known that trc extends to a continuous operator (which we continue to call trc) from $H^1(K)$ onto $H^{1/2}(\partial K)$ [17]. Similarly, the tangential trace trc_τ is well defined on $\boldsymbol{H}(\mathbf{curl}, K)$ and its range is a subspace of $\boldsymbol{H}^{-1/2}(\partial K)$, and the normal trace trc_n is well defined on $\boldsymbol{H}(\mathrm{div}, K)$ [6] with its range equal to $H^{-1/2}(\partial K)$. We want to construct continuous extension operators that map functions on

the boundary of a tetrahedron lying in the range of $\mathrm{trc}, \mathrm{trc}_\tau, \mathrm{trc}_n$ into $H^1(K), \boldsymbol{H}(\mathbf{curl}, K)$, and $\boldsymbol{H}(\mathrm{div}, K)$, respectively.

A *polynomial extension operator* is an extension operator with the additional property that whenever the function on ∂K to be extended is the trace of a polynomial on K, the extended function is also a polynomial.

1.1. **Goal.** Our aim in this series of papers is to construct three polynomial extension operators $\mathcal{E}_K^{\mathrm{grad}}, \mathcal{E}_K^{\mathrm{curl}}$, and $\mathcal{E}_K^{\mathrm{div}}$ on any tetrahedron K such that the following diagram commutes:

$$
(1.4) \qquad
\begin{array}{ccccc}
H^{1/2}(\partial K) & \xrightarrow{\ \mathbf{grad}_\tau\ } & \mathrm{trc}_\tau(\boldsymbol{H}(\mathbf{curl}, K)) & \xrightarrow{\ \mathrm{curl}_\tau\ } & \mathrm{trc}_n(\boldsymbol{H}(\mathrm{div}, K)) \\
\downarrow{\scriptstyle \mathcal{E}_K^{\mathrm{grad}}} & & \downarrow{\scriptstyle \mathcal{E}_K^{\mathrm{curl}}} & & \downarrow{\scriptstyle \mathcal{E}_K^{\mathrm{div}}} \\
H^1(K) & \xrightarrow{\ \mathbf{grad}\ } & \boldsymbol{H}(\mathbf{curl}, K) & \xrightarrow{\ \mathbf{curl}\ } & \boldsymbol{H}(\mathrm{div}, K)
\end{array}
$$

The goal of this part of the series is to construct the first in the sequence of these operators, namely $\mathcal{E}_K^{\mathrm{grad}}$. In the forthcoming parts [12, 13], we will construct the other two polynomial extension operators. Next, we state precisely the properties we require for $\mathcal{E}_K^{\mathrm{grad}}$.

1.2. **The $H^1(K)$ polynomial extension problem.** The problem we occupy ourselves with in this paper is that of constructing a linear operator $\mathcal{E}_K^{\mathrm{grad}} : H^{1/2}(\partial K) \mapsto H^1(K)$ with the following properties:

(1.5)   - The trace of $\mathcal{E}_K^{\mathrm{grad}} u$ on ∂K coincides with u, i.e., $\mathrm{trc}(\mathcal{E}_K^{\mathrm{grad}} u) = u$.

(1.6)   - $\mathcal{E}_K^{\mathrm{grad}}$ is a continuous map from $H^{1/2}(\partial K)$ into $H^1(K)$, i.e., there is a constant C_{grad} independent of u such that

$$\|\mathcal{E}_K^{\mathrm{grad}} u\|_{H^1(K)} \leq C_{\mathrm{grad}} \|u\|_{H^{1/2}(\partial K)} \qquad \text{for all } u \in H^{1/2}(\partial K).$$

(1.7)   - If u is a polynomial of degree at most p on each face of K and continuous on ∂K, then $\mathcal{E}_K^{\mathrm{grad}} u$ is a polynomial of degree at most p on K.

The main result of this paper is Theorem 6.1, which solves the problem as stated above.

1.3. **Existing work.** The concept of extension operators is intimately related with the idea of the trace operators and it has been present in the vast literature on Sobolev Spaces for a long time. For instance, the proof of surjectivity of trace operator for standard Sobolev spaces $H^s(\Omega)$ is based on the Lion's construction of a corresponding extension operator, see e.g. [23, Lemma 3.36]. It is perhaps worth mentioning that, contrary to this trace operator, the extension operator does not break down at half-integers and that the same construction serves the whole Sobolev scale for $s \in \mathbb{R}$.

The subject of polynomial preserving extension operators originates from the convergence analysis for the p- and hp-versions of the Finite Element Method (FEM). The first construction of such an operator is for the H^1-space on a two dimensional region and is due to Babuška and Suri [3]. It contains the origins of many ideas that have been generalized and developed in subsequent contributions. Those include the definition of the primary extension operator (cf. Section 2 of this contribution), the solution of a 'two-edge extension problem' using a system of two integral equations, and the analysis of continuity properties of 'edge-to-edge operators' necessary for the solution of an ultimate 'three edge extension' problem. The analysis was carried out first for a triangle and then extended to the case of a square element by using a bilinear map collapsing a square into a triangle. The construction

from [3] appears again in a later paper [2] by Babuška, Craig, Mandel, and Pitkäranta, in the context of preconditioning for the p version of the FEM.

The two dimensional polynomial preserving extension operator was subsequently utilized by Maday [21, 22] to demonstrate that interpolation between polynomial spaces equipped with Sobolev integer norms yields norms equivalent to the standard fractional Sobolev norms. A recent exposition of this subject is contained in [5]. Maday [22] also studied the continuity properties of the Babuška-Suri operator in weighted Sobolev spaces.

The first construction of a polynomial preserving extension operator in three space dimensions from $H^{1/2}(\partial U)$ into $H^1(U)$ was done for a cube U by Ben Belgacem in [4]. The construction utilized the earlier results of Maday mentioned above. The second construction was done for a tetrahedron by Muñoz-Sola [24]. The idea of Muñoz-Sola is rooted in the construction of "face bubble" shape functions for tetrahedral FEM. If ζ is the product of barycentric coordinates of the vertices of a face, then the face bubble functions have ζ as a factor. Muñoz-Sola applies a three dimensional analogue of the Babuška-Suri extension to the quotient ϕ/ζ, and multiplies the factor ζ back into the extension. The resulting lift vanishes on the remaining faces and displays appropriate continuity properties. The Muñoz-Sola's construction was recently analyzed in [18] in context of boundary elements, wherein it is shown that the norm of the extension as an operator from L^2 into $\tilde{H}^{1/2}$ grows with polynomial order p as $\log^{1/2} p$. (Here $\tilde{H}^{1/2}$ is the stronger intrinsic norm on the subspace of $H^{1/2}$ with weakly vanishing traces – see [18] for its definition.) Although the tetrahedral H^1 polynomial extension problem, as stated in (1.2), was solved by Muñoz-Sola [24], in this paper we want to present an alternate solution to the same problem. The main reason for presenting our new construction is that our techniques can be generalized to give polynomial extensions in the other two Sobolev spaces in (1.2) and (1.3), as will be amply evident from the subsequent parts of this series [12, 13]. Furthermore, with our techniques, we are able to provide polynomial extensions with the commutativity properties shown in (1.4), which we find aesthetic as well as useful. It is not clear to us if the technique in [24] can yield such extensions, but we do recognize the importance of [24], and indeed some of our arguments are motivated by it.

The need for polynomial preserving extension operators, at least in the analysis of p methods in H^1, is well known, as attested by the above mentioned works. In fact they are important also in other areas, such as in the theory of spectral methods even on one element as shown in [15], and for preconditioning as shown in [25]. The importance of polynomial extensions in the context of p and hp approximation theory for Maxwell equations was first recognized by Demkowicz and Babuška in [10], who considered the two dimensional triangular case. For a triangle T, the construction of a polynomial preserving extension operator from $H^{-1/2}(\partial T)$ into $H(\mathrm{curl}, T)$ follows directly from the corresponding construction for the H^1-case, and presents no essential technical difficulties. The utility of the polynomial extension operators in $H^1(T)$ and $H(\mathrm{curl}, T)$ forming a commuting de Rham diagram in developing approximations using "projection-based interpolation operators" is evident from [10]. The projection-based interpolation theory was generalized to three space dimensions (tetrahedra) by Demkowicz and Buffa in [11], under the conjecture of existence of commuting, polynomial preserving extension operators for the three spaces $H^1(K), \boldsymbol{H}(\mathbf{curl}, K), \boldsymbol{H}(\mathrm{div}, K)$ (for a refined version of the theory in context of both tetrahedral and hexahedral elements, see [9]). The extensions we construct in this series of papers establish the truth of this conjecture, thus providing the missing link in the approximation theory of [11].

An alternate technique for constructing commuting $H^1(T)$ and $H(\text{curl}, T)$ polynomial extensions deserves special mention. In two space dimensions, for any triangle T, Ainsworth and Demkowicz [1] constructed polynomial extensions explicitly by solving a system of three integral equations in the spirit of the system of two integral equations analyzed in [3]. The $H^1(T)$ extension operator has smaller norm than the two-dimensional version of the Muñoz-Sola's operator [24] and can be used for constructing optimal shape functions for a triangular element. The Ainsworth-Demkowicz operator was also shown to map $L^2(\partial K)$ into $H^{1/2}(K)$, with a constant independent of polynomial order p (cf. this result with that of [18] mentioned above). Unfortunately, our attempts to extend the Ainsworth-Demkowicz technique to three dimensions failed as the analysis reached prohibitive levels of complexity.

To the best of our knowledge, our commuting polynomial extensions for a tetrahedron is the first result of this kind. To compare with the most recent other work in this direction that we know of, the contribution of Costabel, Dauge and Demkowicz [8] presents an analogous family of commuting extension operators defined on polynomial spaces on a cube. The construction mimics separation of variables for polynomial spaces and is based on the Maday's spectral equivalence results for fractional spaces mentioned earlier. However, these extension operators defined on polynomial spaces change with the polynomial degree, although their norms are shown to be independent of polynomial degree p. In contrast, the expressions defining our extensions do not vary with degree.

1.4. **Overview of our techniques.** Our approach to solve the problem stated above starts with a study of the simpler case of extending a given function from just one face. Then we will analyze how to modify this extension process to solve the case when data is given on more faces. We highlight the main new techniques in the construction of our extension operators:

(1) *Primary extensions:* We start with processes that extend functions given on the plane $\mathbb{R}^2$, which we call "primary extensions". In constructing the final extension from ∂K, the first step is to pick a face of ∂K and apply the primary extension from that face. Of course, such an extended function in general will not have the needed traces in the remaining faces.

(2) *Corrections:* The next step is to "correct" the above mentioned incorrect traces on the remaining faces. This step is divided into the construction of "face correction operators", "edge correction operators", and a "vertex correction operator". Most of the technical aspects of the presented construction are in the design of these corrections.

(3) *Commutativity:* Once we construct the first operator appearing in the commuting diagram (1.4), namely $\mathcal{E}_K^{\text{grad}}$, then the construction of the succeeding operators are motivated by the commutativity properties in (1.4). Indeed, to obtain $\boldsymbol{\mathcal{E}}_K^{\text{curl}}$, we consider each of the primary and correction operators that went into the design of the preceding operator $\mathcal{E}_K^{\text{grad}}$, and find corresponding $\boldsymbol{H}(\textbf{curl})$ operators that commute with it. Similarly, the construction of the $\boldsymbol{H}(\text{div})$ extension will proceed by examining the steps in the construction of the $\boldsymbol{H}(\textbf{curl})$ extension and finding their commuting $\boldsymbol{H}(\text{div})$ analogues. This will be clear from Parts II [12] and III [13] of this series.

(4) *Regular decomposition of traces:* In the $\boldsymbol{H}(\textbf{curl})$ and $\boldsymbol{H}(\text{div})$ cases where traces are in Sobolev spaces of negative index, we will characterize the trace space using a decomposition involving Sobolev spaces of positive indices only. This will feature in

Parts II [12] and III [13] only, as the trace space of $H^1(K)$ is $H^{1/2}(\partial K)$, a Sobolev space of positive index.

(5) *Weighted norm estimates:* The extension operators we construct are all integral operators with polynomial kernels. To establish their continuity properties in appropriate fractional Sobolev norms, we use weighted norm estimates. We are able to do this even when traces are in Sobolev spaces of negative index, because of the above mentioned regular decomposition of the trace space. A number of weighted norm estimates used throughout this series are given in Appendix A of this part.

1.5. **Notations.** For ready reference, we quickly list here some notations that we will use throughout this series.

The "reference tetrahedron" $\hat{K}$ is defined by

$$\hat{K} = \{(x, y, z) : x \geq 0, \ y \geq 0, \ x + y + z \leq 1\},$$

with the following enumeration of vertices:

$$\hat{\boldsymbol{a}}_0 = (0, 0, 0), \ \hat{\boldsymbol{a}}_1 = (1, 0, 0), \ \hat{\boldsymbol{a}}_2 = (0, 1, 0), \ \text{and} \ \hat{\boldsymbol{a}}_3 = (0, 0, 1).$$

The face of $\hat{K}$ opposite to vertex $\hat{\boldsymbol{a}}_i$ is denoted by $\hat{F}_i$. The face $\hat{F}_3$ is distinguished in that the data for the primary extensions is given there, so we shall also denote it simply by $\hat{F}$. The edge connecting $\hat{\boldsymbol{a}}_i$ and $\hat{\boldsymbol{a}}_j$ is denoted by $\hat{E}_{ij}$.

We denote by K a generic tetrahedron (of positive volume). When the notations defined for $\hat{K}$ above are employed without the superscript (hat), they denote the corresponding geometrical objects on a general tetrahedron K, e.g., $\boldsymbol{a}_i$ denote the vertices of K, and F_i is the face of K opposite to the vertex $\boldsymbol{a}_i$.

Let $\lambda_i(\boldsymbol{x})$, for $i = 0, \ldots 3$ be the linear function on K satisfying $\lambda_i(\boldsymbol{a}_j) = \delta_{ij}$. The *affine coordinates* (or barycentric coordinates) of a point in K are the values of λ_i at that point arranged into a 4-tuple $(\lambda_i, \lambda_j, \lambda_k, \lambda_l)$, where the order is not significant. Throughout this series, the indices i, j, k, l are a permutation of $0, 1, 2, 3$.

Many integral operators we consider will require us to integrate over subtriangles of a face F_l of K. We now express these subtriangles using the affine coordinates of F_l, namely $\lambda_m^{F_l} = \lambda_m|_{F_l}$ for $m = i, j$, or k. For any permutation $\{i, j, k, l\}$ of $\{0, 1, 2, 3\}$, we define

$$(1.8) \qquad T_l(r_i, r_j, r_k) = \{\boldsymbol{x} \in F_l : \ \lambda_i^{F_l}(\boldsymbol{x}) \geq r_i, \ \lambda_j^{F_l}(\boldsymbol{x}) \geq r_j, \ \text{and} \ \lambda_k^{F_l}(\boldsymbol{x}) \geq r_k\}.$$

Note that the order of the arguments r_i, r_j, r_k in the notation $T_l(r_i, r_j, r_k)$ is not significant, but the subscripts of these arguments indicate which affine coordinate it corresponds to. This region is illustrated in Figure 1. Also define

$$(1.9) \qquad T_l(0, r_j, r_k) = \{\boldsymbol{x} \in F_l : \ \lambda_j^{F_l}(\boldsymbol{x}) \geq r_j, \ \text{and} \ \lambda_k^{F_l}(\boldsymbol{x}) \geq r_k\},$$

$$(1.10) \qquad T_l(0, 0, r_i) = \{\boldsymbol{x} \in F_l : \ \lambda_i^{F_l}(\boldsymbol{x}) \geq r_i\}.$$

Note that the definition of $T_l(0, r_j, r_k)$ is consistent with (1.8) when $r_i = 0$. Similarly the definition of $T_l(0, 0, r_i)$ is consistent with (1.8) when $r_j = r_k = 0$. In particular, from the indices of the arguments, we judge which barycentric coordinates are zero, e.g., in $T_l(0, r_k, 0)$, since l and k have already appeared, and since $\{i, j, k, l\}$ is a permutation of $\{0, 1, 2, 3\}$, we understand that the affine coordinates with indices that have not appeared, namely $\lambda_i^{F_l}$ and $\lambda_j^{F_l}$, are simply greater than or equal to zero: $T_l(0, r_k, 0) = \{\boldsymbol{x} \in F_l : \ \lambda_k^{F_l}(\boldsymbol{x}) \geq r_k, \ \lambda_i^{F_l}(\boldsymbol{x}) \geq 0, \ \lambda_j^{F_l}(\boldsymbol{x}) \geq 0\}$, which is consistent with (1.8). See Figures 1, 3, and 4 for illustrations of these subtriangles.
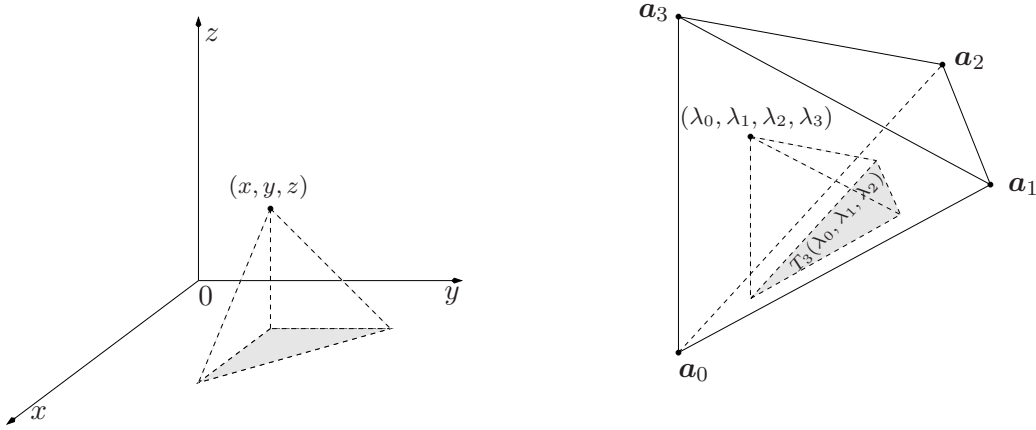
FIGURE 1. The regions of integration that define $\mathcal{E}^{\mathrm{grad}} u$. The left figure shows the integration region in gray for the extension from the x-y plane. The right figure shows the integration region for the mapped extension from a face of a general tetrahedron.

Our notation for Sobolev spaces is standard: Let $H^s(D)$ denote the standard Sobolev space of order s on domain D, e.g., when $s = 1$ the definition is as in (1.1). See e.g. [14, 19, 23] for the more complex definitions in the case when s is fractional. The definitions of $\boldsymbol{H}(\mathbf{curl}, D)$ and $\boldsymbol{H}(\mathrm{div}, D)$ are already given in (1.2)-(1.3) To shorten notation, when the domain is K, we often simply write $\boldsymbol{H}(\mathbf{curl})$ for $\boldsymbol{H}(\mathbf{curl}, K)$ (and similarly $\boldsymbol{H}(\mathrm{div})$).

In inequalities bounding function norms, we denote by C (or C with some subscript) a generic constant whose value at different occurrences may differ but is independent of the functions involved.

2. THE PRIMARY EXTENSION OPERATOR

The primary extension operator for the $H^1(K)$ case follows from the well known two dimensional polynomial extension from a line [2], as generalized to three space dimensions in [24] . Suppose $u(x, y)$ is a smooth function given on the face $\hat{F}$ of $\hat{K}$ (see § 1.5 for notation). The primary extension maps u to a function $\mathcal{E}^{\mathrm{grad}} u$ defined on $\hat{K}$ as follows

$$(2.1) \qquad \mathcal{E}^{\mathrm{grad}} u\,(x, y, z) = \frac{2}{z^2} \int_x^{x+z} \int_y^{x+y+z-\widetilde{x}} u(\widetilde{x}, \widetilde{y})\, d\widetilde{y}\, d\widetilde{x}.$$

Clearly, the value of the extension at the point (x, y, z) is determined by integrating u over the triangle with vertices $(x, y, 0), (x + z, y, 0)$, and $(x, y + z, 0)$, as shown in the first illustration of Figure 1. Note that $\mathcal{E}^{\mathrm{grad}}$ can be thought of either as an extension from the plane $\mathbb{R}^2$ into the adjacent infinite slab $\mathbb{R}^2 \times (0, 1) \equiv \{(x, y, z) : 0 < z < 1\}$, or as an extension from the face $\hat{F}$ into $\hat{K}$.

Theorem 2.1. *The following statements apply to $\mathcal{E}^{\mathrm{grad}}$:*

(1) *The operator $\mathcal{E}^{\mathrm{grad}}$ defined for smooth functions u by (2.1) extends as a continuous operator from $H^{1/2}(\hat{F})$ to $H^1(\hat{K})$, i.e., there is a constant $C > 0$ such that*

$$\|\mathcal{E}^{\mathrm{grad}} u\|_{H^1(\hat{K})} \le C \|u\|_{H^{1/2}(\hat{F})},$$

for all u in $H^{1/2}(\hat{F})$.

(2) *The trace of $\mathcal{E}^{\mathrm{grad}}u$ on $\hat{F}$ equals u for all u in $H^{1/2}(\hat{F})$.*

(3) *If u is a polynomial of degree at most p, then $\mathcal{E}^{\mathrm{grad}}u$ is also a polynomial of degree at most p.*

Proof. The proof of the continuity estimate in the first item proceeds by a standard technique involving the Fourier transform. We have included this proof in Appendix B. (In Part II [12], we will develop an alternate technique.)

The second statement of the theorem is immediately verified for smooth u, once we rewrite (2.1) by a change of variable as

$$(2.2) \qquad \mathcal{E}^{\mathrm{grad}}u\,(x,y,z) = 2\int_0^1 \int_0^{1-s} u(x+sz, y+tz)\,dt\,ds.$$

So, by the standard density argument, it follows for all $u \in H^{1/2}(\hat{F})$.

To prove the last statement of the theorem, observe that if u is a polynomial of degree at most p, then the integrand $u(x+sz, y+tz)$ in (2.2) is a polynomial of degree at most p in x, y, and z, with coefficients depending on s and t. After integrating over s and t, we continue to have a polynomial of degree at most p in x, y, and z. $\qquad\square$

Since any tetrahedron can be mapped one-one onto $\hat{K}$ using an affine map, the above definition of $\mathcal{E}^{\mathrm{grad}}$ automatically defines an extension from a face for any tetrahedron. The expressions for such mapped extensions are greatly simplified by the use of affine coordinates (or barycentric coordinates) of the tetrahedron. Let K be any tetrahedron. Suppose u is a smooth function defined on a face F_l of K and we want to extend it to K. The extension operator on K now involves integration of u over a subtriangle of F_l. We express such subtriangles using the notations established in § 1.5. The extension from F_l into K is defined by

$$(2.3) \qquad \mathcal{E}_l^{\mathrm{grad}}u\,(\lambda_i, \lambda_j, \lambda_k, \lambda_l) = \frac{2}{\lambda_l^2} \iint\limits_{T_l(\lambda_i,\lambda_j,\lambda_k)} u(\boldsymbol{s})\,d\boldsymbol{s},$$

where $\boldsymbol{s}$ is the two dimensional variable of integration running over the subtriangle $T_l(\lambda_i, \lambda_j, \lambda_k)$ defined in (1.8) and $d\boldsymbol{s}$ is the standard (two dimensional) Lebesgue measure on this subtriangle. (The region of integration is illustrated in Figure 1.) Clearly, the definition in (2.3) reduces to (2.1) if we choose $l = 3$ and $K = \hat{K}$. An analogue of Theorem 2.1 obviously holds for the operator in (2.3) by mapping.

3. FACE CORRECTIONS

From the definition of $\mathcal{E}_l^{\mathrm{grad}}u$, it is clear that when u is a smooth function on F_l which vanishes on the boundary ∂F_l, the extended function $\mathcal{E}_l^{\mathrm{grad}}u$ in general does not have zero trace on $\partial K \setminus F_l$. Now we show how to add a correction to $\mathcal{E}_l^{\mathrm{grad}}u$ so that the corrected result will have zero trace on a face other than F_l.

For definiteness, we first consider the situation on the face $\hat{F}_1$ of the reference tetrahedron $\hat{K}$ after the extension $\mathcal{E}^{\mathrm{grad}}u$ has been performed. The correction term for $\hat{F}_1$, namely $\mathcal{E}_{\hat{F}_1}^{\mathrm{grad}}u$, must be so that the corrected result $\mathcal{E}^{\mathrm{grad}}u - \mathcal{E}_{\hat{F}_1}^{\mathrm{grad}}u$ achieves zero trace on $\hat{F}_1$ without altering the trace on the original face $\hat{F}$. Therefore, we design the correction term by linear interpolation between the value of $\mathcal{E}^{\mathrm{grad}}u$ on $\hat{F}_1$ and 0 along the lines where $x + z$
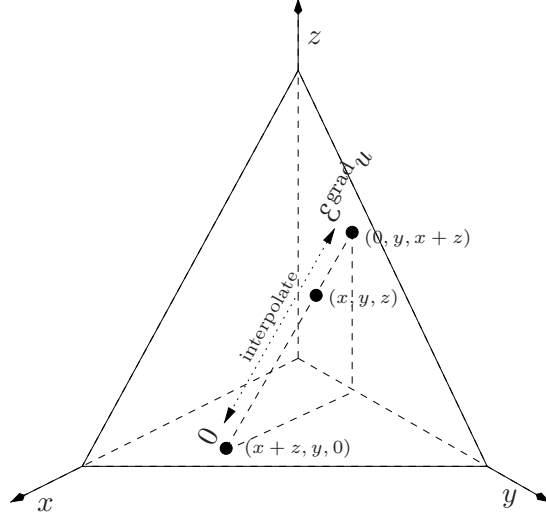
FIGURE 2. Illustration of the face correction process

and y are constant (see Figure 2), i.e., we define

$$(3.1) \qquad \mathcal{E}^{\mathrm{grad}}_{\hat{F}_1} u\,(x,y,z) = \frac{z}{x+z}\, \mathcal{E}^{\mathrm{grad}} u(0,y,x+z).$$

The interpolation process that we employed here has the following interesting continuity property in weighted spaces (whose proof appears in Appendix A). Our notation for weighted spaces is as follows: Let $L^2_w(D)$ denote the weighted space of all Lebesgue measurable functions f on a domain D (in $\mathbb{R}^n$ – in this paper $n = 1, 2,$ or 3) such that $\|f\|^2_{L^2_w(D)} \equiv \int_D w\,|f|^2\,dx < \infty$. Here $w(x)$ is a nonnegative weight function on D and dx is the standard Lebesgue measure on D. Similarly, the weighted Sobolev space $H^1_w(D)$ consists of all functions in $L^2_w(D)$ whose first order distributional derivatives are also in $L^2_w(D)$.

Lemma 3.1. *The map* $\mathcal{B}_{\hat{F}_1}$ *defined by*

$$\phi(y,z)\ \mapsto\ \mathcal{B}_{\hat{F}_1}\phi := \frac{z}{x+z}\phi(y,x+z)$$

is continuous from $L^2_{1/z}(\hat{F}_1) \cap H^1_z(\hat{F}_1)$ *into* $H^1(\hat{K})$.

To study the face correction operator in (3.1), in addition to the above lemma, we shall need some more properties of the primary extension $\mathcal{E}^{\mathrm{grad}} u$ involving weighted spaces.

Lemma 3.2.

(1) *Let* $\mathcal{R}^{\mathrm{grad}}$ *map smooth functions* u *on face* $\hat{F}_3$ *to functions on face* $\hat{F}_1$ *by*

$$u(x,y)\ \mapsto\ \mathcal{R}^{\mathrm{grad}} u\,(y,z) := \mathcal{E}^{\mathrm{grad}} u(0,y,z).$$

$\mathcal{R}^{\mathrm{grad}}$ *extends to a continuous map from* $L^2_{1/x}(\hat{F}_3)$ *into* $L^2_{1/z}(\hat{F}_1) \cap H^1_z(\hat{F}_1)$.

(2) *Let* $\mathcal{L}^{\mathrm{grad}}$ *map smooth functions* u *on face* $\hat{F}_3$ *to functions on edge* $\hat{E}_{03}$ *by*

$$u(x,y)\ \mapsto\ \mathcal{L}^{\mathrm{grad}} u\,(z) := \mathcal{E}^{\mathrm{grad}} u(0,0,z).$$

$\mathcal{L}^{\mathrm{grad}}$ *extends to a continuous map from* $L^2_{1/x}(\hat{F}_3) \cap L^2_{1/y}(\hat{F}_3)$ *into* $L^2(\hat{E}_{03}) \cap H^1_{z^2}(\hat{E}_{03})$.
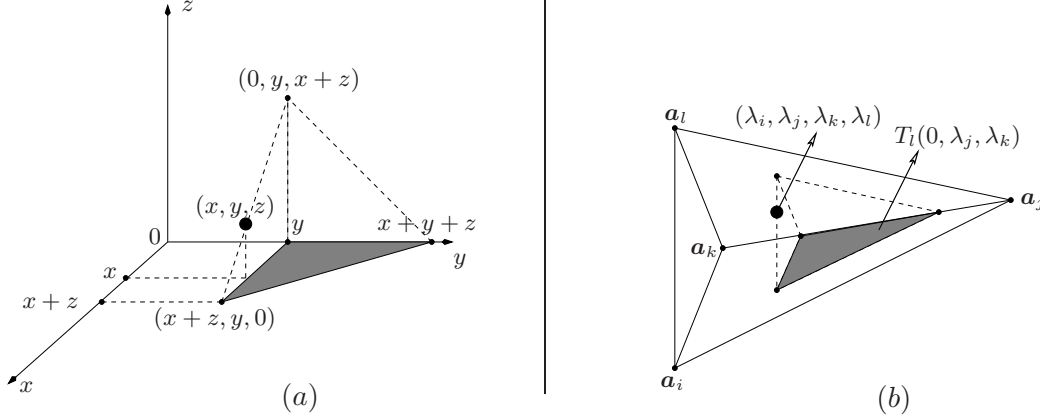
FIGURE 3. Domain of integration for face corrections (a) in the reference domain and (b) in a general tetrahedron

Proofs of this and all other lemmas appear in Appendix A. Note that although Theorem 2.1 needs u to be in $H^{1/2}(\hat{F})$ for $\mathcal{E}^{\mathrm{grad}}u$ to be in $H^1(\hat{K})$, the above lemma shows that the traces of $\mathcal{E}^{\mathrm{grad}}u$ on a face and an edge are well defined even when u is only in a weighted L^2 space.

Before stating the properties of $\mathcal{E}^{\mathrm{grad}}_{\hat{F}}$, let us extend its definition for any face of a general tetrahedron K, using the affine coordinate notation established in § 1.5. The face correction operator for a face F_i to correct the primary extension from face F_l, is defined in affine coordinates by

$$(3.2) \qquad \mathcal{E}^{\mathrm{grad}}_{F_i,l}u\,(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \frac{2\lambda_l}{(\lambda_i + \lambda_l)^3} \iint\limits_{T_l(0,\lambda_j,\lambda_k)} u(\boldsymbol{s})\,d\boldsymbol{s},$$

where $T_l(0, r_j, r_k)$ is as defined in (1.9). The region of integration is illustrated in Figure 3

Now suppose u is a smooth function on F_l which vanishes on one of its edges, say the edge connecting vertices $\boldsymbol{a}_j$ and $\boldsymbol{a}_k$, which we denote by E_{jk}. This edge is shared by F_i and F_l. The *two-face problem* is the problem of finding a polynomial extension of u from F_l that is zero on F_i. With the above defined face correction, we are now able to solve the two face problem. The extension operator that solves the F_i-F_l two-face problem is defined by

$$(3.3) \qquad \mathcal{E}^{\mathrm{grad}}_{i,l}u = \mathcal{E}^{\mathrm{grad}}_l u - \mathcal{E}^{\mathrm{grad}}_{F_i,l}u.$$

It can be extended to an operator on

$$(3.4) \qquad H^{1/2}_{0,i}(F_l) = H^{1/2}(F_l) \cap L^2_{1/\lambda_i}(F_l)$$

with the following properties:

Proposition 3.1. *The two-face extension $\mathcal{E}^{\mathrm{grad}}_{i,l}$ satisfies the following:*

(1) *$\mathcal{E}^{\mathrm{grad}}_{i,l}$ is a continuous operator from $H^{1/2}_{0,i}(F_l)$ into $H^1(K)$.*

(2) *For all $u \in H^{1/2}_{0,i}(F_l)$, the trace of $\mathcal{E}^{\mathrm{grad}}_{i,l}u$ on F_i is zero, while its trace on F_l equals u.*

(3) *If u is a polynomial of degree at most p that vanishes on E_{jk}, then $\mathcal{E}^{\mathrm{grad}}_{i,l}u$ is a polynomial of degree at most p.*

Proof. To prove item (1), observe that since Theorem 2.1 yields

$$\|\mathcal{E}^{\mathrm{grad}}_l u\|_{H^1(K)} \le C\|u\|_{H^{1/2}(F_l)} \le C\|u\|_{H^{1/2}_{0,i}(F_l)},$$

it suffices to prove that $\mathcal{E}_{F_i,l}^{\text{grad}}$ is a continuous map from $H_{0,i}^{1/2}(F_l)$ into $H^1(K)$. In fact, $\mathcal{E}_{F_i,l}^{\text{grad}}$ is continuous on the larger space $L_{1/\lambda_i}^2(F_l)$, as we show now. Because $\mathcal{E}_{F_i,l}^{\text{grad}}$ for any i is obtained by mapping $\mathcal{E}_{\hat{F}_1}^{\text{grad}}$ from $\hat{K}$, it suffices to prove that $\mathcal{E}_{\hat{F}_1}^{\text{grad}} : L_{1/x}^2(\hat{F}_3) \mapsto H^1(\hat{K})$ is continuous. By definition, $\mathcal{E}_{\hat{F}_1}^{\text{grad}}$ is the composition

$$\mathcal{E}_{\hat{F}_1}^{\text{grad}} = \mathcal{B}_{\hat{F}_1} \circ \mathcal{R}^{\text{grad}}.$$

By Lemma 3.2, $\mathcal{R}^{\text{grad}} : L_{1/x}^2(\hat{F}_3) \mapsto L_{1/z}^2(\hat{F}_1) \cap H_z^1(\hat{F}_1)$ is continuous, and by Lemma 3.1, $\mathcal{B}_{\hat{F}_1} : L_{1/z}^2(\hat{F}_1) \cap H_z^1(\hat{F}_1) \mapsto H^1(\hat{K})$ is continuous, hence the continuity of their composition follows.

Proof of (2): By (2.3) and (3.2),

$$\mathcal{E}_{i,l}^{\text{grad}}u = \frac{2}{\lambda_l^2} \iint\limits_{T_l(\lambda_i,\lambda_j,\lambda_k)} u(\boldsymbol{s})\, d\boldsymbol{s} \;-\; \frac{2\lambda_l}{(\lambda_i + \lambda_l)^3} \iint\limits_{T_l(0,\lambda_j,\lambda_k)} u(\boldsymbol{s})\, d\boldsymbol{s}.$$

Since $\lambda_i = 0$ on face F_i, we immediately see after setting $\lambda_i = 0$ above, that the trace on F_i vanishes. The trace on F_l is u because the the last term above vanishes upon setting $\lambda_l = 0$, while the trace of $\mathcal{E}_l^{\text{grad}}u$ on F_l is u (by Theorem 2.1).

Proof of (3): Going back to the reference tetrahedron, observe that if u vanishes on edge along the y-axis, then

$$u(x,y) = x\, u_{p-1}(x,y)$$

for some polynomial u_{p-1} of degree at most $p-1$. Using this in the face correction expression in (3.1), we have

$$(\mathcal{E}_{\hat{F}_1}^{\text{grad}}u)(x,y,z) = \frac{z}{x+z} \int_0^1 \int_0^{1-t} u(s(x+z), y+t(x+z))\, ds\, dt$$

$$= \frac{z}{x+z} \int_0^1 \int_0^{1-t} s(x+z)u_{p-1}(s(x+z), y+t(x+z))\, ds\, dt$$

$$= z \int_0^1 \int_0^{1-t} s u_{p-1}(s(x+z), y+t(x+z))\, ds\, dt.$$

The integral above is a polynomial of degree at most $p-1$ by the same arguments as in the proof of Theorem 2.1(3), hence the result follows. $\qquad\square$

4. Edge corrections

If the function u (to be extended from face F_l) vanishes on ∂F_l, then we want the extended function to vanish on all faces other than F_l. After an application of the face correction operator for face F_i, the extension has zero trace on F_i. To obtain zero trace on another face, say F_j, we can consider applying the face correction operator for F_j. Unfortunately, after this second correction the resulting total trace on F_i will no longer be zero, in general. This necessitates the use of further correction operators which we discuss now.

Let us first consider the case of the reference tetrahedron $\hat{K}$ after the application of the face correction $\mathcal{E}_{\hat{F}_1}^{\text{grad}}$. This operator alters the traces on face $\hat{F}_2$. In order to return this trace to its original setting we use an additional correction operator whose trace coincides
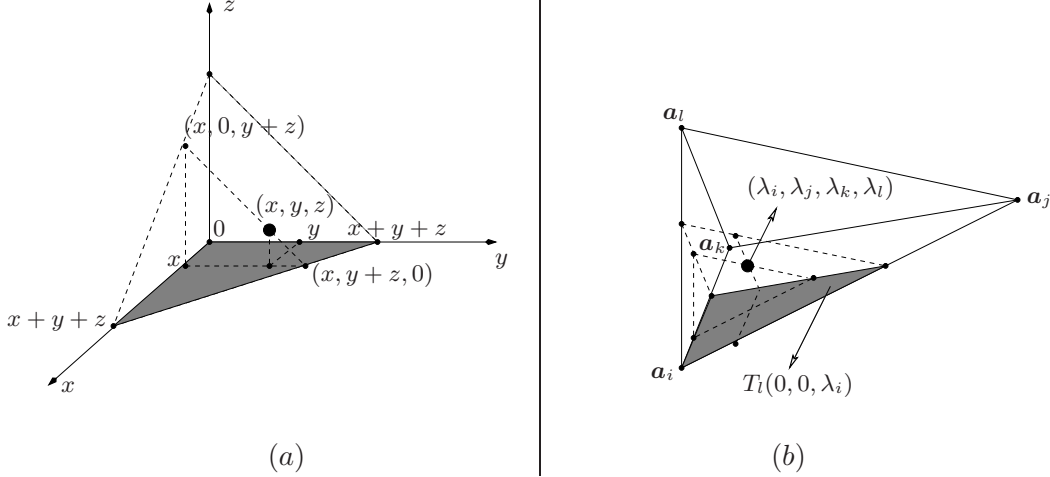
FIGURE 4. Domain of integration for the edge correction operators in (a) reference domain, and (b) general tetrahedron.

with that of of $\mathcal{E}^{\text{grad}}_{\hat{F}_1} u$ on $\hat{F}_2$. It is defined by linearly interpolating the value of $\mathcal{E}^{\text{grad}}_{\hat{F}_1} u$ on this face down to zero along the lines where $y + z$ and x are constant:

$$\mathcal{E}^{\text{grad}}_{\hat{E}_{03}} u(x, y, z) = \frac{z}{y + z} \mathcal{E}^{\text{grad}}_{\hat{F}_1} u(x, 0, y + z),$$

$$(4.1) \qquad = \frac{z}{x + y + z} \mathcal{E}^{\text{grad}} u(0, 0, x + y + z), \qquad \text{by (3.1)}$$

$$(4.2) \qquad = \frac{2z}{(x + y + z)^3} \int_0^{x+y+z} \int_0^{x+y+z-\widetilde{x}} u(\widetilde{x}, \widetilde{y})\, d\widetilde{y}\, d\widetilde{x}, \qquad \text{by (2.1).}$$

We call this operator the "edge correction operator" for the $\hat{E}_{03}$ edge, because as is clear from (4.1), its action only depends on the value of $\mathcal{E}^{\text{grad}} u$ along $\hat{E}_{03}$. The interpolation process from this edge to the tetrahedron implicit in (4.1) has the following continuity property:

Lemma 4.1. *The map* $\mathcal{B}_{\hat{E}_{03}}$ *defined by*

$$\phi(z) \;\mapsto\; \mathcal{B}_{\hat{E}_{03}} \phi := \frac{z}{x + y + z} \phi(x + y + z)$$

is continuous from $L^2(\hat{E}_{03}) \cap H^1_{z^2}(\hat{E}_{03})$ *into* $H^1(\hat{K})$.

Next, we generalize the expression in (4.2) to one in affine coordinates on a general tetrahedron. Let K be a general tetrahedron with the function to be extended given on face F_l. Let E_{il} denote the edge connecting the vertex $\boldsymbol{a}_i$ to $\boldsymbol{a}_l$. Then, the correction operator associated to the edge E_{il} is

$$(4.3) \qquad \mathcal{E}^{\text{grad}}_{E_{il}, l} u\,(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \frac{2\lambda_l}{(1 - \lambda_i)^3} \iint\limits_{T_l(0, 0, \lambda_i)} u(\boldsymbol{s})\, d\boldsymbol{s},$$

where, $T_l(0, 0, r_i)$ is as defined in (1.10). The integration domain is illustrated in Figure 4.

We can now solve the *three-face problem* of finding an extension of u from F_l that is zero on F_i and F_j whenever u is a smooth function that vanishes on edges E_{jk} and E_{ik}. The

F_i-F_j-F_l three-face problem is solved by the extension operator

$$(4.4) \qquad \mathcal{E}_{ij,l}^{\text{grad}} = \mathcal{E}_{l}^{\text{grad}} - \mathcal{E}_{F_i,l}^{\text{grad}} - \mathcal{E}_{F_j,l}^{\text{grad}} + \mathcal{E}_{E_{kl},l}^{\text{grad}},$$

whose properties appear in the next proposition. The space to which $\mathcal{E}_{ij,l}^{\text{grad}}$ can be extended continuously is

$$(4.5) \qquad H_{0,ij}^{1/2}(F_l) = H^{1/2}(F_l) \cap L_{1/\lambda_i}^2(F_l) \cap L_{1/\lambda_j}^2(F_l)$$

as seen below.

Proposition 4.1. *The three-face extension $\mathcal{E}_{ij,l}^{\text{grad}}$ satisfies the following:*

  (1) *$\mathcal{E}_{ij,l}^{\text{grad}}$ is a continuous operator from $H_{0,ij}^{1/2}(F_l)$ into $H^1(K)$.*
  (2) *For all $u \in H_{0,ij}^{1/2}(F_l)$ the traces of the edge and face corrections satisfy*

$$(4.6) \qquad \mathcal{E}_{F_i,l}^{\text{grad}} u \big|_{F_k} = \mathcal{E}_{E_{jl},l}^{\text{grad}} u,$$

$$(4.7) \qquad (\mathcal{E}_{E_{jl},l}^{\text{grad}} u)\big|_{F_j} = 2\lambda_l \iint\limits_{F_l} u(\boldsymbol{s}) \, d\boldsymbol{s},$$

$$(4.8) \qquad (\mathcal{E}_{E_{jl},l}^{\text{grad}} u)\big|_{F_l} = 0,$$

$$(4.9) \qquad \mathcal{E}_{ij,l}^{\text{grad}} u \big|_{F_i \cup F_j} = 0.$$

  (3) *If u is a polynomial of degree at most p that vanishes on $E_{ij} \cup E_{jk}$, then $\mathcal{E}_{E_{jl},l}^{\text{grad}} u$ is a polynomial of degree at most p for all $j = 0, 1, 2$.*

Proof. To prove the first item, in view of Theorem 2.1(1) and Proposition 3.1(1), it suffices to prove that the last term in (4.4), namely $\mathcal{E}_{E_{kl},l}^{\text{grad}}$, is a continuous operator from $H_{0,ij}^{1/2}$ into $H^1(K)$. Since $\mathcal{E}_{E_{kl},l}^{\text{grad}}$ is obtained by mapping $\mathcal{E}_{\hat{E}_{03}}^{\text{grad}}$ from $\hat{K}$, it suffices to prove the continuity of $\mathcal{E}_{\hat{E}_{03}}^{\text{grad}}$. But $\mathcal{E}_{\hat{E}_{03}}^{\text{grad}} = \mathcal{B}_{\hat{E}_{03}} \circ \mathcal{L}^{\text{grad}}$, so by Lemmas 3.2 and 4.1, we conclude that $\mathcal{E}_{\hat{E}_{03}}^{\text{grad}}$ is continuous from $L_{1/x}^2(\hat{F}_3) \cap L_{1/y}^2(\hat{F}_3)$ into $H^1(\hat{K})$. Thus $\mathcal{E}_{E_{kl},l}^{\text{grad}}$ is continuous from $H_{0,ij}^{1/2}$ into $H^1(K)$.

To prove (4.6), we start with

$$\mathcal{E}_{F_i,l}^{\text{grad}} u - \mathcal{E}_{E_{jl},l}^{\text{grad}} u = \frac{2\lambda_l}{(\lambda_i + \lambda_l)^3} \iint\limits_{T_l(0,\lambda_j,\lambda_k)} u(\boldsymbol{s}) \, d\boldsymbol{s} - \frac{2\lambda_l}{(1 - \lambda_j)^3} \iint\limits_{T_l(0,\lambda_j,0)} u(\boldsymbol{s}) \, d\boldsymbol{s}.$$

Since $\lambda_k = 0$ on face F_k, by setting $\lambda_k = 0$ in the first integral above, and observing that $1 - \lambda_j = \lambda_i + \lambda_l$, we see that the right hand side above is zero, thus proving (4.6). Proofs of (4.7) and (4.8) follow immediately by setting $\lambda_i = 0$ and $\lambda_l = 0$, respectively, in (4.3). To prove (4.9), observe that

$$\mathcal{E}_{ij,l}^{\text{grad}} u \big|_{F_i} = \mathcal{E}_{i,l}^{\text{grad}} u \big|_{F_i} - (\mathcal{E}_{F_j,l}^{\text{grad}} u - \mathcal{E}_{E_{kl},l}^{\text{grad}} u) \big|_{F_i}.$$

We see that this trace is zero by using Proposition 3.1(2) and (4.6). By symmetry of $\mathcal{E}_{ij,l}^{\text{grad}}$, the trace on F_j must also vanish.

Proof of (3): This is best seen using the reference tetrahedron expression (4.1). Since u vanishes on the edges of $\hat{K}$ along the x and y axes, we can write, for instance, $u(x,y) =$

$xu_{p-1}(x, y)$ for some polynomials u_{p-1} of degree at most $p - 1$. Then

$$\mathcal{E}_{\hat{E}_{03}}^{\text{grad}} u = \frac{2z}{x+y+z} \int_0^1 \int_0^{1-s} u(s(x+y+z), t(x+y+z)) \, dt \, ds$$

$$= 2z \int_0^1 \int_0^{1-s} s \, u_{p-1}(s(x+y+z), t(x+y+z)) \, dt \, ds,$$

so the polynomial preservation property follows. $\qquad\qquad\qquad\qquad\qquad\qquad$ □

5. EXTENSION OF A FACE BUBBLE

In this section, we consider the problem of extending functions in

$$(5.1) \qquad H_{0,ijk}^{1/2}(F_l) = H^{1/2}(F_l) \cap L_{1/\lambda_i}^2(F_l) \cap L_{1/\lambda_j}^2(F_l) \cap L_{1/\lambda_k}^2(F_l)$$

in such a way that the extension is polynomial preserving and has zero traces on all faces other than F_l. Such extensions have applications in the design of optimal face shape functions for high order finite elements. Note that smooth functions that vanish on the boundary edges of F_l (the so called "face bubble" functions) are in $H_{0,ijk}^{1/2}(F_l)$.

The extension operator from $H_{0,ijk}^{1/2}(F_l)$ into $H^1(K)$ is given by

$$(5.2) \qquad \mathcal{E}_{ijk,l}^{\text{grad}} u = \mathcal{E}_l^{\text{grad}} u \; - \; \mathcal{E}_{V_l}^{\text{grad}} u \; - \sum_{m \in \{i,j,k\}} \left(\mathcal{E}_{F_m,l}^{\text{grad}} u - \mathcal{E}_{E_{ml},l}^{\text{grad}} u \right),$$

where $\mathcal{E}_l^{\text{grad}}$ is the primary extension operator defined in (2.3), $\mathcal{E}_{F_i,l}^{\text{grad}}$ is the face correction operator defined in (3.2), $\mathcal{E}_{E_{il},l}^{\text{grad}}$ is the edge correction operator defined in (4.3), and

$$\mathcal{E}_{V_l}^{\text{grad}} u \, (\lambda_0, \lambda_1, \lambda_2, \lambda_3) = 2\lambda_l \iint_{F_l} u(\boldsymbol{s}) \, d\boldsymbol{s}.$$

This last operator may be thought of as a *vertex correction*, because the right hand side above equals $\lambda_l \mathcal{E}_l^{\text{grad}} u(\boldsymbol{a}_l)$, i.e., it depends only on the value of $\mathcal{E}_l^{\text{grad}} u$ at a vertex. It is needed because the edge corrections alter the traces zeroed by the face corrections.

Proposition 5.1. *The operator $\mathcal{E}_{ijk,l}^{\text{grad}}$ satisfies the following:*

(1) *$\mathcal{E}_{ijk,l}^{\text{grad}}$ is a continuous map from $H_{0,ijk}^{1/2}(F_l)$ into $H^1(K)$.*
(2) *The traces of $\mathcal{E}_{ijk,l}^{\text{grad}} u$ on all faces of the tetrahedron are zero except for the face F_l, where the trace equals u.*
(3) *If u is a polynomial of degree at most p, then $\mathcal{E}_{ijk,l}^{\text{grad}} u$ is a polynomial of degree at most p.*

Proof. The continuity property of $\mathcal{E}_{ijk,l}^{\text{grad}}$ follows immediately by the results established in Propositions 3.1(1) and 4.1(1), since the continuity of the vertex correction is obvious.

Let us now prove that the trace of $\mathcal{E}_{ijk,l}^{\text{grad}} u$ on F_i is zero. Observe that by a rearrangement of the terms in (5.2),

$$\mathcal{E}_{ijk,l}^{\text{grad}} u|_{F_i} = (\mathcal{E}_l^{\text{grad}} u - \mathcal{E}_{F_i,l}^{\text{grad}} u)|_{F_i}$$

$$- (\mathcal{E}_{F_j,l}^{\text{grad}} u - \mathcal{E}_{E_{kl},l}^{\text{grad}} u)|_{F_i} - (\mathcal{E}_{F_k,l}^{\text{grad}} u - \mathcal{E}_{E_{jl},l}^{\text{grad}} u)|_{F_i}$$

$$+ (\mathcal{E}_{E_{il},l}^{\text{grad}} u - \mathcal{E}_{V_l}^{\text{grad}} u)|_{F_i}.$$

The first term on the right hand side is zero by Proposition 3.1(2). The second and third are also zero by Proposition 4.1 – see (4.6). Furthermore, the last term also vanishes by (4.7). Thus the trace of $\mathcal{E}^{\mathrm{grad}}_{ijk,l}u$ on F_i is zero. The traces on F_j and F_k must also be zero because the expression for $\mathcal{E}^{\mathrm{grad}}_{ijk,l}u$ is symmetric in i, j, k. That the trace on F_l is u, as well as the polynomial preservation property, follows by collecting the results in Propositions 3.1 and 4.1. $\qquad\square$

6. The total extension operator

We are now in a position to solve the extension problem as posed in the beginning of this paper in (1.5)–(1.7) by combining the primary extension $\mathcal{E}^{\mathrm{grad}}$ with the face, edge, and vertex corrections.

Let u be any function in $H^{1/2}(\partial K)$. We extend this function into K by selecting the faces of K in some order, say F_i, F_j, F_k and F_l, and defining the following extensions from these faces:

$$
\begin{aligned}
U_i &= \mathcal{E}^{\mathrm{grad}}_i u, \\
U_j &= \mathcal{E}^{\mathrm{grad}}_{i,j} w_j, &&\text{where } w_j = (u - U_i)|_{F_j}, \\
U_k &= \mathcal{E}^{\mathrm{grad}}_{ij,k} w_k, &&\text{where } w_k = (u - U_i - U_j)|_{F_k}, \\
U_l &= \mathcal{E}^{\mathrm{grad}}_{ijk,l} w_l, &&\text{where } w_l = (u - U_i - U_j - U_k)|_{F_l}.
\end{aligned}
$$

Here, the operators $\mathcal{E}^{\mathrm{grad}}_i$, $\mathcal{E}^{\mathrm{grad}}_{i,j}$, $\mathcal{E}^{\mathrm{grad}}_{ij,k}$ and $\mathcal{E}^{\mathrm{grad}}_{ijk,l}$ are as defined in (2.3), (3.3), (4.4), and (5.2), respectively. The total extension operator is then defined by

$$(6.1) \qquad\qquad \mathcal{E}^{\mathrm{grad}}_K u = U_i + U_j + U_k + U_l.$$

This operator is well defined provided w_j, w_k, and w_l are in the domains of the operators $\mathcal{E}^{\mathrm{grad}}_{i,j}$, $\mathcal{E}^{\mathrm{grad}}_{ij,k}$, and $\mathcal{E}^{\mathrm{grad}}_{ijk,l}$, respectively, which is indeed the case as asserted by the following lemma:

Lemma 6.1. *There is a constant $C > 0$ independent of u such that the functions w_j, w_k, and w_l defined above satisfy*

$$
\begin{aligned}
\|w_j\|_{H^{1/2}_{0,i}(F_j)} &\leq C\|u\|_{H^{1/2}(\partial K)}, \\
\|w_k\|_{H^{1/2}_{0,ij}(F_k)} &\leq C\|u\|_{H^{1/2}(\partial K)}, \\
\|w_l\|_{H^{1/2}_{0,ijk}(F_l)} &\leq C\|u\|_{H^{1/2}(\partial K)}.
\end{aligned}
$$

Collecting all the results we have established in the course of the construction of $\mathcal{E}^{\mathrm{grad}}_K$, we have the following theorem:

Theorem 6.1. *The operator $\mathcal{E}^{\mathrm{grad}}_K$ defined by (6.1) satisfies*

(1) *the extension property (1.5),*
(2) *the continuity property (1.6), and*
(3) *the polynomial preservation property (1.7).*

Proof. All the three properties follow from the previous propositions. E.g., to prove the continuity property, we use the previously proved inequalities

$$\|\mathcal{E}^{\mathrm{grad}}_{ijk,l} w_l\|_{H^1(K)} \leq C \|w_l\|_{H^{1/2}_{0,ijk}(F_l)}, \qquad\qquad \text{by Proposition 5.1(1),}$$

$$\|\mathcal{E}^{\mathrm{grad}}_{ij,k} w_k\|_{H^1(K)} \leq C \|w_k\|_{H^{1/2}_{0,ij}(F_k)}, \qquad\qquad \text{by Proposition 4.1(1),}$$

$$\|\mathcal{E}^{\mathrm{grad}}_{i,j} w_j\|_{H^1(K)} \leq C \|w_j\|_{H^{1/2}_{0,i}(F_j)}, \qquad\qquad \text{by Proposition 3.1(1),}$$

$$\|\mathcal{E}^{\mathrm{grad}}_{i} u\|_{H^1(K)} \leq C \|u\|_{H^{1/2}(F_l)}, \qquad\qquad \text{by Theorem 2.1(1),}$$

in

$$\|\mathcal{E}^{\mathrm{grad}}_{K} u\|_{H^1(K)} \leq \|\mathcal{E}^{\mathrm{grad}}_{i} u\|_{H^1(K)} + \|\mathcal{E}^{\mathrm{grad}}_{i,j} w_j\|_{H^1(K)} + \|\mathcal{E}^{\mathrm{grad}}_{ij,k} w_k\|_{H^1(K)} + \|\mathcal{E}^{\mathrm{grad}}_{ijk,l} w_l\|_{H^1(K)}$$

and complete the estimate using Lemma 6.1. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Appendix A. Proofs of the lemmas

In this section, we prove all the previously stated lemmas in the order in which they appeared in Sections 2–5. Before we start proving these lemmas, let us begin with some preliminary results which will turn useful in the proofs. These results are often stated in more generality than we need them in this paper, because we will need the general forms in later parts of this series.

There are two kind of operators that pervade the proofs, namely the averaging type, and the interpolating type. We first collect the continuity properties of some averaging type operators in Lemmas A.1 and A.2. The interpolatory operators are considered next in Lemmas A.3 and A.4, after which we begin proving the lemmas of the previous sections. We start with operators that map functions on the x-y face to functions on the y-z face.

Lemma A.1 (Face-to-face maps)**.** *Let $\theta(s,t)$ be a function in $C(\bar{\hat{F}})$. Define the the following maps for smooth $u(x,y)$:*

$$u(x,y) \longmapsto \quad A^{\theta}_1 u\,(y,z) := \int_0^1 \theta(s,0)\, u(sz,y)\, ds$$

$$u(x,y) \longmapsto \quad A^{\theta}_2 u\,(y,z) := \int_0^1 \theta(s,1-s)\, u(sz, y+(1-s)z)\, ds$$

$$u(x,y) \longmapsto \quad A^{\theta}_3 u\,(y,z) := 2\int_0^1 \int_0^{1-s} \theta(s,t)\, u(sz, y+tz)\, dt\, ds.$$

Then the following continuity properties hold:

(1) *The maps A^{θ}_1, A^{θ}_2, and A^{θ}_3 extend to continuous operators from $L^2_{1/x}(\hat{F}_3)$ into $L^2_{1/z}(\hat{F}_1)$.*

(2) *If in addition, $\theta(s,t)$ is in $C^1(\bar{\hat{F}})$, then A^{θ}_3 is continuous under the a stronger norm, namely*

$$(\mathrm{A.1}) \qquad\qquad \|A^{\theta}_3 u\|_{L^2_{1/z}(\hat{F}_1) \cap H^1_z(\hat{F}_1)} \leq C \|u\|_{L^2_{1/x}(\hat{F}_3)}.$$

Proof. Because of the well known [20] density of $C^\infty(\bar{\hat{F}}_3) \cap L^2_{1/x}(\hat{F}_3)$ in $L^2_{1/x}(\hat{F}_3)$, to prove item (1), it suffices to prove that there is a constant $C > 0$ such that

$$\|A^{\theta}_i u\|_{L^2_{1/z}(\hat{F}_3)} \leq C \|u\|_{L^2_{1/x}(\hat{F}_1)}$$

for all smooth functions $u(x,y)$.

Let us consider the operator A_1^θ acting on a smooth function u. We start with an application of Cauchy-Schwarz inequality to bound the required norm:

$$\|A_1^\theta u\|^2_{L^2_{1/z}(\hat{F}_1)} = \int_0^1 \int_0^{1-y} \frac{1}{z} \left| \int_0^1 \theta(s,0) \, u(sz,y) \, ds \right|^2 dz \, dy$$

$$\leq \int_0^1 \int_0^{1-y} \frac{1}{z} \left(\int_0^1 |\theta(s,0)|^2 ds \right) \left(\int_0^1 |u(sz,y)|^2 ds \right) dz \, dy,$$

by Cauchy-Schwarz inequality. Setting $C_\theta = \int_0^1 |\theta(s,0)|^2 ds$, we continue, and apply Fubini's theorem on a tetrahedral region, as follows.

$$\|A_1^\theta u\|^2_{L^2_{1/z}(\hat{F}_1)} = C_\theta^2 \int_0^1 \int_0^{1-y} \frac{1}{z} \int_0^z |u(x,y)|^2 \frac{dx}{z} \, dz \, dy,$$

$$= C_\theta^2 \int_0^1 \int_0^{1-y} \left(\int_x^{1-y} \frac{1}{z^2} \, dz \right) |u(x,y)|^2 \, dx \, dy,$$

$$= C_\theta^2 \int_0^1 \int_0^{1-y} \left(\frac{1}{x} - \frac{1}{1-y} \right) |u(x,y)|^2 \, dx \, dy,$$

$$\leq C_\theta^2 \|u\|^2_{L^2_{1/x}(\hat{F})}.$$

This establishes the continuity of A_1^θ.

To prove the continuity of the next map,

$$\|A_2^\theta u\|^2_{L^2_{1/z}(\hat{F}_1)} = \int_0^1 \int_0^{1-y} \frac{1}{z} \left| \int_0^1 \theta(s,1-s)u(sz,y+(1-s)z) \, ds \right|^2 dz \, dy$$

$$= \left(\int_0^1 |\theta(s,1-s)|^2 ds \right) \int_0^1 \int_0^{1-y} \frac{1}{z} \int_0^1 |u(sz,y+(1-s)z)|^2 ds \, dz \, dy$$

$$= C_\theta^2 \int_0^1 \int_0^{1-y} \int_0^z \frac{1}{z} |u(x,y+z-x)|^2 \frac{dx}{z} dz \, dy,$$

by the substitution $x = zs$. Here $C_\theta = \int_0^1 |\theta(s,1-s)|^2 ds$. Continuing,

$$\|A_2^\theta u\|^2_{L^2_{1/z}(\hat{F}_1)} = C_\theta^2 \int_0^1 \int_x^1 \int_0^{1-z} \frac{1}{z^2} |u(x,y+z-x)|^2 \, dy \, dz \, dx, \quad \text{by Fubini's theorem,}$$

$$= C_\theta^2 \int_0^1 \int_x^1 \int_{z-x}^{1-x} \frac{1}{z^2} |u(x,y')|^2 dy' \, dz \, dx, \qquad \text{by substitution } y' = y+z-x,$$

$$\leq C_\theta^2 \int_0^1 \int_x^1 \int_0^{1-x} |u(x,y')|^2 dy' \, dz \, dx$$

$$= C_\theta^2 \int_0^1 \int_0^{1-x} \left(\frac{1}{x} - 1 \right) |u(x,y')|^2 dy' \, dx$$

$$\leq C_\theta^2 \|u\|^2_{L^2_{1/x}(\hat{F})}.$$

To prove the continuity of A_3^θ,

$$\left\| \frac{1}{2} A_3^1 u \right\|^2_{L^2_{1/z}(\hat{F}_1)} = \int_0^1 \int_0^{1-y} \frac{1}{z} \left(\int_0^1 \int_0^{1-s} \theta(s,t) \, u(sz,y+tz) \, dt \, ds \right)^2 dz \, dy$$

$$\leq \|\theta\|^2_{L^2(\hat{F})} \int_0^1 \int_0^{1-y} \frac{1}{z} \left(\int_0^1 \int_0^{1-s} |u(sz,y+tz)|^2 \, dt \, ds \right) dz \, dy,$$

where we have applied the Cauchy-Schwarz inequality. Next, we change variables, and then interchange the order of integration, carefully considering the variable integration limits:

$$\left\|\frac{1}{2}A_3^1 u\right\|_{L_{1/z}^2(\hat{F}_1)}^2 \leq \int_0^1 \int_0^{1-y} \frac{1}{z^3} \int_0^z \int_y^{y+z-x'} |u(x',y')|^2 \; dy'dx' \; dz\, dy$$

$$= \int_0^1 \int_0^{1-y'} |u(x',y')|^2 \int_0^{y'} \int_{x'+y'-y}^{1-y} z^{-3} \; dz\, dy \; dx'\, dy',$$

(In the last step we have applied Fubini's theorem to a four dimensional region.) Now the inner z and y integrals can be evaluated and estimated by

$$2\int_0^{y'} \int_{x'+y'-y}^{1-y} z^{-3} \; dz\, dy = \int_0^{y'} \left((x'+y'-y)^{-2} - (1-y)^{-2} \right) dy$$

$$= -\frac{1}{1-y'} + \frac{1}{x'} + 1 - \frac{1}{x'+y'}$$

$$\leq \frac{1}{x'}.$$

Hence the continuity of A_3^θ from $L_{1/x}^2(\hat{F}_3)$ into $L_{1/z}^2(\hat{F}_1)$ follows.

It now only remains to prove (A.1) under the stronger assumption on θ. This is a consequence of the following two identities.

$$(A.2) \qquad \partial_y(A_3^\theta u) = -\frac{1}{z}A_3^{\partial_t\theta}u + \frac{2}{z}(A_1^\theta u - A_2^\theta u)$$

$$(A.3) \qquad \partial_z(A_3^\theta u) = -\frac{1}{z}A_3^{(s\partial_s\theta + t\partial_t\theta)}u + \frac{2}{z}(A_2^\theta u - A_3^\theta u).$$

These identities follow by variable changes and differentiation, e.g., to prove the first one,

$$\partial_y A_3^\theta v = \frac{2}{z^2} \frac{\partial}{\partial y}\left(\int_0^x \int_y^{y+z-x'} \theta(\frac{x'}{z}, \frac{y'-y}{z}) \, u(x',y') \; dy'\, dx' \right)$$

$$= -\frac{1}{z}\int_0^x \int_y^{y+z-x'} \frac{\partial\theta}{\partial t}(\frac{x'}{z}, \frac{y'-y}{z}) \, u(x',y') \; dy'\, dx'$$

$$+ \frac{2}{z^2}\int_0^x \theta(\frac{x'}{z}, \frac{y'-y}{z}) \, u(x', y+z-x') \; dx' - \frac{2}{z^2}\int_0^x \theta(\frac{x'}{z}, 0) \, v(x', y) \; dx'$$

$$= -\frac{1}{z}A_3^{\partial_t\theta}u + \frac{2}{z}\int_0^1 \theta(s, 1-s) \, u(sz, y+(1-s)z) \; ds - \frac{2}{z}\int_0^1 \theta(s, 0) \, u(sz, y) \; ds,$$

which is (A.2). Equation (A.3) is proved similarly. From (A.2) and (A.3), it is clear that if θ satisfies the additional smoothness assumptions, then we can apply continuity results already proved above to obtain (A.1). □

Lemma A.2 (Face-to-edge maps). *Let $\theta(s,t)$ be a function in $C(\bar{F})$. Define the the following maps for smooth $u(x,y)$.*

$$u(x,y) \longmapsto B_1^\theta u\,(z) := \int_0^1 \theta(s, 1-s) \, u(sz, (1-s)z) \; ds,$$

$$u(x,y) \longmapsto B_2^\theta u\,(z) := 2\int_0^1 \int_0^{1-s} \theta(s,t) u(sz, tz) \; dt\, ds.$$

Then the following continuity properties hold:

(1) *B_1^θ and B_2^θ extend to continuous operators from $L_{1/x}^2(\hat{F}_3) \cap L_{1/y}^2(\hat{F}_3)$ into $L^2(\hat{E}_{03})$.*

(2) *If in addition, $\theta(s,t)$ is in $C^1(\bar{\hat{F}})$, then B_2^θ is continuous in a stronger norm:*

(A.4)
$$\|B_2^\theta u\|_{H^2_{z^2}(\hat{E}_{03} \cap L^2(\hat{E}_{03})} \le C \|u\|_{L^2_{1/x}(\hat{F}_3) \cap L^2_{1/y}(\hat{F}_3)}.$$

Proof. As in Lemma A.1, it suffices to consider smooth $u(x,y)$ because of density. To prove the continuity of B_1^θ, set $C_\theta = \int_0^1 |\theta(s, 1-s)|^2 ds$ and observe that

$$\|B_1^\theta u\|_{L^2(\hat{E}_{03})}^2 = \int_0^1 \left| \int_0^1 \theta(s, 1-s) \, u(sz, (1-s)z) \, ds \right|^2 dz,$$
$$\le C_\theta^2 \int_0^1 \int_0^1 |u(sz, (1-s)z)|^2 \, ds \, dz,$$
$$= C_\theta^2 \int_0^1 \int_0^z \frac{1}{z} |u(x', z-x')|^2 \, dx' \, dz,$$
$$= C_\theta^2 \int_0^1 \int_0^{1-x} \frac{1}{x+y} |u(x,y)|^2 \, dy \, dx.$$

The result now follows from

$$\frac{1}{x+y} \le \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right).$$

The continuity of B_2 is proved as follows:

$$\left\|\frac{1}{2} B_2^\theta u\right\|_{L^2(\hat{E}_{03})}^2 = \int_0^1 \left| \int_0^1 \int_0^{1-s} \theta(s,t) \, u(sz, tz) \, dt \, ds \right|^2 dz$$
$$\le \|\theta\|_{L^2(\hat{F})}^2 \int_0^1 \int_0^1 \int_0^{1-s} |u(sz, tz)|^2 \, dt \, ds \, dz$$
$$= \|\theta\|_{L^2(\hat{F})}^2 \int_0^1 \int_0^z \int_0^{z-x'} \frac{1}{z^2} |u(x',y')|^2 \, dy' \, dx' \, dz.$$

Now we apply Fubini's theorem over a tetrahedron to get

$$\left\|\frac{1}{2} B_2^\theta u\right\|_{L^2(\hat{E}_{03})}^2 = \int_0^1 \int_0^{1-x'} |u(x',y')|^2 \int_{x'+y'}^1 \frac{1}{z^2} \, dz \, dy' \, dx'$$
$$= \int_0^1 \int_0^{1-x'} \left(\frac{1}{x'+y'} - 1\right) |u(x',y')|^2 \, dz \, dy' \, dx',$$

from which the result follows since $\dfrac{1}{x'+y'} - 1 \le \dfrac{1}{2}(\dfrac{1}{x'} + \dfrac{1}{y'})$.

It now only remains to prove (A.4). Differentiating,

$$\frac{d}{dz} B_2^\theta u = \frac{d}{dz}\left(\frac{2}{z^2} \int_0^z \int_0^{z-x'} u(x',y') \, dy' \, dx'\right)$$
$$= -\frac{2}{z} B_2^\theta u - \frac{1}{z} B_2^{s\partial_s\theta + t\partial_t\theta} u + \frac{2}{z^2} \int_0^z (\frac{x'}{z}, z - x'z) \, u(x', z - x') \, dx'.$$

In other words,

(A.5)
$$\frac{d}{dz} B_2^\theta u = -\frac{1}{z} B_2^{s\partial_s\theta + t\partial_t\theta} u + \frac{2}{z}(B_1^\theta u - B_2^\theta u).$$

Hence (A.4) follows from the already proved item (1). $\qquad \square$

Lemma A.3 (Face-to-tetrahedron maps)**.** *Let $\theta(x, y, z)$ be a function in $L^\infty(\hat{K})$. The following map, defined for smooth $\phi(y, z)$,*

$$\phi(y, z) \longmapsto J_\theta \phi(x, y, z) := \theta(x, y, z)\, \phi(y, x + z)$$

extend to a continuous operator from $L_z^2(\hat{F}_1)$ into $L^2(\hat{K})$.

Proof. The proof is immediate once we make the change of variable $(x', y', z') = (x, y, x + z)$. Indeed,

$$\begin{aligned}
\|J_\theta \phi\|_{L^2(\hat{K})}^2 &= \int_0^1 \int_0^{1-x} \int_0^{1-x-z} |\theta(x, y, z)\, \phi(y, x + z)|^2 \; dy\, dz\, dx \\
&= \int_0^1 \int_0^{1-y'} \int_0^{z'} |\theta(x', y', z' - x')|^2 \, |\phi(y', z')|^2 \; dx'\, dz'\, dy' \\
&\leq \int_0^1 \int_0^{1-y'} \left(\int_0^{z'} \|\theta\|_{L^\infty(\hat{K})}^2 dx' \right) |\phi(y', z')|^2 \, dz'\, dy' \\
&\leq \|\theta\|_{L^\infty(\hat{K})}^2 \|\phi\|_{L_z^2(\hat{F}_1)}^2 .
\end{aligned}$$

$\square$

Lemma A.4 (Edge-to-tetrahedron maps)**.** *Let $\theta(x, y, z)$ be a function in $L^\infty(\hat{K})$. The following map, defined for smooth $\psi(z)$,*

$$\psi(z) \longmapsto L_\theta \psi(x, y, z) := \theta(x, y, z)\, \psi(x + y + z)$$

extend to continuous operators from $L_{z^2}^2(\hat{E}_{03})$ into $L^2(\hat{K})$.

Proof. Making the variable change $(x', y', z') = (x + y, y, x + y + z)$, we find that

$$\begin{aligned}
\|L_\theta \psi\|_{L^2(\hat{K})}^2 &= \int_0^1 \int_0^{z'} \int_0^{y'} |\theta(x' - y', y', z - x')\, \psi(z')|^2 \; dx'\, dy'\, dz' \\
&\leq \int_0^1 \|\theta\|_{L^\infty(\hat{K})}^2 \frac{(z')^2}{2} |\psi(z')|^2 \; dz',
\end{aligned}$$

thus proving the continuity of L_θ.

$\square$

With the help of the above results, we now start proving all the lemmas that appeared in the previous sections.

Proof of Lemma 3.2. First, we need to prove that

$$\mathcal{R}^{\text{grad}} : \; L_{1/x}^2(\hat{F}_3) \longmapsto L_{1/z}^2(\hat{F}_1) \cap H_z^1(\hat{F}_1)$$

is continuous. But

$$\mathcal{R}^{\text{grad}} u = A_3^\theta u,$$

with $\theta(s, t) \equiv 1$. Hence, we obtain the required continuity properties from Lemma A.1, specifically the estimate in (A.1).

Similarly, the continuity of $\mathcal{L}^{\text{grad}} : L_{1/x}^2(\hat{F}_3) \cap L_{1/y}^2(\hat{F}_3) \longmapsto L^2(\hat{E}_{03}) \cap H_{z^2}^1(\hat{E}_{03})$ follows from Lemma A.2 because

$$\mathcal{L}^{\text{grad}} u = B_2^\theta u,$$

with $\theta(s, t) \equiv 1$ (see (A.4)).

$\square$

Proof of Lemma 3.1. It suffices to prove that

$$(A.6) \qquad \|\mathcal{B}_{\hat{F}_1}\phi\|_{H^1(\hat{K})} \le C\big(\|\phi\|_{L^2_{1/z}(\hat{F}_1)} + \|\phi\|_{H^1_z(\hat{F}_1)}\big)$$

for all $\phi(y, z)$ in $C^\infty(\bar{\hat{F}}_1)$ that vanish in a neighborhood of the y-axis, because it is proven in [16, Lemma 3.1] that such functions are dense in $L^2_{1/z}(\hat{F}_1) \cap H^1_z(\hat{F}_1)$. For such a ϕ, the function $\mathcal{B}_{\hat{F}_1}\phi$ is obviously in $H^1(\hat{K})$ and vanishes on $\hat{F}_3$. Hence the Poincaré inequality gives a constant $\hat{C}$ depending only on $\hat{K}$ such that

$$\|\mathcal{B}_{\hat{F}_1}\phi\|_{L^2(\hat{K})} \le \hat{C}\|\mathbf{grad}(\mathcal{B}_{\hat{F}_1}\phi)\|_{L^2(\hat{K})}.$$

Moreover,

$$\mathbf{grad}(\mathcal{B}_{\hat{F}_1}\phi) = \frac{z}{x+z}\,\mathbf{grad}\Big(\phi(y, x+z)\Big) + \phi(y, x+z)\,\mathbf{grad}\Big(\frac{z}{x+z}\Big)$$

$$= \frac{z}{x+z}\begin{pmatrix} \partial_z\phi(y, x+z) \\ \partial_y\phi(y, x+z) \\ \partial_z\phi(y, x+z) \end{pmatrix} + \frac{1}{(x+z)^2}\begin{pmatrix} -z \\ 0 \\ x \end{pmatrix}\phi(y, x+z)$$

$$= \begin{pmatrix} J_{\theta_2}(\partial_z\phi) \\ J_{\theta_2}(\partial_y\phi) \\ J_{\theta_2}(\partial_z\phi) \end{pmatrix} + \begin{pmatrix} -J_{\theta_2}(\phi/z) \\ 0 \\ J_{\theta_1}(\phi/z) \end{pmatrix},$$

with

$$\theta_1 = \frac{x}{x+z} \quad \text{and} \quad \theta_2 = \frac{z}{x+z}.$$

Thus (A.6) follows from the continuity properties of J_θ established in Lemma A.3 (noting that both θ_1 and θ_2 only take values between 0 and 1). $\qquad\square$

Proof of Lemma 4.1. Since $\mathcal{B}_{\hat{E}_{03}}\psi = L_{\theta_2}\psi$ with

$$\theta_2 = \frac{z}{x+y+z},$$

and since

$$\mathbf{grad}(\mathcal{B}_{\hat{E}_{03}}\psi) = \frac{1}{(x+y+z)^2}\begin{pmatrix} -z \\ -z \\ x+y \end{pmatrix}\psi(x+y+z) + \frac{z}{x+y+z}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\psi'(x+y+z)$$

$$= \begin{pmatrix} -L_{\theta_2}(\psi/z) \\ -L_{\theta_2}(\psi/z) \\ L_{\theta_1}(\psi/z) \end{pmatrix} + \begin{pmatrix} L_{\theta_2}(\psi') \\ L_{\theta_2}(\psi') \\ L_{\theta_2}(\psi') \end{pmatrix},$$

with

$$\theta_1 = \frac{x+y}{x+y+z}$$

the required estimate follows immediately from the continuity properties of L_θ established in Lemma A.4. $\qquad\square$

Proof of Lemma 6.1. By a well known characterization of $H^{1/2}$-norms on polyhedral surfaces (see [6, Theorem 2.5], [17, Lemma 1.3.2.6], or [24, Lemma 1]), there is a constant $C > 0$ such that for all v in $H^{1/2}(\partial K)$,

$$(A.7) \qquad \|v\|^2_{L^2_{1/\lambda_i}(F_j)} \leq C\|v\|^2_{H^{1/2}(F_i \cup F_j)} \qquad \text{if } v|_{F_i} = 0,$$

$$(A.8) \qquad \|v\|^2_{L^2_{1/\lambda_i}(F_k)} + \|v\|^2_{L^2_{1/\lambda_j}(F_k)} \leq C\|v\|^2_{H^{1/2}(F_i \cup F_j \cup F_k)} \qquad \text{if } v|_{F_i \cup F_j} = 0,$$

$$(A.9) \quad \|v\|^2_{L^2_{1/\lambda_i}(F_l)} + \|v\|^2_{L^2_{1/\lambda_j}(F_l)} + \|v\|^2_{L^2_{1/\lambda_k}(F_l)} \leq C\|v\|^2_{H^{1/2}(\partial K)} \qquad \text{if } v|_{F_i \cup F_j \cup F_k} = 0.$$

Applying the first inequality above to the function $v = u - \mathcal{E}_i^{\mathrm{grad}}u|_{\partial K}$, which vanishes on F_i (by Theorem 2.1(2)), and coincides with the given function w_j on F_j, we find that

$$\|w_j\|^2_{H^{1/2}_{0,i}(F_j)} = \|w_j\|^2_{H^{1/2}(F_j)} + \|w_j\|^2_{L^2_{1/\lambda_i}(F_j)} \leq C\|v\|^2_{H^{1/2}(F_i \cup F_j)}$$

$$\leq C\|u\|^2_{H^{1/2}(F_i \cup F_j)} + C\|\mathcal{E}_i^{\mathrm{grad}}u\|^2_{H^{1/2}(F_i \cup F_j)}$$

$$\leq C\|u\|^2_{H^{1/2}(F_i \cup F_j)} + C\|\mathcal{E}_i^{\mathrm{grad}}u\|^2_{H^1(K)} \qquad \text{(by trace theorem)}$$

$$\leq C\|u\|^2_{H^{1/2}(F_i \cup F_j)} + C\|u\|^2_{H^{1/2}(F_i)} \qquad \text{(by Theorem 2.1(1))},$$

which proves the first inequality of the lemma.

To prove the bound for $w_k = (u - U_i - U_j)|_{F_k}$, we apply (A.8) to the function

$$v = (I - \mathcal{E}_{i,j}^{\mathrm{grad}}R_j)(I - \mathcal{E}_i^{\mathrm{grad}})u,$$

where R_j denotes the restriction to face F_j (and I is the identity). Clearly, $w_k = R_k v$, and by Proposition 3.1(2), v vanishes on $F_i \cup F_j$, so (A.8) gives

$$\|w_k\|_{H^{1/2}_{0,ij}(F_k)} \leq C\|v\|_{H^{1/2}(F_i \cup F_j \cup F_k)}.$$

The required bound for w_k follows as in the w_j-case by using the trace theorem and the continuity of the operators $\mathcal{E}_{i,j}^{\mathrm{grad}}$ and $\mathcal{E}_i^{\mathrm{grad}}$. The final estimate for w_l is proved similarly, but using

$$v = (I - \mathcal{E}_{ij,k}^{\mathrm{grad}}R_k)(I - \mathcal{E}_{i,j}^{\mathrm{grad}}R_j)(I - \mathcal{E}_i^{\mathrm{grad}})u$$

and (A.9). $\qquad \square$

Appendix B. Two techniques to study the primary extension

The starting point for constructing the tetrahedral extensions in any of the $H^1(K)$, $\boldsymbol{H}(\mathbf{curl})$ or $\boldsymbol{H}(\mathrm{div})$ case is the study of "primary extensions", i.e., extensions into $H^1(\hat{K})$, $\boldsymbol{H}(\mathbf{curl}, \hat{K})$, and $\boldsymbol{H}(\mathrm{div}, \hat{K})$ from one face:

$$(B.1) \qquad \mathcal{E}^{\mathrm{grad}}u\,(x,y,z) = \frac{2}{z^2}\int_x^{x+z}\int_y^{x+y+z-\widetilde{x}} u(\widetilde{x},\widetilde{y})\,d\widetilde{y}\,d\widetilde{x}$$

$$(B.2) \qquad \boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}\,(x,y,z) = \frac{2}{z^3}\int_x^{x+z}\int_y^{x+y+z-\widetilde{x}} \begin{pmatrix} z & 0 \\ 0 & z \\ x-\widetilde{x} & y-\widetilde{y} \end{pmatrix} \boldsymbol{v}(\widetilde{x},\widetilde{y})\,d\widetilde{y}\,d\widetilde{x}$$

$$(B.3) \qquad \boldsymbol{\mathcal{E}}^{\mathrm{div}}w\,(x,y,z) = \frac{2}{z^3}\int_x^{x+z}\int_y^{x+y+z-\widetilde{x}} \begin{pmatrix} \widetilde{x}-x \\ \widetilde{y}-y \\ -z \end{pmatrix} w(\widetilde{x},\widetilde{y})\,d\widetilde{y}\,d\widetilde{x}$$

We have already seen the primary extension appearing in (B.1) in the previous sections. The other two operators above are the primary extensions for the $\boldsymbol{H}(\mathbf{curl})$ and $\boldsymbol{H}(\mathrm{div})$ case (which will be discussed in more detail in Parts II [12] and III [13]).

The purpose of this appendix is to establish the continuity property of $\mathcal{E}^{\mathrm{grad}}$ stated in Theorem 2.1. But since the arguments required to prove the continuity of the other two operators $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ and $\boldsymbol{\mathcal{E}}^{\mathrm{div}}$ are very similar, we establish the continuity of all of these primary extensions together here. We can exhibit two techniques to establish such continuity properties, one using the Fourier transform, and the other using Peetre's K-functional. The latter approach will be shown in Part II. In this appendix, we outline the more standard Fourier transform technique.

The Fourier transform technique proceeds by viewing the above operators as extending functions given on $\mathbb{R}^2$ (the entire x-y plane) into the infinite slab $\mathbb{R}^2 \times (0,1) \equiv \{(x,y,z) : 0 < z < 1\}$. Define the Fourier transform on $\mathbb{R}^2$ as usual by

$$(\text{B.4}) \qquad \mathcal{F}u\,(\boldsymbol{\omega}) \equiv \widehat{u}(\boldsymbol{\omega}) := \iint\limits_{\mathbb{R}^2} e^{-i2\pi\boldsymbol{\omega}\cdot\boldsymbol{x}} u(\boldsymbol{x})\,d\boldsymbol{x}.$$

We begin by noting some properties of the Fourier transform of an indicator function.

Lemma B.1. *Let $\chi_{_\triangle}$ denote the characteristic function of the unit triangle $\triangle = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, x + y < 1\}$, and $\eta(x,y)$ be a linear function. Then, for all $t > 0$ and all unit vectors $\boldsymbol{\omega}$,*

(1) $\widehat{\chi_{_\triangle}\eta}\,(t\boldsymbol{\omega})$ is bounded uniformly in t and $\boldsymbol{\omega}$, and

(2) the integral $\displaystyle\int_0^\infty |\widehat{\chi_{_\triangle}\eta}\,(t\boldsymbol{\omega})|^2\,dt$ is finite and bounded uniformly in $\boldsymbol{\omega}$.

Proof. We begin with the case of a constant function $\eta \equiv 1$. For this case the first assertion is immediate as the exponential in (B.4) is uniformly bounded by one:

$$(\text{B.5}) \qquad |\widehat{\chi_{_\triangle}\eta}(\boldsymbol{\omega})| = |\widehat{\chi_{_\triangle}}(\boldsymbol{\omega})| \le \iint\limits_{\triangle} \left| e^{-i2\pi t\boldsymbol{\omega}\cdot\boldsymbol{x}} \right| d\boldsymbol{x} \le \frac{1}{2}.$$

To prove the second assertion, it will be convenient to work with a rotated system of coordinates $\widetilde{x}, \widetilde{y}$ in such a way that the $\widetilde{x}$-axis passes through the point $\boldsymbol{\omega}$ (see Figure 5). We only consider the case when $\boldsymbol{\omega}$ is in the first quadrant as the other cases are similar. Then

$$\widehat{\chi_{_\triangle}\eta}(\boldsymbol{\omega}) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i2\pi|\boldsymbol{\omega}|\widetilde{x}} \chi_{_\triangle}(\widetilde{x}, \widetilde{y})\,d\widetilde{x}\,d\widetilde{y} = \int_{\widetilde{y}_1}^{\widetilde{y}_2} \left[\frac{e^{-i2\pi|\boldsymbol{\omega}|\widetilde{x}}}{-i2\pi|\boldsymbol{\omega}|} \right]_{\widetilde{x}=a(\widetilde{y})}^{\widetilde{x}=b(\widetilde{y})} d\widetilde{y}\,,$$

where $a(\widetilde{y})$ and $b(\widetilde{y})$ are as in Figure 5, $\widetilde{y}_1$ and $\widetilde{y}_2$ are such that the interval $[\widetilde{y}_1, \widetilde{y}_2]$ is the projection of $\triangle$ on the $\widetilde{y}$-axis, and the notation $[g(x)]_{x=p}^{x=q}$ denotes the difference $g(q) - g(p)$. Now, for $|\boldsymbol{\omega}| = 1$ and $t > 0$, the second assertion follows from

$$\int_1^\infty |\widehat{\chi_{_\triangle}}(t\boldsymbol{\omega})|^2\,dt = \int_1^\infty \left| \int_{\widetilde{y}_1}^{\widetilde{y}_2} \left[\frac{e^{-i2\pi t\widetilde{x}}}{-i2\pi t} \right]_{\widetilde{x}=a(\widetilde{y})}^{\widetilde{x}=b(\widetilde{y})} d\widetilde{y} \right|^2 dt$$

$$\le \int_1^\infty \frac{1}{4\pi^2} \left| \frac{1}{t} \int_{-1}^1 \left[e^{-i2\pi t\widetilde{x}} \right]_{\widetilde{x}=a(\widetilde{y})}^{\widetilde{x}=b(\widetilde{y})} d\widetilde{y} \right|^2 dt$$

and the fact that the inner integral is bounded uniformly in t. Notice that integrability from 0 to 1 follows from the first assertion.
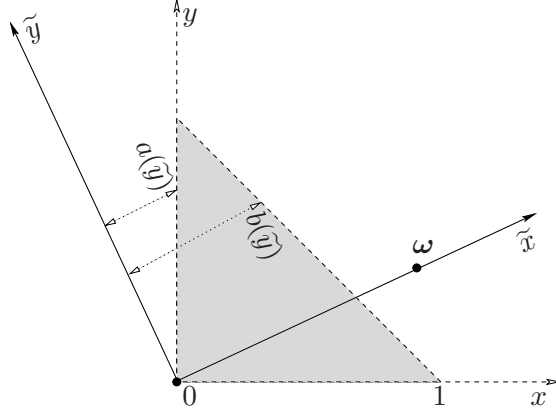
FIGURE 5. Rotated system of coordinates $(\widetilde{x}, \widetilde{y})$ in Lemma B.1

The proofs for $\eta(x,y) = x$ and $\eta(x,y) = y$ are similar. For instance, for $\eta(x,y) = x = (\widetilde{x}, -\widetilde{y})^t \cdot \boldsymbol{\omega}$, we have

$$\widehat{\chi_\Delta x}(t\boldsymbol{\omega}) = \int_{\widetilde{y}_1}^{\widetilde{y}_2} \left(\int_{a(\widetilde{y})}^{b(\widetilde{y})} e^{-i2\pi t \widetilde{x}} \begin{pmatrix} \widetilde{x} \\ -\widetilde{y} \end{pmatrix} \cdot \boldsymbol{\omega} \ d\widetilde{x} \right) d\widetilde{y},$$

so the first assertion follows because the integrand is uniformly bounded. The second assertion also follows because once the inner integral is evaluated, it is immediate that it can be bounded as before by C/t as $t \to \infty$. $\qquad\square$

Remark B.1. In fact, Lemma B.1 extends to a large class of functions $\eta(x,y)$ including polynomials of arbitrary degree.

Lemma B.2. *Let $\eta(s,t)$ be a linear polynomial. Then the map defined for smooth functions $u(x,y)$ on $\mathbb{R}^2$ by*

$$u(x,y) \longmapsto \ \mathcal{K}_\eta u\,(x,y,z) := \int_0^1 \int_0^{1-s} \eta(s,t)\, u(x+sz, y+tz)\, dt\, ds$$

extends to a continuous operator from $H^{-1/2}(\mathbb{R}^2)$ into $L^2\big(\mathbb{R}^2 \times (0,1)\big)$.

Proof. We shall consider the case $\eta \equiv 1$ first. Notice that for each fixed z, the operator is in the form of a convolution in x and y variables:

$$\mathcal{K}_\eta u(x,y,z) = \frac{1}{z^2} \int_{-z}^0 \int_{-z-s'}^0 u(x-s', y-t')\, dt'\, ds' = \frac{1}{z^2}(\chi_{z\triangledown} * u),$$

where $\chi_{z\triangledown}$ is the characteristic function of $\{(x,y) : x < 0, y < 0, x+y > -z\}$ (the unit triangle scaled by $-z$). In what follows, we assume that $u(x,y)$ is a test function from the Schwartz space (see e.g. [19, 23]) and, upon establishing the continuity estimate, tacitly complete the proof using the standard density argument.

Applying the Fourier transform (on the x-y plane) and using its standard properties [19]

$$\widehat{\mathcal{K}_\eta u}\,(\boldsymbol{\omega}, z) \ = \ \frac{1}{z^2}\, \widehat{\chi}_{z\triangledown}(\boldsymbol{\omega})\, \widehat{u}(\boldsymbol{\omega}) \ = \ \widehat{\chi}_\Delta(-z\boldsymbol{\omega})\, \widehat{u}(\boldsymbol{\omega}).$$

By Parseval's identity,

$$\int_0^1 \iint_{\mathbb{R}^2} |\mathcal{K}_\eta u(x,y,z)|^2 \, dx \, dy \, dz = \int_0^1 \iint_{\mathbb{R}^2} |\widehat{\mathcal{K}_\eta u}(\boldsymbol{\omega},z)|^2 \, d\boldsymbol{\omega} \, dz$$

(B.6)
$$= \iint_{\mathbb{R}^2} \left(\int_0^1 |\widehat{\chi}_\triangle(z\boldsymbol{\omega})|^2 \, dz \right) |\widehat{u}(\boldsymbol{\omega})|^2 \, d\boldsymbol{\omega}.$$

Now we split the integral over $\mathbb{R}^2$ into two integrals, one over the unit disk $D = \{\boldsymbol{\omega} \in \mathbb{R}^2 : |\boldsymbol{\omega}| < 1\}$ and the other over $\mathbb{R}^2 \setminus D$, and analyze each separately.

Substituting $z|\boldsymbol{\omega}| = t$ and denoting $\widehat{\boldsymbol{\omega}} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$,

(B.7)
$$\int_0^1 |\widehat{\chi}_\triangle(z\boldsymbol{\omega})|^2 \, dz = \frac{1}{|\boldsymbol{\omega}|} \int_0^{|\boldsymbol{\omega}|} |\widehat{\chi}_\triangle(t\widehat{\boldsymbol{\omega}})|^2 \, dt \le \frac{1}{2}$$

by the first assertion of Lemma B.1 (see (B.5)). By the second assertion of Lemma B.1, we can get another bound for the same term:

(B.8)
$$\int_0^1 |\widehat{\chi}_\triangle(z\boldsymbol{\omega})|^2 \, dz = \frac{1}{|\boldsymbol{\omega}|} \int_0^{|\boldsymbol{\omega}|} |\widehat{\chi}_\triangle(t\widehat{\boldsymbol{\omega}})|^2 \, dt \le \frac{1}{|\boldsymbol{\omega}|} \int_0^\infty |\widehat{\chi}_\triangle(t\widehat{\boldsymbol{\omega}})|^2 \, dt \le \frac{c}{|\boldsymbol{\omega}|}.$$

Using these estimates in (B.6), specifically (B.7) in D and (B.8) in $\mathbb{R}^2 \setminus D$,

$$\int_0^1 \iint_{\mathbb{R}^2} |\mathcal{K}_\eta u(x,y,z)|^2 \, dx \, dy \, dz \le \iint_D \frac{1}{2} |\widehat{u}(\boldsymbol{\omega})|^2 \, d\boldsymbol{\omega} + \iint_{\mathbb{R}^2 \setminus D} \frac{c}{|\boldsymbol{\omega}|} |\widehat{u}(\boldsymbol{\omega})|^2 \, d\boldsymbol{\omega}$$

$$\le C \iint_{\mathbb{R}^2} \left(1 + |\boldsymbol{\omega}|^2\right)^{-1/2} |\widehat{u}(\boldsymbol{\omega})|^2 \, d\boldsymbol{\omega}.$$

Because of a well known characterization of Sobolev norms via the Fourier transform [19, 23], the right hand side above is the square of a norm equivalent to the $H^{-1/2}(\mathbb{R}^2)$-norm, so the proof for the $\eta \equiv 1$ case is finished.

The reasoning for cases $\eta(x,y) = x$ and $\eta(x,y) = y$ is fully analogous. For instance, for the former, we have,

$$\mathcal{K}_\eta(x,y,z) = \frac{1}{z^2} \left(\frac{x}{z} \chi_{z\nabla} * u \right),$$

so

$$\widehat{\mathcal{K}_\eta u}(\boldsymbol{\omega},z) = \frac{1}{z^2} \widehat{\chi_{z\nabla} x}(\boldsymbol{\omega}) \, \widehat{u}(\boldsymbol{\omega}) = \widehat{\chi_\triangle x}(z\boldsymbol{\omega}) \widehat{u}(\boldsymbol{\omega}).$$

The rest of the estimation is fully analogous utilizing the boundedness properties for the Fourier transform of factor $x\chi_\triangle$ proved in Lemma B.1. $\square$

Theorem B.1. *The polynomial extension operators in (B.1)–(B.3) extend to continuous operators on the spaces below:*

$$\mathcal{E}^{\mathrm{grad}} : \quad H^{1/2}(\hat{F}) \quad \longmapsto \quad H^1(\hat{K})$$

$$\boldsymbol{\mathcal{E}}^{\mathrm{curl}} : \quad \boldsymbol{H}^{-1/2}(\mathrm{curl}_\tau, \hat{F}) \longmapsto \quad \boldsymbol{H}(\mathbf{curl}, \hat{K})$$

$$\boldsymbol{\mathcal{E}}^{\mathrm{div}} : \quad H^{-1/2}(\hat{F}) \quad \longmapsto \quad \boldsymbol{H}(\mathrm{div}, \hat{K}).$$

Proof. We quickly sketch the arguments, considering the last operator first.

The $\boldsymbol{H}(\mathrm{div})$ primary extension: According to Lemma B.2, each of the three components of $\boldsymbol{\mathcal{E}}^{\mathrm{div}}$ is a continuous map from $H^{-1/2}(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2 \times (0,1))$. Moreover, $\mathrm{div}(\boldsymbol{\mathcal{E}}^{\mathrm{div}} w) = 0$. Consequently, $\boldsymbol{\mathcal{E}}^{\mathrm{div}}$ maps $H^{-1/2}(\mathbb{R}^2)$ into $\boldsymbol{H}(\mathrm{div}, \mathbb{R}^2 \times (0,1))$. Since there is a continuous

extension operator [7] from $H^{-1/2}(\hat{F})$ into $H^{-1/2}(\mathbb{R}^2)$, and since the restriction operator from $\boldsymbol{H}(\mathrm{div}, \mathbb{R}^2 \times (0,1))$ into $\boldsymbol{H}(\mathrm{div}, \hat{K})$ is obviously continuous, the proof of this case is finished.

The $\boldsymbol{H}(\mathbf{curl})$ primary extension: By the same arguments as above, the three components of $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ are continuous maps from $H^{-1/2}(\hat{F})$ into $L^2(\hat{K})$. Moreover, by the readily established commutativity property

$$\mathbf{curl}\,(\boldsymbol{\mathcal{E}}^{\mathrm{curl}}\boldsymbol{v}) = \boldsymbol{\mathcal{E}}^{\mathrm{div}}(\mathrm{curl}_\tau\,\boldsymbol{v})$$

and the continuity of $\boldsymbol{\mathcal{E}}^{\mathrm{div}}$ on $H^{-1/2}(\hat{F})$, we find that $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$ is a continuous map from $\boldsymbol{H}^{-1/2}(\mathbf{curl}, \hat{F})$ into $\boldsymbol{H}(\mathbf{curl}, T)$.

The H^1 primary extension: The surface gradient $\mathbf{grad}_\tau$ is a continuous map [6] from $H^{1/2}(\hat{F})$ into $\boldsymbol{H}^{-1/2}(\hat{F})$. Combining this with the easily established commutativity property

$$\mathbf{grad}(\mathcal{E}^{\mathrm{grad}}u) = \boldsymbol{\mathcal{E}}^{\mathrm{curl}}(\mathbf{grad}_\tau\,u),$$

and the continuity of $\boldsymbol{\mathcal{E}}^{\mathrm{curl}}$, we find that the left hand side above has its $L^2(\hat{K})$-norm bounded by the $H^{1/2}(\hat{F})$-norm of u. Since square integrability of the gradient $\mathbf{grad}(\mathcal{E}^{\mathrm{grad}}u)$ implies the square integrability of the function $\mathcal{E}^{\mathrm{grad}}u$ (by e.g., a generalization of Stokes theorem [14, Theorem I.2.9]), and since constants are preserved by $\mathcal{E}^{\mathrm{grad}}$, the H^1 extension operator $\mathcal{E}^{\mathrm{grad}}$ is continuous from $H^{1/2}(\hat{F})$ into $H^1(\hat{K})$. $\qquad\square$

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