A MULTILEVEL DECOMPOSITION RESULT IN $H(\text{curl})$

JOACHIM SCHÖBERL

Abstract. Maxwell equations are posed as variational boundary value problems in the function space $H(\text{curl})$ and are discretized by Nédélec finite elements. The arising linear equation systems are usually huge and thus require iterative solvers with good preconditioners. In [10] and [1] multigrid preconditioners for Maxwell equations have been developed and analyzed. In the current paper, we improve the results available so far. The key is to utilize recently proposed Clément type interpolation operators in $H(\text{curl})$ which allow an analysis very similar to the scalar case. The present improvement involves commuting operators which are projections.

1. Introduction

Maxwell equations are partial differential equations describing electromagnetic phenomena. In comparison to other fields, their numerical treatment by finite element methods is relatively new. A reason is that they require the vector valued function space $H(\text{curl})$, what has many consequences for the whole numerical analysis. A recent monograph is [11].

The key for the numerical analysis for Maxwell equations is most often the de Rham complex [6, 2]. It is the basis for the construction of finite elements [12, 13, 9, 8, 20, 22] and the a priori error estimates, preconditioners [10, 1, 21, 14], and eigenvalue problems [3, 4].

Most often, Maxwell equations must be treated in 3D. The arising linear equation systems are usually huge and thus require iterative solvers. Due to the different scaling of the differential operator on the solenoidal and the irrotational sub-spaces, the matrix is very ill conditioned. In [10] and [1] multigrid preconditioners taking care of the different scaling on the sub-spaces are formulated and analyzed.

In the present paper we present a new multigrid convergence proof based on commuting interpolation operators. Now, the most tricky

---

1991 *Mathematics Subject Classification.* 65N30.

Key words and phrases. Clément operator, Maxwell equations, multigrid.

The author acknowledges the support from the Austrian Science Foundation FWF within project grant Start Y-192, “hp-FEM: Fast Solvers and Adaptivity”.

part is the construction of the operators [17, 18, 19]. After that, the multigrid analysis follows the line of the scalar case. We can also sharpen the results available so far. The analysis treats the case of non-convex domains, the estimates are robust with respect to small L_2-coefficients, and the constants do not depend on the global shape of the domain.

Notation: We write $a \preceq b$, when $a \leq cb$, where c is a constant independent of a, b, the coefficients ν and κ of the equation, and the mesh-size h. The constant may and will depend on the shape of the finite elements. We write $a \succeq b$ for $b \preceq a$, and we write $a \simeq$ for $a \preceq b$ and $b \preceq a$.

The rest of the paper is organized as follows. In Section 2, the variational problem, the multigrid algorithm and the main theorem is presented. The commuting projection operators are defined and analyzed in Section 3, The main theorem if proven in Section 4.

2. Multigrid methods for $H(\text{curl})$

Let Ω be a bounded, polyhedral Lipschitz domain in $\mathbb{R}^3$. Its boundary $\Gamma = \partial\Omega$ is decomposed into the Dirichlet part Γ_D and the Neumann part Γ_N. As usual, define the space $H(\text{curl}, \omega) = \{v \in [L_2(\omega)]^3 : \text{curl}\, v \in [L_2(\omega)]^3\}$ for some domain ω, and write $H(\text{curl})$ for $\omega = \Omega$. Let $V := H_D(\text{curl}) := \{v \in H(\text{curl}) : v_t = 0 \text{ on } \Gamma_D\}$. Similarly, we define $H^1_D = \{v \in H^1 : v = 0 \text{ on } \Gamma_D\}$. We write v_t and v_n for the tangential and normal traces, respectively.

Several versions of Maxwell equations lead to the variational problem: find $u \in V$ such that

$$A(u, v) = f(v) \qquad \forall\, v \in V \tag{1}$$

with the bilinear-form

$$A(u, v) := \int_\Omega \text{curl}\, u \ \text{curl}\, v \, dx + \kappa \int_\Omega uv \, dx$$

and the linear form $f(.)$ defined as

$$f(v) := \int_\Omega jv \, dx,$$

where $j \in [L_2]^3$ is the given current density satisfying $\text{div}\, j = 0$ and $j_n = 0$.

We assume that the coefficient $\kappa \in \mathbb{R}$ satisfies $0 < \kappa \preceq 1$. In particular if the magnetostatic problem is regularized by adding a small L_2-term, the coefficient κ is small and the robustness of the algorithm is a must.

The analysis becomes often more transparent if one considers the *exact sequence* of spaces

$$H^1 \xrightarrow{\nabla} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L^2, \tag{2}$$

with the Sobolev space $H^1 := \{v \in L_2 : \nabla v \in [L_2]^3\}$, the vector valued spaces $H(\mathrm{curl})$ and $H(\mathrm{div}) := \{v \in [L_2]^3 : \mathrm{div}\, v \in L_2\}$, and the Lebesgue space L_2. The exact sequence property means that the range of the left operator coincides with the kernel of the right operator. For simplicity we assume simple topologies.

Let the domain Ω be covered with a regular triangulation. We define

$$\begin{aligned} \text{the set of vertices} &\quad \mathcal{V} = \{V_i\}, \\ \text{the set of edges} &\quad \mathcal{E} = \{E = [V_{E_1}, V_{E_2}]\}, \\ \text{the set of faces} &\quad \mathcal{F} = \{F = [V_{F_1}, V_{F_2}, V_{F_3}]\}, \\ \text{the set of tetrahedra} &\quad \mathcal{T} = \{T = [V_{T_1}, V_{T_2}, V_{T_3}, V_{T_4}]\}. \end{aligned}$$

We write V, E, F, and T for the particular vertex, edge, face, and tetrahedron. Let V_h^v denote the finite element space generated by the Lagrangian vertex elements, let V_h^e be the space of Nédélec's edge elements [12], the V_h^f be the space of Raviart-Thomas face elements [15], and V_h^t be the space of piece-wise constant finite elements. These discrete spaces inherit the exact sequence property

$$V_h^v \xrightarrow{\nabla} V_h^e \xrightarrow{\mathrm{curl}} V_h^f \xrightarrow{\mathrm{div}} V_h^t\,. \tag{3}$$

The super-scripts v, e, f, and t will be used to indicate vertex-, edge-, face-, and element-related entities.

The multigrid method is based on a sequence of nested meshes $\mathcal{T}_0, \ldots, \mathcal{T}_L$. On each level l, $0 \leq l \leq L$, we define the corresponding finite element space V_l^v, V_l^e, V_l^f, and V_l^t. These spaces are nested, i.e. there holds

$$V_0^e \subset V_1^e \subset \ldots \subset V_L^e$$

for the Nédélec spaces and as well for the other ones. The Maxwell equations require special smoothing iterations. One possibility is the smoother proposed by Hiptmair. It is an additive or multiplicative sub-space correction method. The finite element space on the level l is decomposed as

$$V_l^e = \sum_{E \in \mathcal{E}_l} \mathrm{span}\{\varphi_E\} + \sum_{V \in \mathcal{V}_l} \mathrm{span}\{\nabla \varphi_V\}.$$

Here, φ_E is the edge-element basis function associated with the edge E, and φ_V is the H^1-conforming nodal basis function associated with the vertex V. Note that $\nabla \varphi_V$ belongs to the Nédélec finite element space. The smoothing steps for the gradient basis functions can be implemented as Jacobi or Gauss-Seidel iteration for a scalar potential problem, see [10].

The smoother by Arnold, Falk, and Winther is based on the subspace decomposition

$$V_l^e = \sum_{V \in \mathcal{V}} V_V,$$

where V_V is the space on the vertex patch

$$V_V = \text{span}\{\varphi_E : E \text{ is an edge connected to the vertex } V\}.$$

We denote the smoothing iteration on level l by

$$\hat{u}_l = u_l + D_l^{-1}(f_l - A_l u_l) \tag{4}$$

Definition 1. *We will need the following norms:*
The energy norm:

$$\|u\|_A^2 := \kappa \, \|u\|_{L_2}^2 + \| \operatorname{curl} u\|_{L_2}^2$$

The 2-norm:

$$|\!|\!|u|\!|\!|_2^2 := \inf_{\varphi \in H^2} \left\{ \kappa \|\varphi\|_{H^2}^2 + \|u - \nabla\varphi\|_{H^2}^2 \right\}$$

The 0-norm:

$$|\!|\!|u|\!|\!|_0^2 := \inf_{\varphi \in L_2} \left\{ \kappa \|\varphi\|_{L_2}^2 + \|u - \nabla\varphi\|_{L_2}^2 \right\}$$

The discrete 0-norm for $u_l \in V_l$:

$$|\!|\!|u_l|\!|\!|_{0,l}^2 := \inf_{\varphi \in V_l^v} \left\{ \kappa \|\varphi\|_{L_2}^2 + \|u_l - \nabla\varphi\|_{L_2}^2 \right\}$$

Lemma 2. *The quadratic forms generated by the smoothers by Hiptmair and by Arnold, Falk, and Winther are equivalent to the discrete 0-norms, i.e. there holds*

$$(D_l u_l, u_l) \simeq |\!|\!|u_l|\!|\!|_{0,l}^2. \tag{5}$$

Follows immediately from [10, 1].

In this paper we introduce a new version of commuting interpolation operators Π_l based on [17, 18]. These new operators are projections onto the finite element spaces. By means of these operators we form the multi-level decomposition.

Theorem 3 (Multilevel decomposition)**.** *Let $u \in V_L$. The multi-level decomposition*

$$u = \Pi_0 u + \sum_{l=1}^{L} (\Pi_l - \Pi_{l-1}) u$$

satisfies the stability estimate

$$\|\Pi_0 u\|_A^2 + \sum_{l=1}^{L} h_l^{-2} \|\!| \Pi_l u - \Pi_{l-1} u \|\!|_{0,l}^2 \preceq \|u\|_A^2. \tag{6}$$

3. Commuting Projection Operators

In [17], regularity-free commuting interpolation operators for the sequence of spaces $H^1 \to H(\mathrm{curl}) \to H(\mathrm{div}) \to L_2$ have been introduced and approximation error estimates have been shown. In [18], additional a posteriori estimates were proven. Now, we modify the operators to obtain also the projection property.

To overcome point evaluation which requires more regularity, local averaging operators [7, 16] for Sobolev spaces have been introduced. For each vertex V_i, let $\omega_i \subset B(0,1)$ be a domain with measure $\mathrm{meas}\{\omega_i\} \simeq 1$.

We define the mapping

$$m_i : y \in \omega_i \mapsto V_i + \rho h_i y,$$

where h_i is the local mesh-size, and $\rho > 0$ is a global parameter which is introduced to scale the size of the patches. We define the possibly enlarged mapped sets

$$\omega_i^v := \mathrm{conv}[V_i, m_i(\omega_i)],$$

and the convex combinations associated with the edges, faces, and tetrahedra

$$\begin{aligned} \omega_{ij}^e &:= \mathrm{conv}[\omega_i^v, \omega_j^v], \\ \omega_{ijk}^f &:= \mathrm{conv}[\omega_i^v, \omega_j^v, \omega_k^v], \\ \omega_{ijkl}^t &:= \mathrm{conv}[\omega_i^v, \omega_j^v, \omega_k^v, \omega_l^v]. \end{aligned}$$

We assume that these sets are contained in the vertex patches $\Omega_i^v := \{T : V_i \in T\}$, the edge patches $\Omega_{ij}^e := \Omega_i^v \cup \Omega_j^v$, and so on. In particular, we assume that all these sets are contained in Ω. This construction is possible for polyhedral Lipschitz domains.

Next, we fix some integer $p \geq 0$, and define for each vertex V_i a function

$$f_i \in L_\infty(\omega_i)$$

such that

$$\int_{\omega_i} f_i(y) w(y)\, dy = w(0) \qquad \text{for all polynomials } w \text{ up to order } p. \tag{7}$$

The generic constants will depend on $\|f_i\|_{L_\infty}$. Note that $\|f_i\|_{L_p} \preceq \|f_i\|_{L_\infty}$. One possibility is to choose f_i as the unique polynomial satisfying (7).

In [17], we have constructed interpolation operators satisfying the commuting diagram property:

$$\begin{array}{ccccccc} H^1 & \xrightarrow{\nabla} & H(\text{curl}) & \xrightarrow{\text{curl}} & H(\text{div}) & \xrightarrow{\text{div}} & L^2 \\ \downarrow \pi^v & & \downarrow \pi^e & & \downarrow \pi^f & & \downarrow \pi^t \\ V_h^v & \xrightarrow{\nabla} & V_h^e & \xrightarrow{\text{curl}} & V_h^f & \xrightarrow{\text{div}} & V_h^t . \end{array} \tag{8}$$

We start with a Clément type quasi-interpolation operators for scalar functions. The interpolant is expanded with respect to the vertex basis $\{\varphi_i^v\}$

$$(\pi^v w)(x) := \sum_{V_i \in \mathcal{V}} \psi_i^v(w)\ \varphi_i^v(x), \tag{9}$$

where the vertex evaluation functionals are

$$\psi_i^v(w) := \int_{\omega_i} f_i(y) w(m_i(y))\, dy.$$

The interpolation operators for the remaining spaces are all derived from the specific choice of the interpolation operator π^v. Like the interpolation point for π^v is smeared out, we now move all the involved vertices of the other operators:

$$(\pi^e v)(x) := \sum_{E_{ij} \in \mathcal{E}} \psi_{ij}^e(v)\ \varphi_{ij}^e(x) \tag{10}$$

with the edge evaluation functionals

$$\psi_{ij}^e(v) := \int_{\omega_i} \int_{\omega_j} f_i(y_1) f_j(y_2) \int_{m_i(y_1)}^{m_j(y_2)} \tau \cdot v\, ds\ dy_2 dy_1.$$

We take line integrals starting in the domain ω_i^v and terminating in ω_j^v, and average them by the weighting functions f_i and f_j. Similarly, we define operators for $H(\text{div})$,

$$(\pi^f q)(x) := \sum_{F_{ijk} \in \mathcal{F}} \psi_{ijk}^f(q)\ \varphi_{ijk}^f(x) \tag{11}$$

with the face evaluation functionals

$$\psi^f_{ijk}(q) = \int\limits_{\omega_i}\int\limits_{\omega_j}\int\limits_{\omega_k} f_i f_j f_k \int\limits_{\substack{[m_i(y_1),m_j(y_2),\\ m_k(y_3)]}} \nu \cdot q\, ds\; dy_3 dy_2 dy_1,$$

and finally for L_2,

$$(\pi^t s)(x) = \sum_{T_{ijkl}\in\mathcal{T}} \psi^t_{ijkl}(s)\; \varphi^t_{ijkl}(x) \tag{12}$$

with the tetrahedral evaluation functionals

$$\Psi^t_{ijkl}(s) = \int\limits_{\omega_i}\int\limits_{\omega_j}\int\limits_{\omega_k}\int\limits_{\omega_l} f_i f_j f_k f_l \int\limits_{\substack{[m_i(y_1),m_j(y_2),\\ m_k(y_3),m_l(y_4)]}} s\, dx\; dy_4 dy_3 dy_2 dy_1.$$

In [17], Theorem 5, the following interpolation error estimates have been proven:

Theorem 4 (L_2-approximation)**.** *Let p be the order of consistency as defined in (7). Then the approximation estimates*

$$\begin{array}{rcll} \|w-\pi^v w\|_{L_2(T)} & \preceq & h_T\,|w|_{H^1(\Omega^T)} & \text{for } p \geq 0,\\ \|v-\pi^e v\|_{L_2(T)} & \preceq & h_T\,|v|_{H^1(\Omega^T)} & \text{for } p \geq 1,\\ \|q-\pi^f q\|_{L_2(T)} & \preceq & h_T\,|q|_{H^1(\Omega^T)} & \text{for } p \geq 2,\\ \|s-\pi^t s\|_{L_2(T)} & \preceq & h_T\,|s|_{H^1(\Omega^T)} & \text{for } p \geq 3 \end{array}$$

are valid.

In [18], new estimates for a posteriori error analysis in $H(\text{curl})$ and $H(\text{div})$ have been established. Thus we call them a posteriori estimates.

Theorem 5 (a posteriori estimates)**.** *For every $u \in H(\text{curl})$ there exists $\varphi \in H^1$ and $z \in [H^1]^3$ such that*

$$u - \pi^e u = \nabla\varphi + z. \tag{13}$$

The decomposition satisfies

$$\begin{array}{rcl} h_T^{-1}\|\varphi\|_{L_2(T)} + \|\nabla\varphi\|_{L_2(T)} & \leq & c\,\|u\|_{L_2(\widetilde{\omega}_T)}\\ h_T^{-1}\|z\|_{L_2(T)} + \|\nabla z\|_{L_2(T)} & \leq & c\,\|\operatorname{curl} u\|_{L_2(\widetilde{\omega}_T)}. \end{array}$$

The constant c depends only on the shape of triangles in the enlarged element patch $\widetilde{\omega}_T$ containing neighbor elements of neighbor elements of T, but does not depend on the global shape of the domain Ω.

Now, we construct new operators $\Pi^v, \Pi^e, \Pi^f, \Pi^t$ sharing most properties of the previous operators π, and which are also projectors onto the finite element spaces. The only property they do not share is locality.

Lemma 6. *On the finite element space, the quasi-interpolation operators converge uniformly to the identity. I.e., there holds*

$$\|u_h - \pi u_h\|_{L_2} \preceq \rho \|u_h\|_{L_2} \qquad \forall\, u_h \in V_h.$$

Here, V_h stands for V_h^v, V_h^e, V_h^f, or V_h^t, and π is the corresponding interpolation operator. The parameter $\rho \in (0, \rho_0)$ scales the size of the averaging domains.

The technical proof is given in [19].

Theorem 7. *There exists a parameter ρ_0 depending only on the shape of the elements and on the $\|\cdot\|_{L_\infty}$-norm of the chosen functionals f_i such that $\pi|_{V_h}$ is invertible for all $\rho \in (0, \rho_0)$.*

Proof. There is a ρ_0 such that

$$\|u_h - \pi u_h\|_{L_2} \leq \frac{1}{2} \|u_h\|_{L_2} \qquad \forall\, u_h \in V_h,\ \forall\, \rho \leq \rho_0.$$

We apply Neumann series $(1-a)^{-1} = \sum_{k=0}^{\infty} a^k$ for $|a| < 1$ to invert π:

$$[\pi|_{V_h}]^{-1} = [I - (I - \pi|_{V_h})]^{-1} = \sum_{k=0}^{\infty} (I - \pi|_{V_h})^k.$$

Since $\|I - \pi|_{V_h}\|_{L_2} \leq \frac{1}{2}$, the Neumann series converges to the inverse $[\pi|_{V_h}]^{-1}$. □

Now, we can define operators

$$\Pi := [\pi|_{V_h}]^{-1} \pi.$$

They are projections onto V_h. There holds the relation

$$I - \Pi = (I - \Pi)(I - \pi),$$

which allows to transfer the approximation properties from π to the projections Π:

$$\|(I - \Pi) u\|_{L_2} \leq \|(I - \Pi)\|_{L_2} \|(I - \pi) u\|_{L_2}.$$

Theorem 8. *The projection operator Π^e is continuous with respect to the norms*

$$\|\Pi^e u\|_0 \preceq \|u\|_0 \tag{14}$$

$$\|\Pi^e u\|_A \preceq \|u\|_A \tag{15}$$

$$|\!|\!|\Pi^e u|\!|\!|_0 \preceq |\!|\!|u|\!|\!|_0 \tag{16}$$

Theorem 9. *Let $\kappa \preceq 1$. The projection operator Π^e satisfies the error estimates*

$$\|\!|u - \Pi^e u|\!\|_0 \preceq h \|u\|_A \tag{17}$$

$$\|u - \Pi^e u\|_A \preceq h |\!\|u\|\!|_2 \tag{18}$$

$$\|\!|u - \Pi^e u|\!\|_0 \preceq h^2 |\!\|u\|\!|_2 \tag{19}$$

Proof. Estimate (17) follows from the a posteriori estimate, Theorem 5. Estimate (18) is a reformulation for the a priori estimates of Theorem 4, namely

$$\begin{aligned}
\|u - \Pi^e u\|_A^2 &= \kappa \|u - \Pi^e u\|_{L_2}^2 + \|\operatorname{curl}(u - \Pi^e u)\|_{L_2}^2 \\
&\preceq h^2 \big\{ \kappa |u|_{H^1}^2 + |\operatorname{curl} u|_{H^1}^2 \big\} \\
&\preceq h^2 \inf_{\varphi \in H^2} \big\{ \kappa |u - \nabla\varphi|_{H^1}^2 + \kappa |\nabla \varphi|_{H^1}^2 + \|\operatorname{curl}(u - \nabla\varphi)\|_{H^1}^2 \big\} \\
&\preceq h^2 \inf_{\varphi \in H^2} \big\{ \kappa \|\varphi\|_{H^2}^2 + \|u - \nabla\varphi\|_{H^2}^2 \big\}
\end{aligned}$$

Estimate (19) uses that $(I - \Pi^e)$ is a projector, and combines estimates (17) and (18). □

4. The Multi-level decomposition result

Now, we are prepared to prove the multi-level decomposition result, Theorem 3.

Lemma 10. *The discrete 0-norm is equivalent to the continuous one, i.e. there holds*

$$\|\!|u_l|\!\|_{0,l} \simeq \|\!|u_l|\!\|_0 \qquad \forall\, u_l \in V_l.$$

Proof. One estimate is trivial, namely

$$\begin{aligned}
\|\!|u_l|\!\|_0^2 &= \inf_{\varphi \in H^1} \big\{ \kappa \|u_l - \nabla\varphi\|_{L_2}^2 + \|\varphi\|_{L_2}^2 \big\} \\
&\leq \inf_{\varphi \in V_l^v} \big\{ \kappa \|u_l - \nabla\varphi\|_{L_2}^2 + \|\varphi\|_{L_2}^2 \big\} = \|\!|u_l|\!\|_{0,l}^2.
\end{aligned}$$

To verify the other one, we insert the continuous, commuting projection operators:

$$\begin{aligned}
|||u_l|||_{0,l}^2 &= \inf_{\varphi_l \in V_l^v} \left\{ \kappa \|u_l - \nabla \varphi_l\|_{L_2}^2 + \|\varphi_l\|_{L_2}^2 \right\} \\
&\leq \inf_{\varphi \in H^1} \left\{ \kappa \|u_l - \nabla \Pi_l^v \varphi\|_{L_2}^2 + \|\Pi_l^v \varphi\|_{L_2}^2 \right\} \\
&= \inf_{\varphi \in H^1} \left\{ \kappa \|\Pi_l^e (u_l - \nabla \varphi)\|_{L_2}^2 + \|\Pi_l^v \varphi\|_{L_2}^2 \right\} \\
&\preceq \inf_{\varphi \in H^1} \left\{ \kappa \|u_l - \nabla \varphi\|_{L_2}^2 + \|\varphi\|_{L_2}^2 \right\} \\
&= |||u_l|||_0^2
\end{aligned}$$

□

From the continuity estimate (16) there follows immediately

$$|||(\Pi_l - \Pi_{l-1})u|||_0 \preceq |||u|||_0 \qquad \forall\, u \in [L_2]^3 + \nabla L_2$$

The approximation estimate (19) of Theorem 9 leads to

$$|||(\Pi_l - \Pi_{l-1})u|||_0 \leq |||\Pi_l u - u|||_0 + |||u - \Pi_{l-1} u|||_0 \preceq h_l^2 |||u|||_2 \qquad \forall\, u \in \nabla H^2 + [H^2]^3.$$

By combining Lemma 10, the triangle inequality, and the two estimates above, and exchanging the order of quantifies, we obtain

$$\begin{aligned}
|||(\Pi_l - \Pi_{l-1})u|||_{0,l}^2 &\preceq \inf_{u = u_1 + u_2} |||(\Pi_l - \Pi_{l-1})u_1|||_0^2 + |||(\Pi_l - \Pi_{l-1})u_2|||_0^2 \\
&\preceq \inf_{u = u_1 + u_2} \left\{ |||u_1|||_0^2 + h_l^4\, |||u_2|||_2^2 \right\} \\
&= \inf_{u = u_1 + u_2} \inf_{\substack{u_1 = \nabla \varphi_1 + z_1 \\ u_2 = \nabla \varphi_2 + z_2}} \left\{ \kappa \|\varphi_1\|_{L_2}^2 + \|z_1\|_{L_2}^2 + \kappa h_l^4 \|\varphi_2\|_{H^2}^2 + h_l^4 \|z_2\|_{H^2}^2 \right\} \\
&= \inf_{u = \nabla \varphi + z} \kappa \inf_{\varphi = \varphi_1 + \varphi_2} \left\{ \|\varphi_1\|_0^2 + h_l^4 \|\varphi_2\|_{H^2}^2 \right\} \\
&\qquad + \inf_{z = z_1 + z_2} \left\{ \|z_1\|_0^2 + h_l^4 \|z_2\|_{H^2}^2 \right\}
\end{aligned}$$

Now, we take the sum for $l = 1 \ldots L$. We use that $\sum \inf[\ldots] \leq \inf \sum [\ldots]$ to obtain

$$\begin{aligned} &\sum_{l=1}^{L} h_l^{-2} |||(\Pi_l - \Pi_{l-1})u|||_{0,l}^2 \\ &\preceq \inf_{u=\nabla\varphi+z} \kappa \sum_{l=1}^{L} \inf_{\varphi=\varphi_1+\varphi_2} \left\{ h_l^{-2}\|\varphi_1\|_{L_2}^2 + h_l^2\|\varphi_2\|_{H^2}^2 \right\} \\ &\qquad + \sum_{l=1}^{L} \inf_{z=z_1+z_2} \left\{ h_l^{-2}\|z_1\|_{L_2}^2 + h_l^2\|z_2\|_{H^2}^2 \right\}. \end{aligned}$$

The sums are dominated by the K-functional interpolation norm $\|\cdot\|_{[L_2,H^2]_{1/2}} = \|\cdot\|_{H^1}$. An elementary proof can be found in [5]. Together with a partition of unity on the coarse grid one can show that

$$\sum_{l=1}^{L} \inf_{\varphi=\varphi_1+\varphi_2} \left\{ h_l^{-2}\|\varphi_1\|_{L_2}^2 + h_l^2\|\varphi_2\|_{H^2}^2 \right\} \preceq h_0^{-2}\|\varphi\|_{L_2}^2 + \|\nabla\varphi\|_{L_2}^2,$$

where the involved constant only depends on the shape of the elements, but not on the global domain.

Thus, the sum is bounded by

$$\begin{aligned} &\sum_{l=1}^{L} h_l^{-2} |||(\Pi_l - \Pi_{l-1})u|||_{0,l}^2 \\ &\preceq \inf_{u=\nabla\varphi+z} \kappa\left\{ h_0^{-2}\|\varphi\|_{L_2}^2 + \|\nabla\varphi\|_{L_2}^2 \right\} + h_0^{-2}\|z\|_{L_2}^2 + \|\nabla z\|_{L_2}^2. \end{aligned} \tag{20}$$

Now, we apply Theorem 5 to bound the coarse grid interpolation error

$$u - \Pi_0^e u = \nabla\varphi + z$$

such that

$$\begin{aligned} \|\nabla\varphi\|_{L_2}^2 + h_0^{-2}\|\varphi\|_{L_2}^2 &\preceq \|u\|_{L_2}^2, \\ \|\nabla z\|_{L_2}^2 + h_0^{-2}\|z\|_{L_2}^2 &\preceq \|\operatorname{curl} u\|_{L_2}^2. \end{aligned}$$

Since Π_l are projections, and $V_0 \subset V_l$, there holds

$$\sum_{l=1}^{L} h_l^{-2} |||(\Pi_l - \Pi_{l-1})u|||_{0,l}^2 = \sum_{l=1}^{L} h_l^{-2} |||(\Pi_l - \Pi_{l-1})(u - \Pi_0^e u)|||_{0,l}^2 \preceq \|u\|_A^2.$$

The bound for the coarse grid term in (6) follows directly from the continuity of the interpolation operators.

References

[1] D. N. Arnold, R. S. Falk, and R. Winther. Multigrid in $H(\mathrm{div})$ and $H(\mathrm{curl})$. *Numer. Math.*, 85:197–218, 2000.

[2] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications *Acta Numerica*, 15, 2006.

[3] D. Boffi. Discrete compactness and fortin operator for edge elements. *Numer. Math.*, 87:229–246, 2000.

[4] D. Boffi. A note on the discrete compactness property and the de Rham diagram. *Appl. Math. Letters*, 14:33–38, 2001.

[5] F. Bornemann and H. Yserentant. A basic norm equivalence for the theory of multilevel methods. *Numer. Math.*, 64:455–476, 1993.

[6] A. Bossavit. Mixed finite elements and the complex of Whitney forms. In J. Whiteman, editor, *The Mathematics of Finite Elements and Applications VI*, 137–144. Academic Press, London, 1988.

[7] P. Clément. Approximation by finite element functions using local regularization. *R.A.I.R.O. Anal. Numer.*, R2:77–84, 1975.

[8] L. Demkowicz, P. Monk, L. Vardapetyan, and W. Rachowicz. De Rham diagram for hp finite element spaces. Technical Report 99-06, TICAM, 1999. (to appear in Math. and Comp. with appl.).

[9] R. Hiptmair. Canonical Construction of Finite Elements. *Math. Comp.*, 68:1325–1346, 1999

[10] R. Hiptmair. Multigrid method for Maxwell's equations. *SIAM J. Numer. Anal.*, 36:204–225, 1999.

[11] P. Monk, Finite Element Methods for Maxwell's Equations. *Oxford University Press*, 2003.

[12] J.-C. Nedelec. Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 35:315–341, 1980.

[13] J.-C. Nédélec. A new family of mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 35:315–341, 1980.

[14] J.Pasciak and J. Zhao. Overlapping Schwarz methods in H(curl) on nonconvex domains. *East West J. Numer. Anal.*, 10:221-234, 2002.

[15] P.-A. Raviart and J.-M. Thomas. A mixed finite element method for second order elliptic problems. In I. Galligani and E. Magenes, editors, *Mathematical Aspects of the Finite Element Method*, Lecture Notes in Mathematics, pages 292–315. Springer, Berlin, 1977.

[16] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[17] J. Schöberl. Commuting quasi-interpolation operators for mixed finite elements. *Report ISC-01-10-MATH*, Texas A&M University, available from www.isc.tamu.edu/iscpubs/iscreports.html, 2001.

[18] J. Schöberl. A posteriori error estimates for Maxwell equations. *Math. Comp* (submitted),

[19] J. Schöberl. Commuting Projection Operators and Applications *Technical report RICAM*, 2006

[20] J. Schöberl and S. Zaglmayr. High Order Nédélec Elements with local complete sequence properties. *Int. J for Computation and Maths in Electrical and Electronic Eng COMPEL*, 24(2), 374-384, 2005.

[21] A. Toselli. Overlapping Schwarz methods for Maxwell's equations in three dimension. *Numer. Math.*, 86:733–752, 2000.

[22] J. P. Webb, Hierarchal vector basis functions of arbitrary order for triangular and tetrahedral finite elements, *IEEE Trans. on Antennas and Propagation*, 47:1244–1253, 1999.

URL: `http://www.hpfem.jku.at/people/joachim`