MULTIGRID METHODS FOR A STABILIZED REISSNER-MINDLIN PLATE FORMULATION

JOACHIM SCHÖBERL* AND ROLF STENBERG†

Abstract. We consider a stabilized finite element formulation for the Reissner-Mindlin plate bending model. The method uses standard bases functions for the deflection and the rotation vector. We apply a standard multigrid algorithm to obtain a preconditioner. We prove that the condition number of the preconditioned system is uniformly bounded with respect to the multigrid level and the thickness parameter. The abstract multigrid theory is applied for carefully chosen norms. We have to prove also some new finite element error estimates. Numerical results confirm the analysis.

Key words. Reissner-Mindlin plate, stabilized finite element method, multigrid method

AMS subject classifications. 65N30, 65N55, 74S05

1. Introduction. In this paper we consider a family of stabilized finite element methods for the Reissner-Mindlin plate model. The origin of the method is in a Galerkin-Least-Squares method introduced by Hughes and Franca [11]. There, the shear force was discretized independently and locally condensed. The method introduced in [20] and further analyzed in [14] avoids the additional variable; it is formulated directly in the displacement variables, the deflection and the rotation vector. The corresponding lowest order method is due to Pitkäranta [17].

This stabilized method has two advantages compared to more traditional methods. First, standard basis functions can be used; in particular, no bubble-functions are needed. Second, the condition number of the stiffness matrix is reduced, which opens the way for using standard multigrid algorithms for the positive definite system matrix.

Classical textbooks on multigrid methods are [10] and [5]. So far, there has been relatively few work on multigrid methods for Reissner-Mindlin plate methods. The first is the work of Peisker, Rust and Stein [16], in which Pitkärantas method is analyzed. In a subsequent paper by Peisker [15] the Hughes-Franca method is analyzed. In this algorithm, the shear force is kept as an independent unknown, and the iteration is based on the indefinite stiffness matrix of the mixed form. The same holds for the multigrid methods analyzed in the papers by Arnold, Falk and Winther [3] and Brenner [8]. In [18, 19] multigrid methods for stiffness matrices in displacement variables were analyzed. There, block-smoothers are required to compensate the ill-conditioning arising from the more classical elements.

The mesh-dependent, stabilized finite element formulation leads to different bilinear-forms on different levels. Thus, the non-nested multigrid analysis has to be applied [4, 6]. Our analysis is based on the abstract framework given in [5].

The paper is organized as follows. In Section 2 we define the method, state available results, and prove some new error estimates as needed for our multigrid analysis. In Section 3 we define and analyze the multigrid method. Numerical results confirming the analysis are given in Section 4.

2. The plate model. Let $\Omega \subset \mathbb{R}^2$ be the midsurface of the plate and suppose that the plate is clamped along the boundary Γ. The variational formulation of the

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Reissner-Mindlin model is: find the deflection $w \in H_0^1(\Omega)$ and the rotation vector $\boldsymbol{\beta} = (\beta_x, \beta_y) \in [H_0^1(\Omega)]^2$ such that

$$(2.1)\quad a(\boldsymbol{\beta}, \boldsymbol{\eta}) + t^{-2}(\nabla w - \boldsymbol{\beta}, \nabla v - \boldsymbol{\eta}) = (f, v) + (\boldsymbol{f}, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2.$$

Here, $t > 0$ is the thickness of the plate and $f \in L_2(\Omega)$ is the transverse load acting on Ω. The additional, non-standard source term $\boldsymbol{f} \in [L_2(\Omega)]^2$ is required for the multigrid analysis. The bilinear form a represents bending energy and is defined as

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) = \frac{1}{6}\Big\{(\boldsymbol{\varepsilon}(\boldsymbol{\beta}), \boldsymbol{\varepsilon}(\boldsymbol{\eta})) + \frac{\nu}{1-\nu}(\operatorname{div}\boldsymbol{\beta}, \operatorname{div}\boldsymbol{\eta})\Big\}, \tag{2.2}$$

where ν is the Poisson ratio, $\boldsymbol{\varepsilon}(\cdot)$ is the small strain tensor and div stands for the divergence, viz.

$$\boldsymbol{\varepsilon}(\boldsymbol{\beta}) = \frac{1}{2}\Big\{\nabla\boldsymbol{\beta} + (\nabla\boldsymbol{\beta})^T\Big\}, \tag{2.3}$$

$$\operatorname{div}\boldsymbol{\beta} = \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y}. \tag{2.4}$$

For $D \subset \mathbb{R}^2$ we define the Sobolev spaces $H^s(D)$ and $H_0^s(D)$, with $s \geq 0$, in the usual way, i.e. first for integral values s and then for nonintegral values by interpolation, cf. [12]. As usual, we define $H^s(D) := [H_0^{-s}(D)]^*$ for $s < 0$. The norms and seminorms will be denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. The L_2-inner products in $L_2(D)$, $[L_2(D)]^2$ or $[L_2(D)]^{2\times 2}$ are denoted by $(\cdot,\cdot)_D$. The subscript D will be dropped when $D = \Omega$.

By taking the scaled shear force

$$\boldsymbol{q} = t^{-2}(\nabla w - \boldsymbol{\beta}) \tag{2.5}$$

as an independent unknown one obtains the following mixed formulation: find $(w, \boldsymbol{\beta}, \boldsymbol{q}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^2$ such that

$$\begin{aligned} &a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) = (f, v) + (\boldsymbol{f}, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2, \\ &(\nabla w - \boldsymbol{\beta}, \boldsymbol{s}) - t^2(\boldsymbol{q}, \boldsymbol{s}) = 0 \qquad \forall \boldsymbol{s} \in [L^2(\Omega)]^2. \end{aligned} \tag{2.6}$$

The distributional differential equations of this system are obtained by integrating by parts:

$$\begin{aligned} \boldsymbol{L}\boldsymbol{\beta} + \boldsymbol{q} &= \boldsymbol{f} && \text{in } \Omega, \\ -\operatorname{div}\boldsymbol{q} &= f && \text{in } \Omega, \\ -t^2\boldsymbol{q} + \nabla w - \boldsymbol{\beta} &= \boldsymbol{0} && \text{in } \Omega, \\ w = 0,\ \boldsymbol{\beta} &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned} \tag{2.7}$$

Here, the differential operator $\boldsymbol{L}$ is defined as

$$\boldsymbol{L}\boldsymbol{\eta} = \frac{1}{6}\mathbf{div}\Big\{\boldsymbol{\varepsilon}(\boldsymbol{\eta}) + \frac{\nu}{1-\nu}\operatorname{div}\boldsymbol{\eta}\boldsymbol{I}\Big\}, \tag{2.8}$$

where we used the notation **div** for the divergence operator applied to a second order tensor:

$$\mathbf{div}\,\boldsymbol{m} = \Big(\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{xy}}{\partial y}, \frac{\partial m_{yx}}{\partial x} + \frac{\partial m_{yy}}{\partial y}\Big). \tag{2.9}$$

For the moment tensor $\boldsymbol{m}$ defined by

$$\boldsymbol{m} = \frac{1}{6}\Big\{\boldsymbol{\varepsilon}(\boldsymbol{\beta}) + \frac{\nu}{1-\nu}\operatorname{div}\boldsymbol{\beta}\boldsymbol{I}\Big\}. \tag{2.10}$$

there holds

$$\boldsymbol{L}\boldsymbol{\beta} = \mathbf{div}\,\boldsymbol{m}. \tag{2.11}$$

The first two equations in (2.7) above are the local equilibrium equations between the moment, shear force and load. The third equation represents the constitutive relation between the shear strain and shear force.

In the limit $t \to 0$ the solution $(w, \boldsymbol{\beta}) = (w_t, \boldsymbol{\beta}_t)$ of the Reissner–Mindlin equations converges to the Kirchhoff solution with

$$\boldsymbol{\beta}_0 = \nabla w_0. \tag{2.12}$$

The limit solution w_0 satisfies the biharmonic equation in the domain Ω and only two boundary conditions on each part of the boundary, cf. [1]. This singularity gives rise to the boundary layers in the solution which complicates the convergence analysis of the methods.

Throughout the rest of the paper we assume that the domain Ω is convex. The following regularity estimate collects results from [2, 13]:

THEOREM 2.1. *Let Ω be a convex polygonal domain. Denote by $(w, \boldsymbol{\beta}, \boldsymbol{q})$ the Reissner–Mindlin solution for the clamped plate and let $w = w_0 + w_r$, where w_0 is the deflection obtained from the Kirchhoff model. Then there holds*

$$\|w_0\|_3 + t^{-1}\|w_r\|_2 + \|\boldsymbol{\beta}\|_2 + \|\boldsymbol{q}\|_0 + t\|\boldsymbol{q}\|_1 \le C(\|f\|_{-1} + t\|f\|_0 + \|\boldsymbol{f}\|_0). \tag{2.13}$$

In our analysis we will utilize the following t-dependent norms. For $(v, \boldsymbol{\eta}) \in L_2(\Omega) \times [L_2(\Omega)]^2$ we define

$$\|(v, \boldsymbol{\eta})\|_{1,t} = \|\boldsymbol{\eta}\|_0 + \inf_{\substack{v = v^0 + v^r \\ v^0 \in H^1(\Omega)}} \Big\{\|v^0\|_1 + t^{-1}\|v^r\|_0\Big\}, \tag{2.14}$$

for $(v, \boldsymbol{\eta}) \in H^2(\Omega) \times [H^2(\Omega)]^2$ we define

$$\|(v, \boldsymbol{\eta})\|_{3,t} = \|\boldsymbol{\eta}\|_2 + \inf_{\substack{v = v^0 + v^r \\ v^0 \in H^3(\Omega) \cap H^2_0(\Omega)}} \Big\{\|v^0\|_3 + t^{-1}\|v^r\|_2\Big\}, \tag{2.15}$$

and for $(f, \boldsymbol{f}) \in L_2(\Omega) \times [L_2(\Omega)]^2$ we define

$$\|(f, \boldsymbol{f})\|_{-1,t} = \|\boldsymbol{f}\|_0 + \|f\|_{-1} + t\|f\|_0. \tag{2.16}$$

The regularity estimate (2.13) immediately gives

$$\|(w, \boldsymbol{\beta})\|_{3,t} \le C\|(f, \boldsymbol{f})\|_{-1,t}. \tag{2.17}$$

Furthermore, the norms $\|(\cdot,\cdot)\|_{-1,t}$ and $\|(\cdot,\cdot)\|_{1,t}$ are dual to each other:

THEOREM 2.2. *There holds*

$$C_1\,\|(f, \boldsymbol{f})\|_{-1,t} \le \sup_{(v,\boldsymbol{\eta})} \frac{(f, v) + (\boldsymbol{f}, \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}} \le C_2\,\|(f, \boldsymbol{f})\|_{-1,t}, \tag{2.18}$$

where C_1 and C_2 do not depend on t.

The proof is given in [19], page 41.

2.1. Finite element subspaces. We will use standard notation from finite element analysis [7]. Let $\mathcal{C}_h$ be the partitioning of $\bar{\Omega}$ into triangles or convex quadrilaterals satisfying the usual compatibility conditions. For generality we allow a mesh consisting of both triangles and quadrilaterals. As usual, h_K denotes the diameter of $K \in \mathcal{C}_h$, and h stands for the global mesh parameter $h = \max_{K\in\mathcal{C}_h} h_K$. We define

$$R_m(K) = \begin{cases} P_m(K) & \text{when } K \text{ is a triangle,} \\ Q_m(K) & \text{when } K \text{ is a quadrilateral.} \end{cases} \tag{2.19}$$

For $k \geq 1$, we define the finite element subspaces for the deflection and the rotation as

$$W_h = \{v \in H_0^1(\Omega) \mid v_{|K} \in R_{k+1}(K),\ \forall K \in \mathcal{C}_h\}, \tag{2.20}$$

$$\boldsymbol{V}_h = \{\boldsymbol{\eta} \in [H_0^1(\Omega)]^2 \mid \boldsymbol{\eta}_{|K} \in [R_k(K)]^2,\ \forall K \in \mathcal{C}_h\}. \tag{2.21}$$

The finite element method of [20, 14] is as follows:

Method 2.1. *Given the loading* $(f, \boldsymbol{f}) \in L^2(\Omega) \times [L^2(\Omega)]^2$, *find* $(w_h, \boldsymbol{\beta}_h) \in W_h \times \boldsymbol{V}_h$ *such that*

$$\mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = \mathcal{F}_h(v, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{2.22}$$

with the bilinear and linear forms defined as

$$\begin{aligned} \mathcal{A}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = a(\boldsymbol{\phi}, \boldsymbol{\eta}) &- \sum_{K\in\mathcal{C}_h} \alpha h_K^2 (\boldsymbol{L}\,\boldsymbol{\phi}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\ &+ \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\nabla z - \boldsymbol{\phi} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\phi}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K. \end{aligned} \tag{2.23}$$

$$\begin{aligned} \mathcal{F}_h(v, \boldsymbol{\eta}) = (f, v) + (\boldsymbol{f}, \boldsymbol{\eta}) &- \sum_{K\in\mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{f}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\ &- \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{f}, \nabla v - \boldsymbol{\eta})_K. \end{aligned} \tag{2.24}$$

Here and throughout the paper α *is a positive parameter lying in the range* $0 < \alpha < C_I$, *where* C_I *is the constant in the inverse inequality*

$$C_I \sum_{K\in\mathcal{C}_h} h_K^2 \|\boldsymbol{L}\,\boldsymbol{\phi}\|_{0,K}^2 \leq a(\boldsymbol{\phi}, \boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \boldsymbol{V}_h.$$

From the solution $(w_h, \boldsymbol{\beta}_h)$ *we then calculate the approximation for the shear by*

$$\boldsymbol{q}_{h|K} = (t^2 + \alpha h_K^2)^{-1} \big(\nabla w_h - \boldsymbol{\beta}_h + \alpha h_K^2 (\boldsymbol{f} - \boldsymbol{L}\,\boldsymbol{\beta}_h)\big)_{|K} \qquad \forall K \in \mathcal{C}_h. \tag{2.25}$$

Note that from (2.7) we see that the exact shear satisfies

$$\boldsymbol{q}_{|K} = (t^2 + \alpha h_K^2)^{-1} \big(\nabla w - \boldsymbol{\beta} + \alpha h_K^2 (\boldsymbol{f} - \boldsymbol{L}\,\boldsymbol{\beta})\big)_{|K} \qquad \forall K \in \mathcal{C}_h. \tag{2.26}$$

Remark 2.1. *For triangular elements with* $k = 1$ *it holds* $\boldsymbol{L}\,\boldsymbol{\phi} = \boldsymbol{0},\ \forall \boldsymbol{\phi} \in \boldsymbol{V}_h$, *and the bilinear form is simply*

$$\mathcal{A}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = a(\boldsymbol{\phi}, \boldsymbol{\eta}) + \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\nabla z - \boldsymbol{\phi}, \nabla v - \boldsymbol{\eta})_K. \tag{2.27}$$

Furthermore, there is no upper limit for the parameter α. This has been first proposed by Fried and Yang [9] and analyzed by Pitkäranta [17]. For $k=1$, this formulation can be used for quadrilaterals as well.

In our previous works [20, 14] we have analyzed the method for $\boldsymbol{f}=\mathbf{0}$. Hence, we will here prove the consistency for a general loading.

THEOREM 2.3. *The solution $(w,\boldsymbol{\beta})$ to* (2.7) *satisfies the equation*

$$\mathcal{A}_h(w,\boldsymbol{\beta};v,\boldsymbol{\eta})=\mathcal{F}_h(v,\boldsymbol{\eta})\quad\forall(v,\boldsymbol{\eta})\in W_h\times\boldsymbol{V}_h. \tag{2.28}$$

Proof. Recalling the first equation in (2.7), the expression (2.26), and the variational form (2.6), we get

$$\begin{aligned}
&\mathcal{A}_h(w,\boldsymbol{\beta};v,\boldsymbol{\eta})\\
&=a(\boldsymbol{\beta},\boldsymbol{\eta})-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(\boldsymbol{L}\,\boldsymbol{\beta},\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&\qquad+\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}(\nabla w-\boldsymbol{\beta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\beta},\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&=a(\boldsymbol{\beta},\boldsymbol{\eta})+\sum_{K\in\mathcal{C}_h}\alpha h_K^2(\boldsymbol{q}-\boldsymbol{f},\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&\qquad+\sum_{K\in\mathcal{C}_h}(\boldsymbol{q}-\alpha h_K^2(t^2+\alpha h_K^2)^{-1}\boldsymbol{f},\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&=a(\boldsymbol{\beta},\boldsymbol{\eta})+\sum_{K\in\mathcal{C}_h}\alpha h_K^2(\boldsymbol{q},\boldsymbol{L}\,\boldsymbol{\eta})_K-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(\boldsymbol{f},\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&\qquad+(\boldsymbol{q},\nabla v-\boldsymbol{\eta})-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(\boldsymbol{q},\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&\qquad-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{f},\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&=a(\boldsymbol{\beta},\boldsymbol{\eta})+(\boldsymbol{q},\nabla v-\boldsymbol{\eta})-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(\boldsymbol{f},\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&\qquad-\sum_{K\in\mathcal{C}_h}\alpha_K h^2(t^2+\alpha h_K^2)^{-1}(,\nabla\mathbf{v}-\boldsymbol{\eta}-\alpha\mathbf{h}_{\mathbf{K}}^{\mathbf{2}}\boldsymbol{L}\,\boldsymbol{\eta})_{\mathbf{K}}\\
&=(f,v)+(\boldsymbol{f},\boldsymbol{\eta})-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(\boldsymbol{f},\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&\qquad-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{f},\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&=(f,v)+(\boldsymbol{f},\boldsymbol{\eta})-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{f},\nabla v-\boldsymbol{\eta})_K\\
&\qquad-\sum_{K\in\mathcal{C}_h}(1-\alpha h_K^2(t^2+\alpha h_K^2)^{-1})(\boldsymbol{f},\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K.\\
&=(f,v)+(\boldsymbol{f},\boldsymbol{\eta})-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{f},\nabla v-\boldsymbol{\eta})_K\\
&\qquad-\sum_{K\in\mathcal{C}_h}t^2(t^2+\alpha h_K^2)^{-1})(\boldsymbol{f},\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K\\
&=\mathcal{F}_h(v,\boldsymbol{\eta}).
\end{aligned}$$

The following norms are natural for the stability and error analysis:

DEFINITION 2.4. *For $(v,\boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2$ we define*

$$\|(v,\boldsymbol{\eta})\|_h^2 = \|v\|_1^2 + \|\boldsymbol{\eta}\|_1^2 + \sum_{K\in\mathcal{C}_h} (t^2+h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta}\|_{0,K}^2, \tag{2.29}$$

and for $\boldsymbol{r} \in [L^2(\Omega)]^2$

$$\|\boldsymbol{r}\|_h = (\sum_{K\in Ch} (t^2+h_K^2)\|\boldsymbol{r}\|_{0,K}^2)^{1/2}. \tag{2.30}$$

The stability of the method was proven in [14]:

THEOREM 2.5. *There is a positive constant C such that*

$$\mathcal{A}_h(v,\boldsymbol{\eta};v,\boldsymbol{\eta}) \ge C\|(v,\boldsymbol{\eta})\|_h^2 \qquad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h. \tag{2.31}$$

Stability, consistency, and regularity imply the following error estimate:

THEOREM 2.6. *For the solution $(w_h,\boldsymbol{\beta}_h)$ of (2.22) it holds*

$$\|(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)\|_h + \|\boldsymbol{q}-\boldsymbol{q}_h\|_h \le Ch\|(f,\boldsymbol{f})\|_{-1,t}. \tag{2.32}$$

The proof follows [14], where also some refined estimates were proven.

For the multigrid analysis we additionally need estimates for the discrete solution with an inconsistent right hand side given by the following method

METHOD 2.2. *Given the loading $(f,\boldsymbol{f}) \in L^2(\Omega) \times [L^2(\Omega)]^2$, find $(w_h^*,\boldsymbol{\beta}_h^*) \in W_h \times \boldsymbol{V}_h$ such that*

$$\mathcal{A}_h(w_h^*,\boldsymbol{\beta}_h^*;v,\boldsymbol{\eta}) = (f,v) + (\boldsymbol{f},\boldsymbol{\eta}) \quad \forall (v,\boldsymbol{\eta}) \in nW_h \times \boldsymbol{V}_h. \tag{2.33}$$

The inconsistent right hand side does not spoil the $O(h)$ convergence in discrete energy norms:

THEOREM 2.7. *For the solution $(w_h^*,\boldsymbol{\beta}_h^*)$ of (2.33) it holds*

$$\|(w-w_h^*,\boldsymbol{\beta}-\boldsymbol{\beta}_h^*)\|_h \le Ch\|(f,\boldsymbol{f})\|_{-1,t}. \tag{2.34}$$

Proof. The difference between the solutions $(w_h,\boldsymbol{\beta}_h)$ of (2.22) and $(w_h^*,\boldsymbol{\beta}_h^*)$ of (2.33) is due to the consistency term. For $(v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h$ there holds

$$\begin{aligned}
&\mathcal{A}_h(w_h-w_h^*,\boldsymbol{\beta}_h-\boldsymbol{\beta}_h^*,v,\boldsymbol{\eta}) \\
&= -\sum_{K\in\mathcal{C}_h} \alpha h_K^2 t^2 (t^2+\alpha h_K^2)^{-1}(\boldsymbol{f},\boldsymbol{L}\,\boldsymbol{\eta})_K - \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\alpha h_K^2(\boldsymbol{f},\nabla v-\boldsymbol{\eta})_K \\
&\le c\sum_{K\in\mathcal{C}_h} h_K\|\boldsymbol{f}\|_{0,K}\,\|\boldsymbol{\eta}\|_{1,K} + c\sum_{K\in\mathcal{C}_h} h_K\|\boldsymbol{f}\|_{0,K}\,(t^2+\alpha h_K^2)^{-1/2}\|\nabla v-\boldsymbol{\eta}\|_{0,K} \\
&\le ch\,\|\boldsymbol{f}\|_0\,\|(v,\boldsymbol{\eta})\|_h.
\end{aligned}$$

Now choose $(v,\boldsymbol{\eta}) = (w_h-w_h^*,\boldsymbol{\beta}_h-\boldsymbol{\beta}_h^*)$, apply the stability estimate (2.31), and divide by one factor $\|(w_h-w_h^*,\boldsymbol{\beta}_h-\boldsymbol{\beta}_h^*\|_h$ to observe

$$\|(w_h-w_h^*,\boldsymbol{\beta}_h-\boldsymbol{\beta}_h^*)\|_h \le c\,h\,\|\boldsymbol{f}\|_0.$$

The rest follows from Theorem 2.6 and the triangle inequality. □

The multigrid analysis will also require the following improved error estimates in weaker norms:

THEOREM 2.8. *For the solutions* $(w_h, \boldsymbol{\beta}_h)$ *of* (2.22) *and* $(w_h^*, \boldsymbol{\beta}_h^*)$ *of* (2.33), *there holds*

$$\|(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{1,t} \le Ch^2 \|(f, \boldsymbol{f})\|_{-1,t}. \tag{2.35}$$

and

$$\|(w - w_h^*, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*)\|_{1,t} \le Ch^2 \|(f, \boldsymbol{f})\|_{-1,t} \tag{2.36}$$

Proof. Step 1. Let For $(l, \boldsymbol{l})$ given, let $(z, \boldsymbol{\theta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2$ be the solution to the problem

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) + t^{-2}(\nabla z - \boldsymbol{\theta}, \nabla v - \boldsymbol{\eta}) = (l, v) + (\boldsymbol{l}, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2. \tag{2.37}$$

Denoting $\boldsymbol{r} = t^{-2}(\nabla z - \boldsymbol{\theta})$, the regularity estimate (2.13) gives

$$\|(z, \boldsymbol{\theta})\|_{3,t} + \|\boldsymbol{r}\|_0 \le C\|(l, \boldsymbol{l})\|_{-1,t}. \tag{2.38}$$

Note also that it holds

$$\boldsymbol{r}_{|K} = (t^2 + \alpha h_K^2)^{-1} \big(\nabla z - \boldsymbol{\theta} + \alpha h_K^2 (\boldsymbol{l} - \boldsymbol{L}\,\boldsymbol{\theta})\big)_{|K} \qquad \forall K \in \mathcal{C}_h. \tag{2.39}$$

As in Theorem 2.3 we now have

$$\mathcal{A}_h(z, \boldsymbol{\theta}; v, \boldsymbol{\eta}) = \mathcal{L}_h(v, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{2.40}$$

with

$$\begin{aligned} \mathcal{L}_h(v, \boldsymbol{\eta}) = (l, v) + (\boldsymbol{l}, \boldsymbol{\eta}) &- \sum_{K \in \mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{l}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\ &- \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{l}, \nabla v - \boldsymbol{\eta})_K. \end{aligned} \tag{2.41}$$

Step 2. Next, we let $\tilde{z} \in W_h$ and $\tilde{\boldsymbol{\theta}} \in \boldsymbol{V}_h$ be the the solution of

$$\mathcal{A}_h(\tilde{z}, \tilde{\boldsymbol{\theta}}; v, \boldsymbol{\eta}) = \mathcal{L}_h(v, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{2.42}$$

and define $\tilde{\boldsymbol{r}}$ by

$$\tilde{\boldsymbol{r}}_{|K} = (t^2 + \alpha h_K^2)^{-1} \big(\nabla \tilde{z} - \tilde{\boldsymbol{\theta}} + \alpha h_K^2 (\boldsymbol{l} - \boldsymbol{L}\,\tilde{\boldsymbol{\theta}})\big)_{|K} \qquad \forall K \in \mathcal{C}_h. \tag{2.43}$$

Hence, it holds

$$\mathcal{A}_h(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}; v, \boldsymbol{\eta}) = 0 \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{2.44}$$

and

$$|||(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}})|||_h + \|\boldsymbol{r} - \tilde{\boldsymbol{r}}\|_h \le Ch\|(l, \boldsymbol{l})\|_{-1,t}. \tag{2.45}$$

From this it also follows that

$$\Big(\sum_{K\in\mathcal{C}_h} h_K^2\|\boldsymbol{L}\,\tilde{\boldsymbol{\theta}}\|_{0.K}^2\Big)^{1/2} + \|\,\tilde{\boldsymbol{r}}\|_h \le Ch\|(l,\boldsymbol{l})\|_{-1,t}. \tag{2.46}$$

Step 3. We choose $v = w - w_h$ and $\boldsymbol{\eta} = \boldsymbol{\beta} - \boldsymbol{\beta}_h$ in (2.40) and obtain

$$\begin{aligned}(l, w - w_h) + (\boldsymbol{l}, \boldsymbol{\beta} - \boldsymbol{\beta}_h) = {}& \mathcal{A}_h(z, \boldsymbol{\theta}; w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)\\ &+ \sum_{K\in\mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{l}, \boldsymbol{L}\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K\\ &+ \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{l}, \nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K.\end{aligned} \tag{2.47}$$

From (2.22) and (2.28) we have

$$\mathcal{A}_h(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h; \tilde{z}, \tilde{\boldsymbol{\theta}}) = 0. \tag{2.48}$$

Using the symmetry of $\mathcal{A}_h$ we then get

$$\begin{aligned}&(l, w - w_h) + (\boldsymbol{l}, \boldsymbol{\beta} - \boldsymbol{\beta}_h)\\ &= \mathcal{A}_h(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}; w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) + \sum_{K\in\mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{l}, \boldsymbol{L}\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K\\ &\quad + \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{l}, \nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K.\end{aligned} \tag{2.49}$$

The first term above we estimate using Theorem 2.6 and (2.45)

$$\begin{aligned}&|\mathcal{A}_h(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}; w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|\\ &\le C\|(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}})\|_h |\!|\!|(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|\!|\!|_h\\ &\le Ch^2\|(l, \boldsymbol{l})\|_{-1,t}\|(f, \boldsymbol{f})\|_{-1,t}.\end{aligned}$$

The second term is treated as follows

$$|\sum_{K\in\mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{l}, \boldsymbol{L}\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K| \tag{2.50}$$

$$\le Ch\|\boldsymbol{l}\|_0 \Big(\sum_{K\in\mathcal{C}_h} h_K^2 \|\boldsymbol{L}\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{0,K}^2\Big)^{1/2} \tag{2.51}$$

$$\le Ch^2\|(l, \boldsymbol{l})\|_{-1,t}\|(f, \boldsymbol{f})\|_{-1,t},$$

where the last step follows from Theorem 2.6 and a scaling argument. The last term is readily estimated

$$\begin{aligned}&|\sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{l}, \nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K|\\ &\le Ch\|\boldsymbol{l}\|_0 |\!|\!|(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|\!|\!|_h\\ &\le Ch^2\|(l, \boldsymbol{l})\|_{-1,t}\|(f, \boldsymbol{f})\|_{-1,t}.\end{aligned} \tag{2.52}$$

The estimate (2.35) now follows by combining (2.49) – (2.52).

Step 4. Finally, we turn to the estimate for $(w-w_h^*,\boldsymbol{\beta}-\boldsymbol{\beta}_h^*)$. From (2.40) we get

$$(2.53)\quad \begin{aligned}(l,w-w_h^*)+(\boldsymbol{l},\boldsymbol{\beta}-\boldsymbol{\beta}_h^*) &= \mathcal{A}_h(z,\boldsymbol{\theta};w-w_h^*,\boldsymbol{\beta}-\boldsymbol{\beta}_h^*)\\ &+\sum_{K\in\mathcal{C}_h}\alpha h_K^2t^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{l},\boldsymbol{L}\,(\boldsymbol{\beta}-\boldsymbol{\beta}_h^*))_K\\ &+\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\alpha h_K^2(\boldsymbol{l},\nabla(w-w_h)-(\boldsymbol{\beta}-\boldsymbol{\beta}_h^*))_K.\end{aligned}$$

Next, adding and subtracting $\mathcal{A}_h(\tilde z,\tilde{\boldsymbol{\theta}};w-w_h^*,\boldsymbol{\beta}-\boldsymbol{\beta}_h^*)$ gives

$$(2.54)\quad \begin{aligned}&(l,w-w_h^*)+(\boldsymbol{l},\boldsymbol{\beta}-\boldsymbol{\beta}_h^*)=\mathcal{A}_h(z-\tilde z,\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}};w-w_h^*,\boldsymbol{\beta}-\boldsymbol{\beta}_h^*)\\ &+\sum_{K\in\mathcal{C}_h}\alpha h_K^2t^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{l},\boldsymbol{L}\,(\boldsymbol{\beta}-\boldsymbol{\beta}_h^*))_K\\ &+\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\alpha h_K^2(\boldsymbol{l},\nabla(w-w_h)-(\boldsymbol{\beta}-\boldsymbol{\beta}_h^*))_K\\ &+\mathcal{A}_h(\tilde z,\tilde{\boldsymbol{\theta}};w-w_h^*,\boldsymbol{\beta}-\boldsymbol{\beta}_h^*).\end{aligned}$$

Using (2.45), (2.46) all except the last term are estimated as in Step 3. This last term we treat using (2.42), (2.33) and (2.24)

$$(2.55)\quad \begin{aligned}\mathcal{A}_h(\tilde z,\tilde{\boldsymbol{\theta}};w-w_h^*,\boldsymbol{\beta}-\boldsymbol{\beta}_h^*)&=\mathcal{A}_h(\tilde z,\tilde{\boldsymbol{\theta}};w,\boldsymbol{\beta})-\mathcal{A}_h(\tilde z,\tilde{\boldsymbol{\theta}};w_h^*,\boldsymbol{\beta}_h^*)\\ &=\mathcal{F}_h(\tilde z,\tilde{\boldsymbol{\theta}})-(f,\tilde z)+(\boldsymbol{f},\tilde{\boldsymbol{\theta}})\\ =-\sum_{K\in\mathcal{C}_h}\alpha h_K^2t^2(t^2+\alpha h_K^2)^{-1}&(\boldsymbol{f},\boldsymbol{L}\,\tilde{\boldsymbol{\theta}})_K-\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\alpha h_K^2(\boldsymbol{f},\nabla\tilde z-\tilde{\boldsymbol{\theta}})_K.\end{aligned}$$

Next, we use (2.46)

$$(2.56)\quad \begin{aligned}&\Big|\sum_{K\in\mathcal{C}_h}\alpha h_K^2t^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{f},\boldsymbol{L}\,\tilde{\boldsymbol{\theta}})_K\Big|\\ &\le Ch\|\boldsymbol{f}\|_0\Big(\sum_{K\in\mathcal{C}_h}h_K^2\|\boldsymbol{L}\,\tilde{\boldsymbol{\theta}}\|_{0.K}^2\Big)^{1/2}\le Ch^2\|(l,\boldsymbol{l})\|_{-1,t}\|(f,\boldsymbol{f})\|_{-1,t}.\end{aligned}$$

From (2.43) and (2.46) we get

$$(2.57)\quad \begin{aligned}&\Big|\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\alpha h_K^2(\boldsymbol{f},\nabla\tilde z-\tilde{\boldsymbol{\theta}})_K\Big|\\ &=\Big|\sum_{K\in\mathcal{C}_h}\alpha h_K^2\big(\boldsymbol{f},\tilde{\boldsymbol{r}}+\alpha h_K^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{L}\,\tilde{\boldsymbol{\theta}}-\boldsymbol{l})\big)_K\Big|\\ &\le Ch\|\boldsymbol{f}\|_0\|\tilde{\boldsymbol{r}}\|_h+Ch\|\boldsymbol{f}\|_0\Big(\sum_{K\in\mathcal{C}_h}h_K^2\|\boldsymbol{L}\,\tilde{\boldsymbol{\theta}}\|_{0,K}^2\Big)^{1/2}+Ch^2\|\boldsymbol{f}\|_0\|\boldsymbol{l}\|_0\end{aligned}$$

$$(2.58)\quad \le Ch^2\|(l,\boldsymbol{l})\|_{-1,t}\|(f,\boldsymbol{f})\|_{-1,t}.$$

The asserted estimate (2.36) now follows by combining the estimates in this step. □

3. The multigrid method. In this section we prove that a simple multigrid method leads to a solver with optimal complexity and which is robust with respect to the parameter t.

The stabilized bilinear-form $\mathcal{A}_h$ depends on the underlying mesh. Thus, the sequence of meshes lead to different operators on each level. Hence, we apply the non-nested framework from [6] and adapt the notation from [5], Section 4.

We assume that we have a sequence of hierarchically refined meshes which we denote by $\mathcal{C}_1, \mathcal{C}_2, \ldots \mathcal{C}_J$. On each level k, $1 \leq k \leq J$, the finite element spaces are denoted by $W_k \times \boldsymbol{V}_k$. We note that the spaces are nested, i.e.,

$$W_{k-1} \times \boldsymbol{V}_{k-1} \subset W_k \times \boldsymbol{V}_k$$

such that no special grid transfer operators have to be defined. On each level we denote the bilinear form by

$$\mathcal{A}_k : (W_k \times \boldsymbol{V}_k) \times (W_k \times \boldsymbol{V}_k) \to \mathbb{R}$$

in accordance to (2.23).

Since, we assume a hierarchy of meshes they are all uniform and we denote the corresponding mesh-size with h_k (or h when it is irrelevant which level is in question).

On each level k, an inner product $(\cdot;\cdot)_k : (W_k \times \boldsymbol{V}_k) \times (W_k \times \boldsymbol{V}_k) \to \mathbb{R}$ is defined as

$$(z, \boldsymbol{\delta}; v, \boldsymbol{\eta})_k := h_k^2 (h_k + t)^{-2} \, (z, v) + h_k^2 \, (\boldsymbol{\delta}, \boldsymbol{\eta}),$$

and $\| \cdot \|_k$ denotes the corresponding norm. We define the operator $A_k : W_k \times \boldsymbol{V}_k \to W_k \times \boldsymbol{V}_k$ by

$$(A_k(z, \boldsymbol{\delta}); v, \boldsymbol{\eta})_k = \mathcal{A}_k(z, \boldsymbol{\delta}; v, \boldsymbol{\eta}) \qquad \forall (v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k.$$

Furthermore, we define the projections $P_{k-1} : W_k \times \boldsymbol{V}_k \to W_{k-1} \times \boldsymbol{V}_{k-1}$ and $Q_{k-1} : W_k \times \boldsymbol{V}_k \to W_{k-1} \times \boldsymbol{V}_{k-1}$ by

$$\mathcal{A}_{k-1}(P_{k-1}(z, \boldsymbol{\delta}); v, \boldsymbol{\eta}) = \mathcal{A}_k(z, \boldsymbol{\delta}; v, \boldsymbol{\eta}) \qquad \forall (v, \boldsymbol{\eta}) \in W_{k-1} \times \boldsymbol{V}_{k-1},$$

and

$$(Q_{k-1}(z, \boldsymbol{\delta}); v, \boldsymbol{\eta})_{k-1} = (z, \boldsymbol{\delta}; v, \boldsymbol{\eta})_k \qquad \forall (v, \boldsymbol{\eta}) \in W_{k-1} \times \boldsymbol{V}_{k-1}.$$

Finally, let $R_k : W_k \times \boldsymbol{V}_k \to W_k \times \boldsymbol{V}_k$ be the smoothing operator defined by a scaled Jacobi iteration or by a Gauss-Seidel iteration. A symmetrized smoothing iteration is defined by setting $R_k^{(l)} = R_k$ if l is odd, and $R_k^{(l)} := R_k^t$ if l is even. Here, $(\cdot)^t$ denotes the adjoint operator with respect to $(\cdot;\cdot)_k$.

We define the multigrid operator B_J by induction. Set $B_1 = A_1^{-1}$. For $k = 2, \ldots, J$ we define $B_k : W_k \times \boldsymbol{V}_k \to W_k \times \boldsymbol{V}_k$ as follows.

ALGORITHM 3.1. *With $g_k \in W_k \times \boldsymbol{V}_k$ we define $B_k g_k$ by the following algorithm.*

Initialize $x_k^0 \in W_k \times V_k$ *as* $x_k^0 = 0$
for $l = 0 \ldots m_k - 1$ *do*
  $x_k^{l+1} := x_k^l + R_k^{(l)}(g_k - A_k x_k^l)$
$x_k^{m_k+1} = x_k^{m_k} + B_{k-1} Q_{k-1}(g_k - A_k x_k^{m_k})$
for $l = m_{k+1} \ldots 2m_k$ *do*
  $x_k^{l+1} := x_k^l + R_k^{(l-1)}(g_k - A_k x_k^l)$
$B_k g_k := x_k^{2m_k+1}$.

We assume that the number of smoothing steps depends on the level as $m_k = 2^{J-k}$. This is the so called variable V-cycle multigrid algorithm.

THEOREM 3.1. *Assume that the Reissner Mindlin plate problem satisfies the regularity estimate* (2.13). *Then the multigrid algorithm provides an optimal preconditioner B_J, i.e.*

$$\mathit{cond}(B_J A_J) \leq C.$$

The constant C does neither depend on the number of levels, nor on the parameter t.

Proof. We apply Theorem 4.6 from [5]. It is easily checked that there holds an inverse inequality

$$\mathcal{A}_k(v_k, \boldsymbol{\eta}_k; v_k, \boldsymbol{\eta}_k) \leq \lambda_k \, \|(v_k, \boldsymbol{\eta}_k)\|_k^2 \qquad \forall (v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k$$

with $\lambda_k \simeq h_k^{-4}$. We have to check that the following conditions hold for all $(v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k$:

- (A.4) :

$$(3.1) \qquad (\overline{R}_k(v_k, \boldsymbol{\eta}_k); v_k, \boldsymbol{\eta}_k)_k \geq c h_k^4 \, \|(v_k, \boldsymbol{\eta}_k)\|_k,$$

  where $\overline{R}_k := (I - R_k A_k)(I - R_k^t A_k) A_k^{-1}$ is the symmetrized smoother.
- (A.10) with the choice $\alpha = 1/2$:

$$(3.2) \quad \mathcal{A}_k((I - P_{k-1})(v_k, \boldsymbol{\eta}_k); v_k, \boldsymbol{\eta}_k) \leq c h_k^2 \, \|A_k v\|_k \, \mathcal{A}_k(v_k, \boldsymbol{\eta}_k; v_k, \boldsymbol{\eta}_k)^{1/2}$$

for all $(v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k$.
These conditions are proven in Lemma 3.2 and Lemma 3.5 below. □

LEMMA 3.2 (Smoothing Property). *Let the smoother be defined by a properly scaled Jacobi iteration, or by the symmetrized Gauss-Seidel iteration. Then the condition* (3.1) *is satisfied.*

Proof. We apply Theorem 5.1 and Theorem 5.2 from [5], respectively. For this we have to show that the decomposition

$$(v_k, \boldsymbol{\eta}_k) = \sum_{i=1}^{\dim W_k} (v_i, 0) + \sum_{i=1}^{\dim \boldsymbol{V}_k} (0, \boldsymbol{\eta}_i)$$

into the one dimensional spaces generated by the finite element basis functions is stable with respect to the $\|\cdot\|_k$-norm, i.e.,

$$\sum_{i=1}^{\dim W_k} \|(v_i, 0)\|_k^2 + \sum_{i=1}^{\dim \boldsymbol{V}_k} \|(0, \boldsymbol{\eta}_i)\|_k^2 \leq c \, \|(v, \boldsymbol{\eta})\|_k^2.$$

This holds since both components of $\|\cdot\|_k$ are simply scaled L^2-norms. Furthermore, we need that the number of overlapping finite element functions is uniformly bounded. □

LEMMA 3.3 (Approximation property). *Let $(z_k, \boldsymbol{\delta}_k) \in W_k \times \boldsymbol{V}_k$ be given. Define the coarse grid functions $(z_{k-1}, \boldsymbol{\delta}_{k-1}) \in W_{k-1} \times \boldsymbol{V}_{k-1}$ by the projection*

$$(3.3) \qquad \mathcal{A}_{k-1}(z_{k-1}, \boldsymbol{\delta}_{k-1}; v_{k-1}, \boldsymbol{\eta}_{k-1}) = \mathcal{A}_k(z_k, \boldsymbol{\delta}_k; v_{k-1}, \boldsymbol{\eta}_{k-1})$$

for all $(v_{k-1}, \boldsymbol{\eta}_{k-1}) \in W_{k-1} \times \boldsymbol{V}_{k-1}$. Then the following approximation estimate holds:

$$\|(z_k - z_{k-1}, \boldsymbol{\delta}_k - \boldsymbol{\delta}_{k-1})\|_{1,t} \leq C h_k^2 \sup_{(v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; v, \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}}$$

Proof. Let $(z_k, \boldsymbol{\delta}_k) \in W_k \times \boldsymbol{V}_k$ be given. Let Π_k and $\boldsymbol{\Pi}_k$ be Clément projection operators. Define $g \in L^2(\Omega)$ and $\boldsymbol{g} \in [L^2(\Omega)]^2$ by

(3.4) $$(g, v) + (\boldsymbol{g}, \boldsymbol{\eta}) := \mathcal{A}_k(z_k, \boldsymbol{\delta}_k; \Pi_k v, \boldsymbol{\Pi}_k \boldsymbol{\eta}). \qquad \forall\, v \in L^2(\Omega),\ \forall\, \boldsymbol{\eta} \in [L^2(\Omega)]^2.$$

There holds

$$\begin{aligned}
\|(g, \boldsymbol{g})\|_{-1,t} &:= \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{(g, v) + (\boldsymbol{g}, \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}} \\
&= \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; \Pi_k v, \boldsymbol{\Pi}_k \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}} \\
&= \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; \Pi_k v, \boldsymbol{\Pi}_k \boldsymbol{\eta})}{\|(\Pi_k v, \boldsymbol{\Pi}_k \boldsymbol{\eta})\|_{1,t}} \frac{\|(\Pi_k v, \boldsymbol{\Pi}_k \boldsymbol{\eta})\|_{1,t}}{\|(v, \boldsymbol{\eta})\|_{1,t}} \\
&= \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; v, \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}} \|(\Pi_k, \boldsymbol{\Pi}_k)\|_{1,t}
\end{aligned}$$

Since Π_k is bounded in the L^2-norm as well as in the H^1-seminorm, and $\boldsymbol{\Pi}_k$ is bounded in the L^2-norm, the compound operator is bounded with respect to $\|\cdot\|_{1,t}$.

We pose the plate problem: find $(z, \boldsymbol{\delta}) \in H^1_0(\Omega) \times [H^1_0(\Omega)]^2$ such that

$$\mathcal{A}(z, \boldsymbol{\delta}; v, \boldsymbol{\eta}) = (g, v) + (\boldsymbol{g}, \boldsymbol{\eta})$$

Using that $(\Pi_k, \boldsymbol{\Pi}_k)$ is a projection on $W_k \times \boldsymbol{V}_k$, we recast (3.4) as

$$\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; v, \boldsymbol{\eta}) = (g, v) + (\boldsymbol{g}, \boldsymbol{\eta}) \qquad \forall\, (v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k.$$

This means that $(z_k, \boldsymbol{\delta}_k)$ is the finite element solution obtained by Method 2.2 (where the consistency terms on the right hand side were dropped.) Theorem 2.8 provides the estimate

$$\|(z - z_k, \boldsymbol{\delta} - \boldsymbol{\delta}_k)\|_{1,t} \le c h_k^2\ \|(g, \boldsymbol{g})\|_{-1,t}.$$

Using (3.3), we observe that

$$\mathcal{A}_{k-1}(z_{k-1}, \boldsymbol{\delta}_{k-1}; v, \boldsymbol{\eta}) = (g, v) + (\boldsymbol{g}, \boldsymbol{\eta}) \qquad \forall\, (v, \boldsymbol{\eta}) \in W_{k-1} \times \boldsymbol{V}_{k-1},$$

and again Theorem 2.8 proves

$$\|(z - z_{k-1}, \boldsymbol{\delta} - \boldsymbol{\delta}_{k-1})\|_{1,t} \le c h_{k-1}^2\, \|(g, \boldsymbol{g})\|_{-1,t}.$$

From the triangle inequality we obtain the result

$$\begin{aligned}
\|(z_{k-1} - z_k, \boldsymbol{\delta}_{k-1} - \boldsymbol{\delta}_k)\|_{1,t} &\le c h_k^2\, \|(g, \boldsymbol{g})\|_{-1,t} \\
&\le c h_k^2 \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; v, \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}}.
\end{aligned}$$

□

The norms $\|\cdot\|_k$ and $\mathcal{A}_k(\cdot,\cdot)^{1/2}$ can be embedded into a scale of norms. For this we set

$$|||(v, \boldsymbol{\eta})|||_0 := \|(v, \boldsymbol{\eta})\|_k \qquad \text{and} \qquad |||(v, \boldsymbol{\eta})|||_2 := \mathcal{A}_k(v, \boldsymbol{\eta}; v, \boldsymbol{\eta})^{1/2}.$$

Intermediate norms are defined by interpolation [7, Chapter 12], i.e.

$$|||(v,\boldsymbol{\eta})|||_s := \|(v,\boldsymbol{\eta})\|_{[|||\cdot|||_0,|||\cdot|||_2]_{s/2}} \qquad s\in(0,2).$$

Furthermore, the scale is extended by duality to the range $(2,4]$

$$|||(v,\boldsymbol{\eta})|||_{2+s} := \sup_{(z,\boldsymbol{\delta})} \frac{\mathcal{A}_l(v,\boldsymbol{\eta};z,\boldsymbol{\delta})}{|||(z,\boldsymbol{\delta})|||_{2-s}} \qquad s\in(0,2].$$

In particular there holds

$$|||(v,\boldsymbol{\eta})|||_4 = \sup_{(z,\boldsymbol{\delta})} \frac{\mathcal{A}_k(v,\boldsymbol{\eta};z,\boldsymbol{\delta})}{|||(z,\boldsymbol{\delta})|||_0} = \sup_{(z,\boldsymbol{\delta})} \frac{(A_k(v,\boldsymbol{\eta});z,\boldsymbol{\delta})_k}{\|(z,\boldsymbol{\delta})\|_k} = \|A_k(v,\boldsymbol{\eta})\|_k.$$

LEMMA 3.4. *The discrete* 1*-norm and the continuous* 1*-norm satisfy the following relation:*

$$(3.5)\qquad |||(v,\boldsymbol{\eta})|||_1 \le C\|(v,\boldsymbol{\eta})\|_{1,t} \qquad \forall\,(v,\boldsymbol{\eta})\in W_k\times \boldsymbol{V}_k.$$

Proof. Let $(v,\boldsymbol{\eta})\in W_k\times\boldsymbol{V}_k$. By the definition of the $\|\cdot\|_{1,t}$ norm, there exists a decomposition $v=v_0+v_r$ such that

$$\|v_0\|_1 + t^{-1}\|v_r\|_0 + \|\boldsymbol{\eta}\|_0 \le \|(v,\boldsymbol{\eta})\|_{1,t}.$$

Although v is a finite element function, its decomposition will in general not remain in the finite element space. To return to the finite element space, we define Clément projection operators $\Pi_k : L^2(\Omega)\to W_k$ and $\mathbf{\Pi}_k : [L^2(\Omega)]^2 \to \boldsymbol{V}_k$ with the following approximation properties:

$$\begin{aligned}\|v-\Pi_k v\|_s &\le h_k^{m-s}\|v\|_m, \quad 0\le s\le 1,\ 0\le m\le 2,\ s\le m,\\ \|\boldsymbol{\eta}-\mathbf{\Pi}_k\boldsymbol{\eta}\|_s &\le h_k^{m-k}\|\boldsymbol{\eta}\|_m, \quad 0\le s\le m\le 1.\end{aligned}$$

Now, the finite element function $(v,\boldsymbol{\eta})$ is decomposed into two finite element functions via

$$(3.6)\qquad (v,\boldsymbol{\eta}) = (\Pi_k v_0, \mathbf{\Pi}_k\nabla\Pi_k v_0) + (\Pi_k v_r, \boldsymbol{\eta}-\mathbf{\Pi}_k\nabla\Pi_k v_0).$$

Applying the triangle inequality leads to

$$(3.7)\qquad |||(v,\boldsymbol{\eta})|||_1 \le |||(\Pi_k v_0, \mathbf{\Pi}_k\nabla\Pi_k v_0)|||_1 + |||(\Pi_k v_r, \boldsymbol{\eta}-\mathbf{\Pi}_k\nabla\Pi_k v_0)|||_1.$$

We estimate both terms by using that $|||\cdot|||_1$ is the interpolation norm between $|||\cdot|||_0$ and $|||\cdot|||_2$ with parameter $1/2$. For $v_0\in H_0^2(\Omega)$, the continuity and approximation properties of Π_k and an inverse inequality implies

$$\begin{aligned}|||(\Pi_k v_0, \Pi_k\nabla\Pi_k v_0)|||_2^2 &= \|\mathbf{\Pi}_k\nabla\Pi_k v_0\|_1^2 + (h+t)^{-2}\|(\boldsymbol{I}-\mathbf{\Pi}_k)\nabla\Pi_k v_0\|_0^2\\ &\le \|\mathbf{\Pi}_k\nabla v_0\|_1^2 + \|\mathbf{\Pi}_k\nabla(v_0-\Pi_k v_0)\|_1^2\\ &\quad +(h+t)^{-2}\|(\boldsymbol{I}-\mathbf{\Pi}_k)\nabla v_0\|_0^2 + (h+t)^{-2}\|(\boldsymbol{I}-\mathbf{\Pi}_k)\nabla(I-\Pi_k)v_0\|_0^2\\ &\le \|v_0\|_2^2.\end{aligned}$$

With an inverse inequalities and L^2-continuity we obtain

$$\begin{aligned}\|\!|(\Pi_k v_0, \Pi_k \nabla \Pi_k v_0)|\!\|_0^2 &= h^2 \|\Pi_k \nabla \Pi_k v_0\|_0^2 + h^2(h+t)^{-2}\|\Pi_k v_0\|_0^2 \\ &\leq \|v_0\|_0^2.\end{aligned}$$

The interpolation space $[L^2(\Omega), H_0^2(\Omega)]_{1/2}$ is $H_0^1(\Omega)$. Thus, we apply operator interpolation to the linear operator $v \mapsto (\Pi_k v, \mathbf{\Pi}_k \nabla \Pi_k v)$ and obtain

$$\|\!|(\Pi_k v_0, \mathbf{\Pi}_k \nabla \Pi_k v_0)|\!\|_1 \leq \|v_0\|_1 \leq \|(v, \boldsymbol{\eta})\|_{1,t}^2. \tag{3.8}$$

We continue with the second term of (3.7). From

$$\begin{aligned}\|\!|(\Pi_k v_r, \boldsymbol{\eta} - \mathbf{\Pi}_k \nabla \Pi_k v_0)|\!\|_2^2 &= \|(\boldsymbol{\eta} - \mathbf{\Pi}_k \nabla \Pi_k v_0)\|_1^2 + (h+t)^{-2}\|\nabla \Pi_k v_r - \boldsymbol{\eta} + \mathbf{\Pi}_k \nabla \Pi_k v_0\|_0^2 \\ &\leq h^{-2}\{\|\boldsymbol{\eta}\|_0^2 + \|v_0\|_1^2\} + (h+t)^{-2}\{h^{-2}\|v_r\|_0^2 + \|\boldsymbol{\eta}\|_0^2 + \|v_0\|_1^2\} \\ &\leq h^{-2}\{\|v_0\|_1^2 + t^{-2}\|v_r\|_0^2 + \|\boldsymbol{\eta}\|_0^2\} \\ &\leq h^{-2}\|(v, \boldsymbol{\eta})\|_{1,t}^2\end{aligned}$$

and

$$\begin{aligned}\|\!|(\Pi_k v_r, \boldsymbol{\eta} - \mathbf{\Pi}_k \nabla \Pi_k v_0)|\!\|_0^2 &= h^2\|\boldsymbol{\eta} - \mathbf{\Pi}_k \nabla \Pi_k v_0\|_0^2 + h^2(h+t)^{-2}\|\Pi_k v_r\|_0^2 \\ &\leq h^2\left\{\|\boldsymbol{\eta}\|_0^2 + t^{-2}\|v_r\|_0^2 + \|v_0\|_1^2\right\} \\ &\leq h^2\,\|(v, \boldsymbol{\eta})\|_{1,t}^2\end{aligned}$$

we can conclude that

$$\|\!|(\Pi_k v_r, \boldsymbol{\eta} - \mathbf{\Pi}_k \nabla \Pi_k v_0)|\!\|_1^2 \leq \|(v, \boldsymbol{\eta})\|_{1,t}^2.$$

□

LEMMA 3.5. *The approximation property* (3.2) *holds.*

Proof. Applying Lemma 3.4 twice and Lemma 3.3 we obtain

$$\begin{aligned}\|\!|(w_k - w_{k-1}, \boldsymbol{\beta}_k - \boldsymbol{\beta}_{k-1})|\!\|_1 &\leq c\,\|(w_k - w_{k-1}, \boldsymbol{\beta}_k - \boldsymbol{\beta}_{k-1})\|_{1,t} \\ &\leq ch^2 \sup_{(v,\boldsymbol{\eta})} \frac{\mathcal{A}_k(w_k, \boldsymbol{\beta}_k; v; \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}} \\ &\leq ch^2 \sup_{(v,\boldsymbol{\eta})} \frac{\mathcal{A}_k(w_k, \boldsymbol{\beta}_k; v; \boldsymbol{\eta})}{\|\!|(v, \boldsymbol{\eta})|\!\|_1} \\ &= ch^2\,\|\!|(w_k, \boldsymbol{\beta}_k)|\!\|_3.\end{aligned}$$

This combined with

$$\begin{aligned}\mathcal{A}_k(w_k - w_{k-1}, \boldsymbol{\beta}_k - \boldsymbol{\beta}_{k-1}; w_k, \boldsymbol{\beta}_k) &\leq \|\!|(w_k - w_{k-1}, \boldsymbol{\beta}_k - \boldsymbol{\beta}_{k-1})|\!\|_1\,\|\!|(w_k, \boldsymbol{\beta}_k)|\!\|_3 \\ &\leq ch^2\,\|\!|(w_k, \boldsymbol{\beta}_k)|\!\|_3^2 \\ &\leq ch^2\,\|\!|(w_k, \boldsymbol{\beta}_k)|\!\|_2\,\|\!|(w_k, \boldsymbol{\beta}_k)|\!\|_4 \\ &= ch^2\,\mathcal{A}_k(w_k, \boldsymbol{\beta}_k; w_k, \boldsymbol{\beta}_k)^{1/2}\,\|A_k(w_k, \boldsymbol{\beta}_k)\|_k\end{aligned}$$

gives the asserted estimate (3.2) □

Level	Elements	$t = 0.1$ cond.numb.	cg	$t = 0.0001$ cond.numb.	cg
2	8	1.51	7	2.91	7
4	128	2.63	13	7.28	21
6	2048	3.32	14	8.88	27
8	32768	3.20	13	7.60	23

TABLE 4.1
Computational results

4. Computational results. We applied the proposed multigrid algorithm to a unit-square model problem. The plate is fully clamped on the boundary. The right hand side is the uniform load $f = 1$. The first mesh $\mathcal{C}_1$ consists of two triangles; the subsequent meshes $\mathcal{C}_2, \ldots, \mathcal{C}_J$ are obtained by regular refinement of one triangle into four.

We applied a conjugate gradient iteration with a multigrid preconditioner. We used the variable V-cycle with 2^{k-J} alternating Gauss-Seidel presmoothing and post-smoothing steps on the k^{th} level. Furthermore, we have computed the condition number of the preconditioned system matrix by the Lanczos algorithm.

Table 4.1 shows the condition number, and the required number of cg iterations for a relative reduction of the error by a factor of 10^{-8}. The error reduction was measured in the norm $(Br, r)^{1/2}$. We clearly see that the condition numbers and iteration numbers are bounded uniformly with respect to h and t. Note that the condition number of the matrix A behaves like $h^{-2}(h+t)^{-2}$ which was as high as 10^9.

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