EDGE ELEMENT COMPUTATION OF MAXWELL'S EIGENVALUES ON GENERAL QUADRILATERAL MESHES

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Abstract. Recent results prove that Nédélec edge elements do not achieve optimal rate of approximation on general quadrilateral meshes. In particular, lowest order edge elements provide stable but non convergent approximation of Maxwell's eigenvalues. In this paper we analyze a modification of standard edge element that restores the optimality of the convergence. This modification is based on a projection technique that can be interpreted as a reduced integration procedure.

1. Introduction

In this paper we consider the finite element approximation on general quadrilateral meshes of the Maxwell eigenvalue problem: find $\lambda \in \mathbb{R}$ such that for a nonvaninshing $\mathbf{u} : \Omega \to \mathbb{R}^2$ it holds

$$
\begin{aligned}
&\mathbf{curl}\,\mathrm{curl}\,\mathbf{u} = \lambda \mathbf{u} && \text{in } \Omega\\
&\mathrm{div}\,\mathbf{u} = 0 && \text{in } \Omega\\
&\mathbf{u}\cdot\mathbf{t} = 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{1}
$$

where Ω is a polygonal domain and $\mathbf{t}$ denotes a unit vector tangent to its boundary $\partial\Omega$.

It is well known that standard finite elements are not well suited for the approximation of problem (1) and that the use of edge finite elements provides good results in two and three dimensions when *affine* meshes are employed. We refer the reader to the papers [10, 5, 3, 4, 12, 6] (see also [9, 11] for a survey). When the mesh is non affine, i.e., when the actual elements cannot be obtained by affine transformation from a reference element, the situation is more complicated. A recent result (see [1]) implies that on general quadrilateral meshes (such meshes are in general *bilinear*, hence nonaffine) edge elements do

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not achieve optimal approximation properties. Lowest order edge element, in particular, do not provide convergent approximation to (1) on general quadrilateral meshes. We shall show numerical results confirming this claim in Section 4. When using general order edge element, a suboptimal rate of convergence is to be expected; in Section 4 we find numerical demonstrations of this behavior as well.

In this paper, we present a modification of edge finite elements which restores the optimal rate of convergence for any order of approximation. This modification can be interpreted as a reduced integration procedure or as a projection technique. In particular it is worth mentioning that, when implemented as a reduced integration scheme, our method does not require any additional cost with respect to the standard one. On *affine* meshes, modified edge elements coincide with the stantard ones.

The structure of the paper is as follows: in Section 2 we present the standard variational formulation (VF) of problem (1) and its equivalent mixed formulation (MF). The finite element formulation (VFh) associated with (VF) is known to be not convergent on general quadrilateral meshes. Moreover, we consider the discretization (MFh) of (MF), together with the projection scheme (EPh) and the reduced integration procedure (RIh). In Section 3, which is the core part of this paper, we proof that actually the three formulations (MFh), (EPh), and (RIh) are equivalent. As a consequence of the theory developed in [1], this shows that they provide optimally convergent solutions to our problem on general distorted quadrilateral meshes. Finally, in Section 4 we show numerical results fully confirming our theory.

2. Discrete eigenvalue problems

Let Ω be a polygonal domain in $\mathbb{R}^2$. We are dealing with the Hilbert spaces

$$\begin{aligned} V &= H_0(\mathrm{curl}) = \{\mathbf{v} \in [L^2(\Omega)]^2 : \mathrm{curl}\,\mathbf{v} \in L^2,\ \mathbf{v}\cdot\mathbf{t} = 0 \text{ on } \partial\Omega\} \\ Q &= L^2(\Omega) \end{aligned}$$

and the following standard Variational Formulation for problem (1). Find $0 \neq \lambda \in \mathbb{R}$ such that for $0 \neq \mathbf{u} \in V$ it holds

$$(\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v}) = \lambda(u, v) \quad \forall \mathbf{v} \in V. \tag{VF}$$

An equivalent Mixed Formulation reads: find $\lambda \in \mathbb{R}$ such that for $0 \neq p \in Q$ and $\mathbf{u} \in V$ it holds

$$\begin{aligned} &(\mathbf{u}, \mathbf{v}) + (\mathrm{curl}\,\mathbf{v}, p) = 0 &\quad \forall \mathbf{v} \in V, \\ &(\mathrm{curl}\,\mathbf{u}, q) = -\lambda(p, q) &\quad \forall q \in Q. \end{aligned} \tag{MF}$$

For the discretization of (1) we consider a sequence of *regular* meshes of quadrilateral elements. The definition of regularity in this case differs from the standard one for triangular meshes, since we have to make sure that the quadrilaterals do not degenerate to triangles. A possible definition, according to [8, A.2, pp. 104–105] or [13], is the following one. From a generic quadrilateral K we obtain four triangles by the four possible choices of three vertices from the vertices of K, and we define ρ_K as the smallest diameter of the inscribed circles to these four triangles. The *shape constant* of K is then $\sigma_K := h_K/\rho_K$ where $h_K = \mathrm{diam}(K)$. A mesh sequence is said *regular* if the shape constants for the meshes can be uniformly bounded.

The *bilinear* transformation from the reference element $\hat{K}$ (here and in the following, a hat supscript denotes quantities related to the reference element) to the actual one K is denoted by

$$F_K : \hat{K} \to K.$$

Let $\mathcal{N}_k$ denote the Nédélec edge finite element space on the reference element $\hat{K}$ and $\mathcal{Q}_k = \mathrm{curl}(\mathcal{N}_k)$ the space of polynomials of separate degree at most k in $\hat{x}$ and $\hat{y}$, for $k \geq 0$. Then the finite element approximations of V and Q are given by

$$V_h = \{\mathbf{v} \in V : v|_K \circ F_K \in (F_K)^{-T}\mathcal{N}_k\}$$

and

$$Q_h = \{q \in Q : q|_K \circ F_K \in \mathcal{Q}_k\}.$$

We explicitly notice that, on general quadrilateral meshes, $\mathrm{curl}\, V_h \not\subset Q_h$. In particular the inclusion is false when the Jacobian of F_K is not constant.

It is known that a finite element scheme based on formulation (VF) cannot converge optimally on general meshes, due to the bad approximation properties of the space V_h. For the sake of completeness, we write down such scheme explicitly: find $0 \neq \lambda_h \in \mathbb{R}$ such that for $0 \neq \mathbf{u}_h \in V_h$ it holds

$$(\mathrm{curl}\,\mathbf{u}_h, \mathrm{curl}\,\mathbf{v}) = \lambda_h(u_h, v) \quad \forall \mathbf{v} \in V_h. \tag{VFh}$$

Numerical results showing the bad behavior of scheme (VFh) are presented in Section 4.

The approximation of the mixed formulation (MF) is: find $\lambda_h \in \mathbb{R}$ such that for $0 \neq p_h \in Q_h$ and $\mathbf{u}_h \in V_h$ it holds

$$\begin{aligned} &(\mathbf{u}_h, \mathbf{v}) + (\mathrm{curl}\,\mathbf{v}, p_h) = 0 && \forall \mathbf{v} \in V_h,\\ &(\mathrm{curl}\,\mathbf{u}_h, q) = -\lambda_h(p_h, q) && \forall q \in Q_h. \end{aligned} \tag{MFh}$$

Theorem 1. *The eigenpairs computed with scheme* (MFh) *converge to Maxwell's eigenvalues on general quadrilateral meshes.*

Proof. This is an immediate consequence of the third estimate in Theorem 12 of [1], which can be used to show the uniform convergence of the discrete resolvent operator associated with (MFh) towards the resolvent operator of (MF). □

Let us introduce the L^2 projection operator $P_{Q_h} : L^2 \to Q_h$

$$(P_{Q_h} p, q_h) = (p_h, q_h) \quad \forall q_h \in Q_h.$$

We the help of P_{Q_h} we can now define the primal finite element method with Explicit Projection: find $0 \neq \lambda_h \in \mathbb{R}$ such that for $0 \neq \mathbf{u}_h \in V_h$ it holds

$$\text{(EPh)} \qquad (P_{Q_h} \operatorname{curl} \mathbf{u}_h, P_{Q_h} \operatorname{curl} \mathbf{v}_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h$$

Let us consider an integration rule[1] on the reference square $\hat{K}$ given by k^2 nodes $\{x_i\}$ and weights $\{\omega_i\}$ such that for $q \in \mathcal{Q}_{2k-1}(\hat{K})$ it holds

$$\int_{\hat{K}} q(\hat{\mathbf{x}})\, d\hat{\mathbf{x}} = \sum_{i=1}^{k^2} \omega_i q(x_i).$$

We then define the discrete inner product associated to the quadrature rule

$$(p, q)_h = \sum_K \sum_{i=1}^{k^2} \omega_i p(F_K(x_i)) q(F_K(x_i)) \det(F'_K(x_i)).$$

The primal formulation with Reduced Integration then reads: find $0 \neq \lambda_h \in \mathbb{R}$ such that for $0 \neq \mathbf{u}_h \in V_h$ it holds

$$\text{(RIh)} \qquad (\operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v}_h)_h = \lambda_h(\mathbf{u}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h$$

3. Equivalence of discrete formulations

Theorem 2. *The three discrete formulations discussed in Section 2 and presented in* (MFh), (EPh)*, and* (RIh) *provide the same solutions* $(\lambda_h, \mathbf{u}_h)$.

Proof. We start to prove the equivalence between (MFh) and (EPh). Let (u_h, p_h) solve (MFh) for a certain (nonvanishing) λ_h. From the second line of (MFh) there follows

$$p_h = \lambda_h^{-1} P_{Q_h} \operatorname{curl} \mathbf{u}_h,$$

[1] We notice that such rule can be obtained, for instance, by tensor product of one dimensional Gauss rule

and plugging this relation into the first line shows that $\mathbf{u}_h$ actually solves (EPh):

$$(\mathbf{u}_h, \mathbf{v}_h) - \lambda_h^{-1}(P_{Q_h} \operatorname{curl} \mathbf{u}_h, \operatorname{curl} \mathbf{v}_h) = 0.$$

Viceversa, if $\mathbf{u}_h \in V_h$ is a solution of (EPh), then

$$(\mathbf{u}_h, p_h) = \lambda_h^{-1} P_{Q_h} \operatorname{curl} \mathbf{u}_h$$

solves (MFh).

Finally, we show that (MFh) and (RIh) provide the same solution.

We observe that there is a one-to-one relation between finite element functions in Q_h and function values at the integration points

$$q_h \in Q_h \leftrightarrow \{q(F_K(x_i))\}. \tag{2}$$

Namely, both spaces have the same dimension (k^2 times the number of elements), and $q(F_R(x_i)) = 0\ \forall K\ \forall i$ implies $q = 0$:

$$\|q_h\|_{L_2}^2 = \sum_K \sum_{i=1}^{k^2} \omega_i J (q_h(x_i)^2)^2 = 0.$$

Thanks to (2) and the fact that the mapping from the reference square to a generic quadrilateral is bilinear, the numerical integration rule is exact for evaluating $(\operatorname{curl} \mathbf{u}_h, q_h)$ and (p_h, q_h). Then

$$\begin{aligned}\int_K \operatorname{curl} \mathbf{u}_h q_h \, d\mathbf{x} &= \int_{\hat{K}} J^{-1} \widehat{\operatorname{curl}} \hat{\mathbf{u}}_h \hat{q}_h J \, d\hat{\mathbf{x}} = \int_{\hat{R}} \widehat{\operatorname{curl}} \hat{\mathbf{u}}_h \hat{q}_h \, d\hat{\mathbf{x}} \\ &= \sum_{i=1}^{k^2} \omega_i \widehat{\operatorname{curl}} \hat{\mathbf{u}}_h(x_i) \hat{q}_h(x_i).\end{aligned}$$

Thus, the discrete problem MFh is equivalent to

$$\begin{aligned} (\mathbf{u}_h, \mathbf{v}_h) + (\operatorname{curl} \mathbf{v}_h, p_h)_h &= 0 \qquad \forall \mathbf{v}_h \in V_h \\ (\operatorname{curl} \mathbf{u}_h, q_h)_h &= -\lambda_h (p_h, q_h)_h \qquad \forall q_h \in Q_h. \end{aligned} \tag{3}$$

Then (3) is equivalent to

$$\begin{aligned} (\mathbf{u}_h, \mathbf{v}_h) + (\operatorname{curl} \mathbf{v}_h, p_h)_h &= 0 \qquad \forall \mathbf{v}_h \in V_h \\ \operatorname{curl} \mathbf{u}_h(F_K(x_i)) &= \lambda_h p_h(F_K(x_i)) \quad \forall K\ \forall i. \end{aligned} \tag{4}$$

The second equation in (4) allows us to express p_h at the integration points; plugging this expression into the first equation of (4) gives (RIh). □

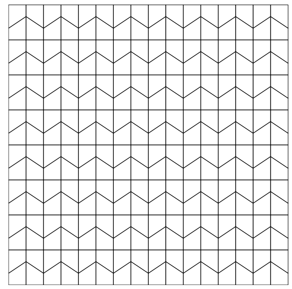

FIGURE 1. A mesh of distorted quadrilaterals

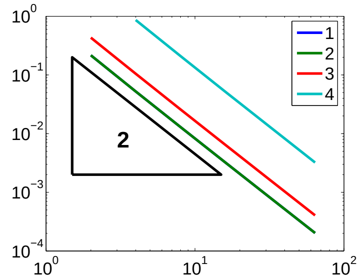

FIGURE 2. Lowest order edge elements on uniform mesh

4. Numerical results

In this section we report on some basic numerical results, which fully confirm the theory of Section 3. Our domain Ω is the square of size π, so that the eigenvalues of (1) are $\lambda = m^2 + n^2$, $m, n \in \mathbb{N}$ ($\lambda = 1, 1, 2, 4, 4, \dots$). For our computations we use two mesh sequences. The first one is a uniform decomposition of Ω into subsquares, while the second one (presented in Figure 1) is made of distorted trapezoids in such a way that the distortion factor is constant as h goes to zero.

4.1. **Lowest order edge elements.** The computations performed with lowest order edge elements show in a clear way the lack of convergence on distorted meshes. In Figure 2, the optimal second order of convergence on the uniform mesh sequence is apparent (different lines correspond to the convergence history for different eigenvalues[2]; in each graph we plotted the first four or five eigenvalues; some of them may be overlapping on symmetric meshes), while in Figure 3 is is clear that the discrete eigenvalues converge to a wrong value (see [7, 2]).

[2]the rank of each eigenvalue is listed in the legend

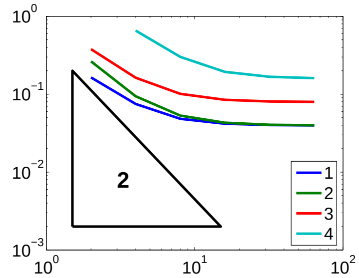

FIGURE 3. Lowest order edge elements on distorted mesh

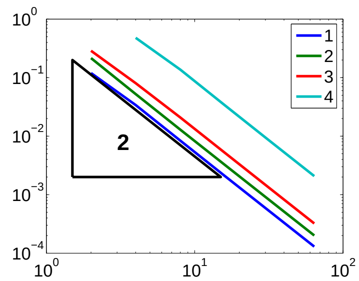

FIGURE 4. Lowest order edge elements with reduced integration on distorted mesh

In Figure 4, however, it is evident that the optimal convergence order is recovered when scheme (RIh) (or, equivalently, (EPh)) is used[3].

4.2. **Second order edge elements.** When higher order edge elements are used on distorted meshes, we still observe converge but with a lower order than the optimal one. This behavior is well understood from Figures 5 (optimal order on uniform mesh) and 6 (suboptimal order on distorted mesh).

The optimal order is restored (see Figure 7) when scheme (RIh) (or, equivalently, (EPh)) is used[4].

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[3]In this case the reduced integration scheme compounds in evaluating the integral of the stiffness matrix at the barycenter

[4]In this case we used as reduced integration rule the 2 by 2 Gauss rule, while for the full integration we used the 4 by 4 Gauss rule

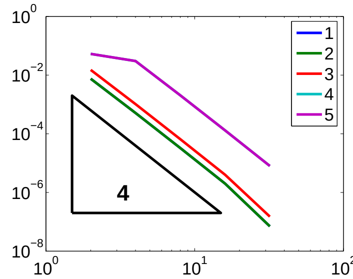

FIGURE 5. Second order edge elements on uniform mesh

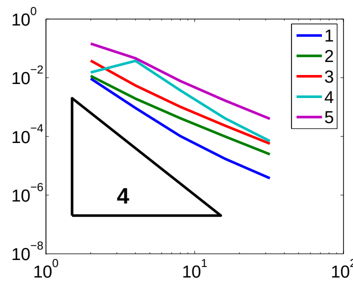

FIGURE 6. Second order edge elements on distorted mesh

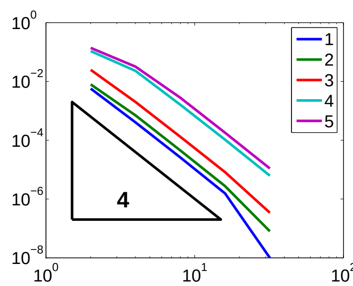

FIGURE 7. Second order edge elements with reduced integration on distorted mesh

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