Technische Universität Chemnitz

Sonderforschungsbereich 393

Numerische Simulation auf massiv parallelen Rechnern

Thomas Apel   Serge Nicaise   Joachim Schöberl

Crouzeix-Raviart type finite elements on anisotropic meshes

Preprint SFB393/99-10

Abstract The paper deals with a non-conforming finite element method on a class of anisotropic meshes. The Crouzeix-Raviart element is used on triangles and tetrahedra. For rectangles and prismatic (pentahedral) elements a novel set of trial functions is proposed. Anisotropic local interpolation error estimates are derived for all these types of element and for functions from classical and weighted Sobolev spaces. The consistency error is estimated for a general differential equation under weak regularity assumptions. As a particular application, an example is investigated where anisotropic finite element meshes are appropriate, namely the Poisson problem in domains with edges. A numerical test is described.

Key Words Anisotropic mesh, Crouzeix-Raviart element, non-conforming finite element method, anisotropic interpolation error estimate, consistency error, edge singularity.

AMS(MOS) subject classification 65N30; 65N15, 65N50, 65D05.

Preprint-Reihe des Chemnitzer SFB 393

SFB393/99-10 **Mai 1999**

Contents

Author's addresses:

Thomas Apel
TU Chemnitz
Fakultät für Mathematik
D-09107 Chemnitz, Germany

apel@mathematik.tu-chemnitz.de
http://www.tu-chemnitz.de/~tap

Serge Nicaise
Université de Valenciennes et du Hainaut Cambrésis
LIMAV, Institut des Sciences et Techniques de Valenciennes
B.P. 311
F–59304 - Valenciennes Cedex, France

snicaise@univ-valenciennes.fr
http://www.univ-valenciennes.fr/macs/nicaise

Joachim Schöberl
Johannes Kepler Universität Linz
Freistädterstrasse 313
A-4020 Linz, Austria

joachim@numa.uni-linz.ac.at
http://www.numa.uni-linz.ac.at/Staff/joachim/schoeberl.html

1 Introduction

The solution of elliptic boundary value problems may have *anisotropic behaviour* in parts of the domain. That means that the solution varies significantly only in certain directions. Examples include diffusion problems in domains with edges and singularly perturbed convection-diffusion-reaction problems where boundary or interior layers appear. In such cases it is an obvious idea to reflect this anisotropy in the discretization by using *anisotropic meshes* with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction. Anisotropic meshes can also be advantageous if surfaces with strongly anisotropic curvature (the front side of a wing of an airplane, for example [31, Figure 6]) or thin layers of different material are to be discretized.

In order to describe the elements of anisotropic meshes mathematically, consider an elliptic boundary value problem posed over a polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. We study the discretization error of the finite element method on a family of meshes $\mathcal{T}_h = \{K\}$ with the usual admissibility conditions (see, for example, Conditions ($\mathcal{T}_h$1)–($\mathcal{T}_h$5) in [17, Chapter 2]). Denote by $h_{L,K}$ the diameter of the finite element K, and by ϱ_K the supremum of the diameters of all balls contained in K. Then it is assumed in the classical finite element theory that $h_{L,K} \lesssim \varrho_K$. The notation $a \lesssim b$ means the existence of a positive constant C (which is independent of $\mathcal{T}_h$ and of the function under consideration) such that $a \leq Cb$. This assumption is no longer valid in the case of anisotropic meshes. Conversely, anisotropic elements K are characterized by

$$\frac{h_{L,K}}{\varrho_K} \to \infty$$

where the limit can be considered as $h \to 0$ (as in the present paper) or $\varepsilon \to 0$ where ε is some (small perturbation) parameter of the problem.

Local interpolation error estimates for anisotropic elements are widely developed in the literature [2, 3, 4, 6, 8, 11, 12, 13, 14, 19, 21, 22, 23, 26, 29, 30, 33, 35]. In particular the improved estimates in [2, 4, 12, 13, 14, 26] are applied, for example, for the investigation of Laplace type problems in domains with edges [3, 4, 7, 8, 25], layers in singularly perturbed problems [5, 6, 20], and anisotropic phenomena in the solution of the Stokes problem [14]. However, all these applications are restricted to conforming finite element methods.

Non-conforming methods are hardly treated. Such methods are of particular interest in mixed methods for problems like the Stokes problem or the Mindlin-Reissner plate problem. The aim of this paper is to provide basic results for a simple class of non-conforming elements, namely the Crouzeix-Raviart element [18] and modifications thereof. We apply them here to the simplest model problem, the Poisson problem. Other applications are postponed to the upcoming papers [9, 10].

In Section 2 we describe a family of anisotropically graded finite element meshes which turned out to be suited for the treatment of edge singularities in the context of conforming $\mathcal{P}_1$ elements [2, 4, 7]. We show in this paper that this family is also suited for non-conforming $\mathcal{P}_1$ elements.

The finite element error of the non-conforming method can be estimated via the second Strang lemma by the sum of an interpolation error and a consistency error. These errors are considered in Sections 3 and 4. In particular, we derive for Crouzeix-Raviart triangular and tetrahedral elements K [18] the interpolation error estimate

$$|u - \mathrm{I}_h u; W^{1,2}(K)| \lesssim |u; W^{1,2}(K)| \tag{1.1}$$

from which we can derive easily

$$|u - \mathrm{I}_h u; W^{1,2}(K)| \lesssim \sum_{i=1}^{d} h_{i,K} |\partial_i u; W^{1,2}(K)|. \tag{1.2}$$

We denote by d the space dimension, by $h_{i,K}$ suitably defined element sizes, by ∂_i the partial derivative $\partial/\partial x_i$, and by $\|\cdot\,; X\|$ and $|\cdot\,; X|$ the usual norm and seminorm in the Banach space X. A similar estimate is obtained for functions u from weighted Sobolev spaces. Both estimates hold for a general triangle/tetrahedron, in particular without an angle condition. We remark that related results were obtained in [1].

Note that (1.1) is not valid for Lagrangian interpolation on the conforming $\mathcal{P}_1$ element. Even (1.2) is not valid for the conforming tetrahedral element [3, 4]. Modified interpolants of Scott-Zhang type have been developed to overcome these deficiencies [2], but until now they are restricted to a special class of mesh. This is clearly an advantage of the non-conforming element.

In Section 4 we prove for the more general equation $-\nabla\eta = f$ estimates of the consistency error. The proof made certain new ideas necessary since the standard proof [18] cannot be applied to anisotropic elements. The reason for the generality is that hence these estimates can be applied in the papers [9, 10] to the Stokes problem and the Reissner-Mindlin plate problem, respectively.

Crouzeix-Raviart type rectangular elements, called *parametric rotated $\mathcal{Q}_1$ element* and *non-parametric rotated $\mathcal{Q}_1$ element* were defined and investigated in [27] for isotropic meshes. The anisotropic case was discussed in [15]. These authors proved that the non-parametric element, together with the $\mathcal{P}_0$ element for the pressure, yield a Stokes element pairing that is stable independently of the aspect ratio. However, the estimation of the consistency error was not addressed. We give in Sections 3 and 4 a complete treatment of a modified Crouzeix-Raviart type rectangular element. The modified element generalizes easily to a class of prismatic three-dimensional elements (pentahedra).

The results of Sections 3 and 4 are applied in Section 5 in order to prove the finite element error estimate for the model Laplace problem in the presence of edge singularities. We obtain the optimal finite element error estimate

$$\|u - u_h\|_{1,h} \lesssim h\|f; L^2(\Omega)\|,$$

where $h := \max_K h_{L,K}$, $h_{L,K} := \max_i h_{i,K}$, and

$$\|\cdot\|_{m,h}^2 := \sum_K |\cdot\,; W^{m,2}(K)|, \quad m \geq 0,$$

are mesh dependent (semi-)norms. For the assessment of this result it is essential to point out that the number of elements/degrees of freedom is of the order h^{-3}, that means, it is asymptotically not larger than that for uniform meshes where only a reduced convergence order h^λ is obtained.

In the final section of the paper we show by a numerical test example that these asymptotical convergence orders can be observed in calculations with practical mesh sizes. Furthermore, we compare the non-conforming with the conforming $\mathcal{P}_1$ element.

Throughout the paper we use the following convention concerning indices. When all indices play the same role we use the index set $\{1,\dots,d\}$ (recall that d is the space dimension). In anisotropic elements, however, one direction is distinguished, that is the stretching direction of the element. Since in two space dimensions this direction is usually indexed by 1, and in three space dimensions by 3, we try to avoid confusion by using the indices L (long, large) and S (short, small), in three dimensions $S1, S2$. In this sense we denote the element sizes by h_L and h_S and the components of the vector function η by η_L and η_S. The aim is to compensate large norms of η_S by small element sizes h_S in direction x_S.

2 Discretization of the model problem

Consider the Poisson problem with Dirichlet boundary conditions in a three-dimensional polyhedral domain Ω,

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

with a right hand side $f \in L^2(\Omega)$. It is well known that the solution has in general singularities near corners and edges and near the lines where the type of the boundary condition changes. As a result, the finite element method on quasi-uniform meshes loses accuracy. The rate of convergence is smaller in comparison with that for problems with smooth solutions. It has been shown under different assumptions that anisotropic mesh grading is appropriate to compensate this effect and to obtain the optimal order of convergence for the conforming first order element [2, 4, 7, 8].

In [2, 4, 7] we considered in particular a prismatic domain

$$\Omega = G \times Z \tag{2.2}$$

where $G \subset I\!R^2$ is a bounded polygonal domain and $Z := (0, z_0) \subset I\!R$ is an interval. This restriction was made there because we wanted to focus on *edge singularities*, and such domains do not introduce additional corner singularities [32, 34]. The finite element meshes were graded perpendicularly to the edge and quasi-uniform in the edge direction. In this section we state first the regularity of the solution of problem (2.1), (2.2), and introduce then the family of non-conforming finite element spaces. The estimation of the finite element error is postponed to Section 5.

Denote by $V_0 \subset W^{1,2}(\Omega)$ the space of all $W^{1,2}(\Omega)$-functions which vanish at the boundary. The variational form of problem (2.1) is given by

$$\textit{Find } u \in V_0 \textit{ such that } (\nabla u, \nabla v) = (f, v) \textit{ for all } v \in V_0. \tag{2.3}$$

The existence of a unique variational solution u follows from the Lax-Milgram lemma.

Let us assume that the cross-section G has only one corner with interior angle $\omega > \pi$ at the origin; thus Ω has only one "singular edge" which is part of the x_L-axis. The case of more than one singular edge introduces no additional difficulties because the edge singularities are of local nature. The properties of the solution u can be described favourably by using weighted Sobolev spaces

$$V_\beta^{\ell,p}(\Omega) := \{v \in \mathcal{D}'(\Omega) : \|v; V_\beta^{\ell,p}(\Omega)\| < \infty\}, \quad \ell \in I\!N, \ p \in [1,\infty], \ \beta \in I\!R.$$

The norm is defined for $p \in [1,\infty)$ by

$$\|v; V_\beta^{\ell,p}(\Omega)\|^p := \sum_{i+j+k\le\ell} \|r^{\beta-\ell+i+j+k}\partial_1^i\partial_2^j\partial_3^k v; L^p(\Omega)\|^p$$

with the usual modification for $p = \infty$.

Lemma 2.1 *The solution u of problem (2.1), (2.2) satisfies*

$$\frac{\partial u}{\partial x_i} \in V_\beta^{1,2}(\Omega), \qquad \left\|\frac{\partial u}{\partial x_i}; V_\beta^{1,2}(\Omega)\right\| \lesssim \|f; L^2(\Omega)\|, \quad i \in \{S1, S2\}, \ \beta > 1 - \frac{\pi}{\omega}, \tag{2.4}$$

$$\frac{\partial u}{\partial x_L} \in V_0^{1,2}(\Omega), \qquad \left\|\frac{\partial u}{\partial x_L}; V_0^{1,2}(\Omega)\right\| \lesssim \|f; L^2(\Omega)\|. \tag{2.5}$$

Proof See for example [7, Section 2]. □

We define now families of meshes $\mathcal{Q}_h = \{Q\}$ and $\mathcal{T}_h = \{K\}$ by introducing in G the standard mesh grading for two-dimensional corner problems, see for example [24, 28]. Let $\{T\}$ be a regular isotropic triangulation of G; the elements are triangles. With h being the global mesh parameter, $\mu \in (0,1]$ being the grading parameter, r_T being the distance of T to the corner,

$$r_T := \inf_{(x_1,x_2)\in T} (x_1^2 + x_2^2)^{1/2},$$

and with some constant $R > 0$, we assume that the element size $h_T := \operatorname{diam} T$ satisfies

$$h_T \sim \begin{cases} h^{1/\mu} & \text{for } r_T = 0, \\ h r_T^{1-\mu} & \text{for } 0 < r_T \le R, \\ h & \text{for } r_T > R. \end{cases}$$

This graded two-dimensional mesh is now extended in the third dimension using a uniform mesh size, h. In this way we obtain a pentahedral triangulation $\mathcal{Q}_h$ or, by dividing each pentahedron, a tetrahedral triangulation $\mathcal{T}_h$ of Ω, see Figure 2.1 for an illustration. Note that the number of elements is of the order h^{-3} for the full range of μ. The notation is extended to the three-dimensional case as follows. Let r_Q and r_K be the distance of an element Q or K to the edge (x_3-axis), respectively. Then the element sizes satisfy

$$h_{L,Q} \sim h, \quad h_{S1,Q} \sim h_{S2,Q} \sim \begin{cases} h^{1/\mu} & \text{for } r_Q = 0, \\ h r_Q^{1-\mu} & \text{for } 0 < r_Q \le R, \\ h & \text{for } r_Q > R. \end{cases} \tag{2.6}$$

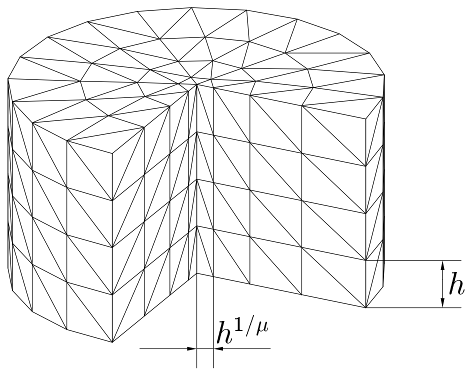

Figure 2.1: Example for an anisotropic mesh.

The element sizes $h_{i,K}$ are used by analogy for tetrahedral elements, $h_{i,K} := h_{i,Q}$ if $K \subset Q$.

On $\mathcal{T}_h$ we introduce the Crouzeix-Raviart finite element space

$$V_h := \{v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}_1 \ \forall K, \int_F [v_h] = 0 \ \forall F\} \tag{2.7}$$

where we denote faces of elements by F and by $[v_h]$ the jump of the function v_h on the faces F. For boundary faces we identify $[v_h]$ with v_h. An appropriate choice of V_h for pentahedral meshes $\mathcal{Q}_h$ is

$$V_h := \{v_h \in L^2(\Omega) : v_h|_Q \in \mathcal{P}_1 \oplus \operatorname{span}\{x_L^2\} \ \forall Q, \int_F [v_h] = 0 \ \forall F\}. \tag{2.8}$$

We note that $V_h \not\subset V_0$, that means the method is non-conforming. Thus ∇v_h is not defined on inter-element boundaries and we define the finite element solution u_h by using the weaker scalar product

$$(u,v)_h := \sum_K \int_K uv \qquad \text{or} \qquad (u,v)_h := \sum_Q \int_Q uv,$$

respectively, namely:

$$\textit{Find } u_h \in V_{0h} \textit{ such that } (\nabla u_h, \nabla v_h)_h = (f, v_h) \textit{ for all } v_h \in V_{0h}. \tag{2.9}$$

The finite element error $u - u_h$ can be estimated in the norm $\|\cdot\|_{1,h}$ by using the second Lemma of Strang,

$$\|u-u_h\|_{1,h} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_{1,h} + \sup_{v_h \in V_h} \frac{|(\nabla u, \nabla v_h)_h - (f, v_h)|}{\|v_h\|_{1,h}}. \tag{2.10}$$

The terms are called approximation error and consistency error, respectively. The approximation error is estimated by using $v_h = \mathrm{I}_h v$ with a suitably defined interpolation operator I_h, see the next section. A general discussion of the consistency error is given in Section 4. We continue the estimation of the finite element error for this model problem in Section 5.

3 Local interpolation error estimates

Consider first simplicial (triangular or tetrahedral) elements $K \subset \mathbb{R}^d$, $d = 2, 3$, with faces (sides) F. The Crouzeix-Raviart interpolant I_h, $\mathrm{I}_h \in \mathcal{P}_1$, is defined by

$$\int_F u = \int_F \mathrm{I}_h u \qquad \forall F \subset \partial K. \tag{3.1}$$

Contrary to the Lagrangian interpolant (nodal values) this interpolant is defined for $u \in W^{1,p}(K)$ for all $p \in [1, \infty]$. Note further that

$$\mathrm{I}_h w = w \qquad \forall w \in \mathcal{P}_1. \tag{3.2}$$

We start with a stability estimate from which the desired local interpolation error estimates can be derived easily.

Lemma 3.1 *For all $p, q \in [1, \infty]$ and $u \in W^{1,p}(K)$ the estimate*

$$\|\partial_j \mathrm{I}_h u; L^q(K)\| \leq (\mathrm{meas}_d K)^{1/q-1/p} \|\partial_j u; L^p(K)\|, \quad j = 1, \ldots, d$$

holds.

Proof The essential ingredient is that $\partial_j \mathrm{I}_h u$ is constant. Let n be the outward unit normal to ∂K and n_j be the projections of n to the x_j-axis, $j = 1, \ldots, d$. By Green's formula and (3.1) we obtain

$$\begin{aligned} \partial_j \mathrm{I}_h u &= (\mathrm{meas}_d K)^{-1} \int_K \partial_j \mathrm{I}_h u = (\mathrm{meas}_d K)^{-1} \sum_F \left(\int_F \mathrm{I}_h u \right) n_j \\ &= (\mathrm{meas}_d K)^{-1} \sum_F \left(\int_F u \right) n_j = (\mathrm{meas}_d K)^{-1} \int_K \partial_j u. \end{aligned} \tag{3.3}$$

The desired estimate is then a consequence of the Hölder inequality,

$$\begin{aligned} \|\partial_j \mathrm{I}_h u; L^q(K)\| &= (\mathrm{meas}_d K)^{1/q} \, |\partial_j \mathrm{I}_h u| \\ &\leq (\mathrm{meas}_d K)^{1/q-1} \|\partial_j u; L^1(K)\| \\ &\leq (\mathrm{meas}_d K)^{1/q-1/p} \|\partial_j u; L^p(K)\|. \end{aligned}$$

□

Corollary 3.2 *For $p, q \in [1, \infty]$, $p \geq q$, and $u \in W^{1,p}(K)$ the estimate*

$$\|\partial_j (u - \mathrm{I}_h u); L^q(K)\| \lesssim (\mathrm{meas}_d K)^{1/q-1/p} \|\partial_j u; L^p(K)\|, \quad j = 1, \ldots, d$$

holds.

Note that Lemma 3.1 and Corollary 3.2 hold true for arbitrary elements K, without any restriction to angles.

For the error estimate against second derivatives of u we utilize two ingredients which need a condition on the elements K and a definition of element sizes $h_{i,K}$. The first is the validity of the embedding $W^{1,p}(K) \hookrightarrow L^q(K)$ in the form

$$\|v; L^q(K)\| \lesssim (\mathrm{meas}_d K)^{1/q-1/p} \left(\|v; L^p(K)\| + \sum_{i=1}^{d} h_{i,K} \|\partial_i v; L^p(K)\| \right). \tag{3.4}$$

The second is a Deny-Lions or Bramble-Hilbert type argument, namely

$$\forall v \in W^{1,p}(K) \; \exists w \in \mathcal{P}_0 : \quad \|v - w; L^p(K)\| \lesssim \sum_{i=1}^{d} h_{i,K} \|\partial_i v; L^p(K)\|, \tag{3.5}$$

which is, with $w = M_K v$,

$$M_G v := (\mathrm{meas}_{\dim G} G)^{-1} \int_G v, \tag{3.6}$$

in this simple case also a conclusion of the Poincaré-Friedrichs inequality.

Both estimates are clearly satisfied on a reference element $\hat{K}$ with $h_{i,K} = 1$. If K is a triangle with two sides parallel to the coordinate axes then the estimates are satisfied with $h_{i,K}$ being the lengths of these sides. If K is a tetrahedron as constructed in Section 2 then the estimates are also satisfied. We will omit the discussion of more general situations here.

Lemma 3.3 *Let K be a simplicial element with element sizes $h_{i,K}$ such that (3.4) and (3.5) are valid where the numbers $p, q \in [1, \infty]$ are such that $W^{1,p}(K) \hookrightarrow L^q(K)$. Then for $u \in W^{2,p}(K)$ the estimate*

$$\|\partial_j (u - \mathrm{I}_h u); L^q(K)\| \lesssim (\mathrm{meas}_d K)^{1/q-1/p} \sum_{i=1}^{d} h_{i,K} \|\partial_i \partial_j u; L^p(K)\|, \quad j = 1, \ldots, d,$$

holds.

Proof From (3.5) we get the existence of a polynomial $w \in \mathcal{P}_1$ such that

$$\|\partial_j (u - w); L^p(K)\| \lesssim \sum_{i=1}^{d} h_{i,K} \|\partial_i \partial_j u; L^p(K)\|. \tag{3.7}$$

Using this polynomial, equation (3.2), the triangle inequality, (3.4) with $v = \partial_j (u - w)$, and Lemma 3.1, we obtain

$$\begin{aligned} &\|\partial_j (u - \mathrm{I}_h u); L^q(K)\| \\ &\quad \leq \|\partial_j (u - w); L^q(K)\| + \|\partial_j \mathrm{I}_h (u - w); L^q(K)\| \\ &\quad \lesssim (\mathrm{meas}_d K)^{1/q-1/p} \left(\|\partial_j (u - w); L^p(K)\| + \sum_{i=1}^{d} h_{i,K} \|\partial_i \partial_j (u - w); L^p(K)\| \right). \end{aligned}$$

With (3.7) and $\partial_i\partial_j w = 0$ we conclude the desired estimate. □

A similar result, but with derivatives in the direction of edges, was derived in [1] by writing (3.3) as $\partial_j \mathrm{I}_h u|_K = M_K \partial_j u$ and using estimates for $\|v - M_K v; L^q(K)\|$. The fact $\int_{\hat{K}} \partial_j(u - \mathrm{I}_h u) = 0$ was already observed in [4, Table 2, No. 5] for the two-dimensional case.

Since the solution of problems with edge singularities are well described in terms of weighted Sobolev spaces, see Lemma 2.1, we will derive also an estimate for such functions.

Lemma 3.4 *Let K be a tetrahedron with $r_K = 0$ and with element sizes $h_{S,K}$ and $h_{L,K}$ as described in Section 2. For $p, q \in [1, \infty]$, $\beta_j \in (-\infty, 1]$ and $\partial_j u \in V^{1,p}_{\beta_j}(K)$ the estimate*

$$\|\partial_j(u - \mathrm{I}_h u); L^q(K)\| \lesssim (\mathrm{meas}_3 K)^{1/q-1/p} h_{S,K}^{1-\beta_j} \|\partial_j u; V^{1,p}_{\beta_j}(K)\|, \quad j = 1, \ldots, 3,$$

holds.

Proof Corollary 3.2 implies

$$\begin{aligned}\|\partial_j(u - \mathrm{I}_h u); L^q(K)\| &\lesssim (\mathrm{meas}_3 K)^{1/q-1/p} \|\partial_j u; L^p(K)\| \\ &\lesssim (\mathrm{meas}_3 K)^{1/q-1/p} \|r^{1-\beta}; L^\infty(K)\| \, \|r^{\beta-1}\partial_j u; L^p(K)\|.\end{aligned}$$

By observing $\|r^{1-\beta}; L^\infty(K)\| \lesssim h_{S,K}^{1-\beta}$ and $\|r^{\beta-1}\partial_j u; L^p(K)\| \le \|\partial_j u; V^{1,p}_\beta(K)\|$ the desired estimate is obtained. □

We will now investigate rectangular (quadrilateral) elements K. It has been known for a long time that the space $\mathcal{Q}_1 = \mathrm{span}\,\{1, x_1, x_2, x_1 x_2\}$ is not unisolvent when the integral on sides is prescribed as in (3.1). Therefore so-called *rotated $\mathcal{Q}_1$ elements* have been investigated [27] where the polynomial space on the reference element $\hat{K}$ is $\mathrm{span}\,\{1, x_1, x_2, x_1^2 - x_2^2\}$. One property is that this space is preserved under a rotation of the coordinate system by 90 degrees. However, estimates as in Lemmata 3.1–3.4 are not valid, see Example 3.5. In [15, 27] also the so-called *non-parametric* version of the rotated $\mathcal{Q}_1$ element was investigated where the polynomial space is $\mathrm{span}\,\{1, x_1, x_2, x_1^2 - x_2^2\}$ on the element K. It was proved in [15] that $|\mathrm{I}_h u; W^{1,2}(K)| \lesssim |u; W^{1,2}(K)|$ holds for elements with arbitrary aspect ratio. However, the consistency error was not analyzed.

Example 3.5 Consider the element $K = (0, h_L) \times (0, h_S)$ and the reference element $\hat{K} = (0,1)^2$. For the function $u = x_L^2$ we obtain by direct calculation

$$\begin{aligned}\hat{u} &= h_L^2 \hat{x}_L^2, \\ \hat{\mathrm{I}}_h \hat{u} &= h_L^2 \left(\tfrac{1}{2}(\hat{x}_L^2 - \hat{x}_S^2) + \tfrac{1}{2}\hat{x}_L + \tfrac{1}{2}\hat{x}_S - \tfrac{1}{12}\right), \\ \mathrm{I}_h u &= \tfrac{1}{2}x_L^2 - \tfrac{1}{2}h_L^2 h_S^{-2} x_S^2 + \tfrac{1}{2}h_L x_L + \tfrac{1}{2}h_L^2 h_S^{-1} x_S - \tfrac{1}{12}h_L^2, \\ \partial_S(u - \mathrm{I}_h u) &= h_L^2 h_S^{-2} x_S - \tfrac{1}{2}h_L^2 h_S^{-1}, \\ \|\partial_S(u - \mathrm{I}_h u); L^2(K)\| &= h_L^2 h_S^{-2} \left(h_L \int_0^{h_S} \left(x_S - \tfrac{1}{2}h_S\right)^2 \,\mathrm{d}x_S\right)^{1/2} \sim h_L^2 h_S^{-1} (h_L h_S)^{1/2},\end{aligned}$$

$$
\begin{aligned}
|u;W^{1,2}(K)| &= \left(\int_Q (2x_L)^2\right)^{1/2} \sim h_L(h_Lh_S)^{1/2},\\
\sum_{i\in\{L,S\}} h_i|\partial_i u;W^{1,2}(K)| &= h_L\left(\int_Q 2^2\right)^{1/2} \sim h_L(h_Lh_S)^{1/2},
\end{aligned}
$$

and, consequently,

$$
\frac{\|\partial_S(u-\mathrm{I}_h u);L^2(K)\|}{|u;W^{1,2}(K)|} \sim \frac{\|\partial_S(u-\mathrm{I}_h u);L^2(K)\|}{\sum_{i\in\{L,S\}} h_i|\partial_i u;W^{1,2}(K)|} \sim \frac{h_L}{h_S}
$$

which can become arbitrary large.

We propose to use the space

$$
\mathcal{P} := \mathrm{span}\,\{1, x_L, x_S, x_L^2\} = \mathcal{P}_1 \oplus \mathrm{span}\,\{x_L^2\}
$$

which has the key property $\partial_S w = \text{const.}$ for $w\in\mathcal{P}$. Since the element K is anisotropic anyway, the space can be anisotropic as well. We could try to unify both types of trial functions by including a dependence on the aspect ratio, for example by using the function $\hat{x}_L^2 - h_L^{-2}h_S^2\hat{x}_S^2$ [15], but we try to keep the explanations as simple as possible. We prove now estimates similar to the ones above. The interpolant is again defined by (3.1).

Lemma 3.6 *A function $v\in\mathcal{P}$ is well defined when the values $\int_F v$ are prescribed on the four sides F of a rectangle K. The faces F are assumed to be parallel to the coordinate axes.*

Proof Since the space is invariant with respect to translation it is sufficient to consider the rectangle $K=(0,h_L)\times(0,h_S)$. Set $v = a_0 + a_Lx_L + a_Sx_S + a_{LL}x_L^2$, then the coefficients are the solution of the system

$$
\begin{pmatrix}
h_L & \frac12 h_L^2 & 0 & \frac13 h_L^3\\
h_S & h_Lh_S & \frac12 h_S^2 & h_L^2h_S\\
h_L & \frac12 h_L^2 & h_Lh_S & \frac13 h_L^3\\
h_S & 0 & \frac12 h_S^2 & 0
\end{pmatrix}
\begin{pmatrix} a_0\\ a_L\\ a_S\\ a_{LL}\end{pmatrix}
=
\begin{pmatrix}
\int_0^{h_L} v(x,0)\,\mathrm{d}x\\
\int_0^{h_S} v(h_L,x)\,\mathrm{d}x\\
\int_0^{h_L} v(x,h_S)\,\mathrm{d}x\\
\int_0^{h_S} v(0,x)\,\mathrm{d}x
\end{pmatrix}.
$$

The determinant of the matrix is $\frac16 h_L^5h_S^3 \neq 0$. □

Lemma 3.7 *Let K be a rectangular element with sides of length h_L and h_S being parallel to the coordinate axes x_L and x_S. For $p,q\in[1,\infty]$, $p\ge q$, and $u\in W^{1,p}(K)$ the estimates*

$$
\|\partial_S(u-\mathrm{I}_h u);L^q(K)\| \lesssim (\mathrm{meas}_2 K)^{1/q-1/p}\|\partial_S u;L^p(K)\|, \tag{3.8}
$$

$$
\|\partial_L(u-\mathrm{I}_h u);L^q(K)\| \lesssim (\mathrm{meas}_2 K)^{1/q-1/p}|u;W^{1,p}(K)| \tag{3.9}
$$

hold. If $p, q \in [1, \infty]$ are such that $W^{1,p}(K) \hookrightarrow L^q(K)$, and if $u \in W^{2,p}(K)$ then the estimates

$$\|\partial_S(u - \mathrm{I}_h u); L^q(K)\| \lesssim (\mathrm{meas}_2 K)^{1/q-1/p} \sum_{i\in\{L,S\}} h_{i,K} \|\partial_i \partial_S u; L^p(K)\|, \tag{3.10}$$

$$\|\partial_L(u - \mathrm{I}_h u); L^q(K)\| \lesssim (\mathrm{meas}_2 K)^{1/q-1/p} \sum_{i,j\in\{L,S\}} h_{i,K} \|\partial_i \partial_j u; L^p(K)\| \tag{3.11}$$

hold.

Proof As in the proof Lemma 3.1 we derive

$$\|\partial_S \mathrm{I}_h u; L^q(K)\| \le (\mathrm{meas}_2 K)^{1/q-1/p} \|\partial_S u; L^p(K)\|. \tag{3.12}$$

For $\partial_L \mathrm{I}_h u$ we get only a weaker (yet sufficient) estimate since this term is not constant. By using the definition of $\mathrm{I}_h u$ we get for any $p, q \in [1, \infty]$

$$\|\hat\partial_L \mathrm{I}_h \hat u; L^q(\hat K)\| \lesssim \|\hat u; W^{1,p}(\hat K)\|.$$

Consequently,

$$\|\partial_L \mathrm{I}_h u; L^q(K)\| \lesssim h_L^{-1} (\mathrm{meas}_2 K)^{1/q-1/p} \left(\|u; L^p(K)\| + \sum_{i\in\{L,S\}} h_{i,K} \|\partial_i u; L^p(K)\| \right). \tag{3.13}$$

Estimate (3.8) is obtained by the triangle inequality from (3.12). For (3.9) we choose $w \in \mathcal{P}_0$ such that (3.5) is satisfied with $v = u$ and conclude with (3.13) and by analogy to the proof of Lemma 3.3

$$\begin{aligned}
&\|\partial_L(u - \mathrm{I}_h u); L^q(K)\| \\
&\quad \le \|\partial_L(u - w); L^q(K)\| + \|\partial_L \mathrm{I}_h(u - w); L^q(K)\| \\
&\quad \lesssim h_L^{-1} (\mathrm{meas}_2 K)^{1/q-1/p} \left(\|u - w; L^p(K)\| + \sum_{i\in\{L,S\}} h_{i,K} \|\partial_i (u - w); L^p(K)\| \right) \\
&\quad \le h_L^{-1} (\mathrm{meas}_2 K)^{1/q-1/p} \sum_{i\in\{L,S\}} h_{i,K} \|\partial_i u; L^p(K)\|
\end{aligned}$$

which is even slightly sharper than (3.9).

The estimates (3.10) and (3.11) are proved as the the corresponding ones in Lemma 3.3. The additional terms appear in (3.11) due to the weaker estimate (3.13). □

In full analogy we treat prismatic elements $Q = T \times I$, where T is an isotropic triangle of diameter $h_{S,Q}$ and I is an interval of length $h_{L,Q}$. We use the polynomial space

$$\mathcal{P} := \mathcal{P}_1 \oplus \mathrm{span}\,\{x_L^2\}, \tag{3.14}$$

prove unisolvence and the following error estimates. For convenience of notation they are formulated slightly weaker (yet sufficient for the application later on) than the corresponding estimates in Lemma 3.7.

Lemma 3.8 *Let Q be a prismatic element as described above. For $p, q \in [1, \infty]$, $p \geq q$, and $u \in W^{1,p}(Q)$ the estimate*

$$|u - \mathrm{I}_h u; W^{1,q}(Q)| \lesssim (\mathrm{meas}_3 Q)^{1/q-1/p} |u; W^{1,p}(Q)| \tag{3.15}$$

holds. If $p, q \in [1, \infty]$ are such that $W^{1,p}(Q) \hookrightarrow L^q(Q)$, and if $u \in W^{2,p}(Q)$ then the estimate

$$|u - \mathrm{I}_h u; W^{1,q}(Q)| \lesssim (\mathrm{meas}_3 Q)^{1/q-1/p} \sum_{i \in \{S_1, S_2, L\}} h_{i,Q} |\partial_i u; W^{1,p}(Q)| \tag{3.16}$$

holds. If $r_Q = 0$, $p, q \in [1, \infty]$, $\beta_j \in (-\infty, 1]$ and $\partial_j u \in V^{1,p}_{\beta_j}(Q)$, $j \in \{S_1, S_2, L\}$, then the estimate

$$|u - \mathrm{I}_h u; W^{1,q}(Q)| \lesssim (\mathrm{meas}_3 Q)^{1/q-1/p} \sum_{j \in \{S_1, S_2, L\}} h^{1-\beta_j}_{S,Q} \|\partial_j u; V^{1,p}_{\beta_j}(Q)\| \tag{3.17}$$

holds.

Proof The first two estimates are proved as Lemma 3.7. Estimate (3.15) can be written as

$$|u - \mathrm{I}_h u; W^{1,q}(Q)| \lesssim (\mathrm{meas}_3 Q)^{1/q-1/p} \sum_{j \in \{S_1, S_2, L\}} \|\partial_j u; L^p(Q)\|,$$

and we obtain (3.17) in analogy to the proof of Lemma 3.4. □

4 Consistency error estimates

4.1 General considerations in the two-dimensional case

The aim of this subsection is to explain the main difficulties and the ideas for the estimation of the consistency error. Therefore we concentrate on the two-dimensional case and, for later use in other applications [9, 10], on the general differential equation

$$-\nabla \cdot \eta = f \qquad \text{in } \Omega, \tag{4.1}$$

with $f \in L^2(\Omega)$. For simplicity, let Ω be a union of rectangles with sides parallel to the axes of a Cartesian coordinate system (x_L, x_S).

Let us consider a family $\{\mathcal{T}_h\}_{h \to 0}$ of triangulations $\mathcal{T}_h = \{K\}$ of rectangular elements K of size $h_{L,K} \times h_{S,K}$, see Figure 4.1, left hand side, for an illustration. By dividing each rectangle we obtain a triangular mesh, see Figure 4.1, right hand side. Since we need for the considerations in this subsection only one element type at one time we denote both

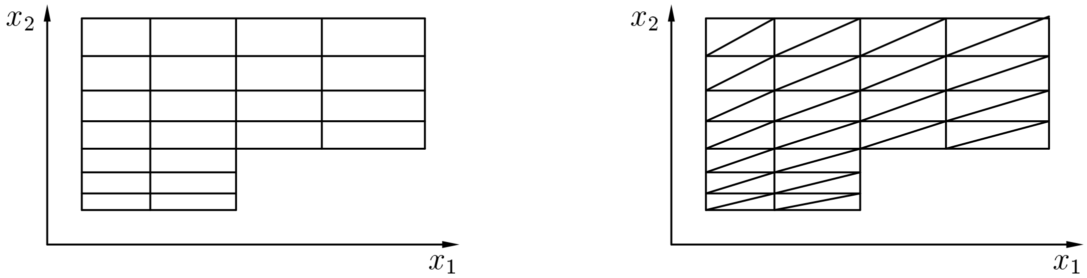

Figure 4.1: Meshes in two dimensions. Left: rectangular elements. Right: triangular elements.

types of element by K. Faces (sides) of the elements are denoted by F. According to Section 3 the corresponding finite element spaces are

$$V_h := \{v_h \in L^2(\Omega) :\ v_h|_K \in \mathcal{P}\ \forall K, \int_F [v_h] = 0\ \forall F\}, \tag{4.2}$$

$$\mathcal{P} := \begin{cases} \mathcal{P}_1 & \text{for triangular elements,} \\ \mathcal{P}_1 \oplus \operatorname{span}\{x_L^2\} & \text{for rectangular elements.} \end{cases} \tag{4.3}$$

In the sense of (2.10) it is our aim to derive an estimate for

$$\sup_{v_h \in V_h} \frac{(\eta, \nabla v_h)_h - (f, v_h)}{\|v_h\|_{1,h}}.$$

Let us start in the usual way in order to see where difficulties arise. Denoting by $n = (n_L, n_S)$ the outward unit normal to ∂K we obtain by Green's formula and (4.1)

$$\begin{aligned} (\eta, \nabla v_h)_h - (f, v_h) &= \sum_K \int_K (\eta \cdot \nabla v_h - f v_h) \\ &= \sum_K \left[\int_{\partial K} (\eta \cdot n)\, v_h - \int_K (\nabla \cdot \eta + f) v_h \right] \\ &= \sum_K \sum_{F \subset \partial K} \int_F (\eta \cdot n)\, v_h. \end{aligned} \tag{4.4}$$

Let $M_F : L^1(F) \to \mathcal{P}_0$ be the averaging operator on the face F which preserves polynomials of degree zero, as defined in (3.6). Since

$$\sum_K \sum_{\substack{F \subset \partial K \\ F \not\subset \partial\Omega}} \int_F \eta \cdot n = 0$$

and

$$M_F v_h = (\operatorname{meas}_1 F)^{-1} \int_F v_h = 0 \quad \text{for all } F \subset \partial\Omega \tag{4.5}$$

we can reformulate (4.4) by

$$(\eta, \nabla v_h)_h - (f, v_h) = \sum_K \sum_{F \subset \partial K} \int_F (\eta \cdot n)(v_h - M_F v_h). \tag{4.6}$$

Furthermore, since $\int_F (v_h - M_F v_h) = 0$ for all F we continue with

$$(\eta, \nabla v_h)_h - (f, v_h) = \sum_K \sum_{F \subset \partial K} \int_F (\eta - M_F \eta) \cdot n \, (v_h - M_F v_h). \tag{4.7}$$

For the estimation of such terms the following lemma is useful.

Lemma 4.1 *Let F be a face of an element K. Then the estimate*

$$\left| \int_F (v - M_F v)(v_h - M_F v_h) \right| \lesssim \frac{\operatorname{meas}_1 F}{\operatorname{meas}_2 K} \left(\sum_{i \in \{L,S\}} h_{i,K}^2 \| \partial_i v; L^2(K) \|^2 \right)^{1/2} \left(\sum_{i \in \{L,S\}} h_{i,K}^2 \| \partial_i v_h; L^2(K) \|^2 \right)^{1/2}$$

holds for any $v \in W^{1,2}(K)$, $v_h \in \mathcal{P}$.

Proof We obtain by transformation to the reference face $\hat{F} \subset \overline{\hat{K}}$, the trace theorem, and the Bramble-Hilbert lemma

$$\| v - M_F v; L^2(F) \| = (\operatorname{meas}_1 F)^{1/2} \| \hat{v} - M_{\hat{F}} \hat{v}; L^2(\hat{F}) \| \lesssim (\operatorname{meas}_1 F)^{1/2} |\hat{v}; W^{1,2}(\hat{K})|.$$

The transformation from $\hat{K}$ to K leads to

$$\| v - M_F v; L^2(F) \| \lesssim (\operatorname{meas}_1 F)^{1/2} (\operatorname{meas}_2 K)^{-1/2} \left(\sum_{i \in \{L,S\}} h_{i,K}^2 \| \partial_i v; L^2(K) \|^2 \right)^{1/2}.$$

The application of the Cauchy-Schwarz inequality and twice the previous estimate yields the desired result. □

Consider now a small face $F_S \subset \partial K$. Then we obtain by applying Lemma 4.1 the estimate

$$\begin{aligned} \left| \int_{F_S} (\eta - M_{F_S} \eta) \cdot n \, (v_h - M_{F_S} v_h) \right| &= \left| \int_{F_S} (\eta_L - M_{F_S} \eta_L)(v_h - M_{F_S} v_h) \right| \\ &\lesssim h_{L,K}^{-1} \left(\sum_{i \in \{L,S\}} h_{i,K}^2 \| \partial_i \eta_L; L^2(K) \|^2 \right)^{1/2} \left(\sum_{i \in \{L,S\}} h_{i,K}^2 \| \partial_i v_h; L^2(K) \|^2 \right)^{1/2} &(4.8) \\ &\le \left(\sum_{i \in \{L,S\}} h_{i,K}^2 \| \partial_i \eta_L; L^2(K) \|^2 \right)^{1/2} |v_h; W^{1,2}(K)|. &(4.9) \end{aligned}$$

This may be a sufficiently good estimate for small faces, however, for large faces we would get a term of order $h_{S,K}^{-1} h_{L,K}^2$.

The idea is to introduce an auxiliary finite element space

$$\tilde{V}_h := \{\tilde{v}_h \in L^2(\Omega) : \ \tilde{v}_h|_K \in \operatorname{span}\{1, x_S\} \ \forall K, \int_{F_L} [\tilde{v}_h] = 0 \ \forall F_L\} \tag{4.10}$$

which is sufficiently close to V_h but the above mentioned term will not appear.

For an arbitrary but fixed $v_h \in V_h$ we define $\tilde{v}_h \in \tilde{V}_h$ such that

$$\int_{F_L} v_h = \int_{F_L} \tilde{v}_h \quad \forall F_L. \tag{4.11}$$

Since triangles and rectangles have exactly two large faces F_L this definition is meaningful for both types of element.

Both $\partial_S v_h$ and $\partial_S \tilde{v}_h$ are constant. Even better, by Green's formula and (4.11) we get

$$\partial_S v_h = \partial_S \tilde{v}_h \tag{4.12}$$

since

$$\partial_S (v_h - \tilde{v}_h) = (\operatorname{meas}_2 K)^{-1} \int_K \partial_S (v_h - \tilde{v}_h) = (\operatorname{meas}_2 K)^{-1} \sum_{F_L \subset \partial K} \int_{F_L} (v_h - \tilde{v}_h)\, n_S = 0$$

holds. We are now prepared to prove an estimate for the consistency error.

Lemma 4.2 *For rectangular and triangular meshes the estimate*

$$\sup_{v_h \in V_h} \frac{(\eta, \nabla v_h)_h - (f, v_h)}{\|v_h\|_{1,h}}$$
$$\lesssim \left(\sum_K \sum_{i,j \in \{L,S\}} h_{i,K}^2 \|\partial_i \eta_j; L^2(K)\|^2 \right)^{1/2} + \left(\sum_K h_{L,K}^2 \|f + \partial_L \eta_L; L^2(K)\|^2 \right)^{1/2}$$

holds provided that

$$\eta \in [W^{1,2}(\Omega)]^2 \tag{4.13}$$

and η, f satisfy (4.1).

Proof We introduce $\tilde{v}_h$ as above and modify (4.4) by using (4.12) and (4.1) as follows,

$$\begin{aligned} & (\eta, \nabla v_h)_h - (f, v_h) \\ = \ & \sum_K \int_K (\eta_L \partial_L v_h + \eta_S \partial_S \tilde{v}_h - f v_h) \\ = \ & -\sum_K \int_K (\partial_L \eta_L\, v_h + \partial_S \eta_S\, \tilde{v}_h + f v_h) + \sum_K \int_{\partial\Omega} \eta_L n_L\, v_h + \sum_K \int_{\partial\Omega} \eta_S n_S\, \tilde{v}_h \\ = \ & -\sum_K \int_K (f + \partial_L \eta_L)(v_h - \tilde{v}_h) + \sum_K \sum_{F \subset \partial K} \int_F \eta_L v_h\, n_L + \sum_K \sum_{F \subset \partial K} \int_F \eta_S \tilde{v}_h\, n_S \end{aligned} \tag{4.14}$$

The reason of writing $f + \partial_L \eta_L$ instead of $-\partial_S \eta_S$ will become clear in the proof of Lemma 4.6. We will now treat the three terms separately.

Due to (4.11) we can apply the Poincaré inequality. On the reference element we get

$$\|\hat{v}_h - \tilde{\hat{v}}_h; L^2(\hat{K})\| \lesssim |\hat{v}_h - \tilde{\hat{v}}_h; W^{1,2}(\hat{K})|.$$

After transformation to K and using (4.10), (4.12) and twice the Cauchy-Schwarz inequality we obtain

$$\|v_h - \tilde{v}_h; L^2(K)\| \lesssim h_{L,K} \|\partial_L v_h; L^2(K)\| \tag{4.15}$$

$$\left| \int_K (f + \partial_L \eta_L)(v_h - \tilde{v}_h) \right| \lesssim h_{L,K} \|f + \partial_L \eta_L; L^2(K)\| \, \|\partial_L v_h; L^2(K)\|$$

$$\sum_K \left| \int_K (f + \partial_L \eta_L)(v_h - \tilde{v}_h) \right| \lesssim \left(\sum_K h_{L,K}^2 \|f + \partial_L \eta_L; L^2(K)\|^2 \right)^{1/2} \|v_h\|_{1,h}. \tag{4.16}$$

The second term of (4.14) can be estimated in the way described above, see (4.4)–(4.9) and Lemma 4.1. Indeed, we get

$$\sum_K \sum_{F \subset \partial K} \int_F \eta_L v_h \, n_L = \sum_K \sum_{F \subset \partial K} n_L \int_F (\eta_L - M_F \eta_L)(v_h - M_F v_h)$$

$$\lesssim \sum_K \sum_{F \subset \partial K} \frac{\text{meas}_1 F}{\text{meas}_2 K} n_L \left(\sum_{i \in \{L,S\}} h_{i,K}^2 \|\partial_i \eta_L; L^2(K)\|^2 \right)^{1/2} \left(\sum_{i \in \{L,S\}} h_{i,K}^2 \|\partial_i v_h; L^2(K)\|^2 \right)^{1/2}.$$

The point is that the factor $\text{meas}_1 F (\text{meas}_2 K)^{-1} n_L$ is for all faces of order h_L^{-1} or even zero, so that we get

$$\sum_K \sum_{F \subset \partial K} \int_F \eta_L v_h \, n_L \lesssim \left(\sum_K \sum_{i \in \{L,S\}} h_{i,K}^2 \|\partial_i \eta_L; L^2(K)\|^2 \right)^{1/2} \|v_h\|_{1,h} \tag{4.17}$$

by using the discrete version of the Cauchy-Schwarz inequality.

The third term can also be estimated in the same way. We mention only two new points. The first is that $M_F \tilde{v}_h = 0$ is in general only satisfied for large faces $F_L \subset \partial\Omega$, compare (4.5). For small faces $F_S \subset \partial\Omega$ we have to use that $n_S = 0$. Second, since $\partial_L \tilde{v}_h = 0$ the term $h_{L,K}^2 \|\partial_L \tilde{v}_h; L^2(K)\|^2$ vanishes such that we can extract a factor $h_{S,K} \|v_h\|_{1,h}$ which is used to compensate the factor $\text{meas}_1 F (\text{meas}_2 K)^{-1}$ for all types of face. Hence the estimate reads

$$\sum_K \sum_{F \subset \partial K} \int_F \eta_S \tilde{v}_h \, n_S = \sum_K \sum_{F \subset \partial K} \int_F (\eta_S - M_F \eta_S)(\tilde{v}_h - M_F \tilde{v}_h) \, n_S$$

$$\lesssim \sum_K \sum_{F \subset \partial K} h_{S,K}^{-1} \left(\sum_{i \in \{L,S\}} h_{i,K}^2 \|\partial_i \eta_S; L^2(K)\|^2 \right)^{1/2} h_{S,K} \|\partial_S \tilde{v}_h; L^2(K)\|$$

$$\lesssim \left(\sum_K \sum_{i\in\{L,S\}} h_{i,K}^2 \|\partial_i \eta_S; L^2(K)\|^2 \right)^{1/2} \|v_h\|_{1,h} \tag{4.18}$$

where we have also used (4.12).

Combining (4.14) and (4.16)–(4.18) we conclude the desired estimate. □

4.2 The three-dimensional case under specific assumptions

In this subsection we want to extend the considerations of the previous one into three space dimensions. The following two points are taken into account.

First, while the extension to prismatic elements is straightforward this is not the case for tetrahedral elements. The main reason is that rectangular, triangular and prismatic elements have exactly d (d is the space dimension) large faces which are used to define $\tilde{v}$ in (4.11). One out of three tetrahedral elements has, however, four large sides. Therefore the approach has to be modified slightly.

Second, we assume in Lemma 4.2 that $\eta \in [W^{1,2}(\Omega)]^2$. In view of Lemma 2.1 we will now weaken this assumption to

$$\eta_{S1}, \eta_{S2} \in V_\beta^{1,2}(\Omega), \quad \beta \in [0,1), \tag{4.19}$$

$$\eta_L \in V_0^{1,2}(\Omega) \hookrightarrow W^{1,2}(\Omega). \tag{4.20}$$

Note that due to (4.1), (4.19), and (4.20) in general

$$\partial_{S1}\eta_{S1}, \partial_{S2}\eta_{S2} \notin L^2(\Omega), \quad \text{but} \quad \partial_{S1}\eta_{S1} + \partial_{S2}\eta_{S2} \in L^2(\Omega). \tag{4.21}$$

In the sense of Section 2, but slightly more general, consider a family of pentahedral triangulations $\mathcal{Q}_h = \{Q\}$. The triangular faces $F_{S,Q}$ of each element Q are parallel to the x_{S1}, x_{S2}-plane. They are isotropic with diameter $h_{S,Q}$. When necessary we will also use the notation $h_{S1,Q}$ and $h_{S2,Q}$ which are both identical with with $h_{S,Q}$. The rectangular faces $F_{L,Q}$ are parallel to the x_L-axis and have a size of order $h_{L,Q} \times h_{S,Q}$.

Each element $Q \in \mathcal{Q}_h$ can be divided into three tetrahedra K such that an admissible tetrahedral triangulation $\mathcal{T}_h = \{K\}$ is obtained. We denote the faces of the tetrahedra by F_K and introduce the element sizes $h_{L,K}$, $h_{S,K}$, $h_{S1,K}$, and $h_{S2,K}$ by analogy to above.

Let us first prove a lemma which is analogous to Lemma 4.1.

Lemma 4.3 *Let F be a face of a tetrahedral element K. Then the estimate*

$$\left| \int_F (v - M_F v)(v_h - M_F v_h) \right| \lesssim \frac{\text{meas}_2 F}{\text{meas}_3 K} \times$$
$$\left(\sum_{i\in\{L,S1,S2\}} h_{S,K}^{-2\beta_{i,K}} h_{i,K}^2 \|r^{\beta_{i,K}} \partial_i v; L^2(K)\|^2 \right)^{1/2} \left(\sum_{i\in\{L,S1,S2\}} h_{i,K}^2 \|\partial_i v_h; L^2(K)\|^2 \right)^{1/2}$$

holds for any $v \in V^{1,2}_{\beta_{i,K}}(K)$*,* $\beta_{i,K} \in [0,1)$*,* $v_h \in \mathcal{P} = \mathcal{P}_1$*. By* $r = (x_{S1}^2 + x_{S2}^2)^{1/2}$ *we denote the distance to the* x_L*-axis.*

With adapted notation the statement holds for pentahedral elements Q *with* $\mathcal{P} := \mathcal{P}_1 \oplus \operatorname{span}\{x_L^2\}$ *as well.*

Proof We modify the proof of Lemma 4.1 slightly. Instead of the Cauchy-Schwarz inequality we apply the Hölder inequality to obtain

$$\left|\int_F (v - M_F v)(v_h - M_F v_h)\right| \le \|v - M_F v; L^1(F)\| \, \|v_h - M_F v_h; L^\infty(F)\|. \tag{4.22}$$

For the first factor we get in analogy to the proof of Lemma 4.1

$$\|v - M_F v; L^1(F)\| \lesssim (\mathrm{meas}_3 K)^{-1}(\mathrm{meas}_2 F) \sum_{i\in\{L,S1,S2\}} h_{i,K}\|\partial_i v; L^1(K)\|.$$

The L^1-norm can be estimated by a weighted L^2-norm by using the Cauchy-Schwarz inequality and a direct calculation,

$$\|w; L^1(K)\| \le \|r^{-\beta_{i,K}}; L^2(K)\| \, \|r^{\beta_{i,K}} w; L^2(K)\| \le h_{S,K}^{-\beta_{i,K}} (\mathrm{meas}_3 K)^{1/2} \|r^{\beta_{i,K}} w; L^2(K)\|.$$

Note that $\|r^{-\beta_{i,K}}; L^2(K)\|$ is not finite for $\beta_{i,K} \ge 1$ and zero distance of K to the x_L-axis. Note further that the estimate is very coarse when K has non-zero distance to the x_L-axis and $\beta_{i,K} > 0$. But this is not the interesting case.

The second factor of (4.22) is estimated by using that norms in finite spaces are equivalent,

$$\begin{aligned}
\|v_h - M_F v_h; L^\infty(F)\| &= \|\hat v_h - M_{\hat F}\hat v_h; L^\infty(\hat F)\| \\
&\le \|\hat v_h - M_{\hat F}\hat v_h; L^\infty(\hat K)\| \\
&\lesssim \|\hat v_h - M_{\hat F}\hat v_h; W^{1,2}(\hat K)\|.
\end{aligned}$$

Since $\int_{\hat F} \hat v_h - M_{\hat F}\hat v_h = 0$ we can use the Poincaré inequality to get rid of the L^2-part of the norm on the right hand side. Using further that $\hat\partial_i M_{\hat F}\hat v_h = 0$ and transforming from $\hat K$ to K we get

$$\|v_h - M_F v_h; L^\infty(F)\| \lesssim |\hat v_h; W^{1,2}(\hat K)| \lesssim (\mathrm{meas}_3 K)^{-1/2} \left(\sum_{i\in\{L,S1,S2\}} h_{i,K}^2 \|\partial_i v_h; L^2(K)\|^2\right)^{1/2}.$$

Combining all these estimates leads to the desired result. □

The finite element space V_h is defined in (2.7) and (2.8) for tetrahedral and pentahedral meshes. Similarly to (4.10) we introduce an auxiliary finite element space

$$\tilde V_h := \{\tilde v_h \in L^2(\Omega) : \ \tilde v_h|_Q \in \operatorname{span}\{1, x_{S1}, x_{S2}\} \ \forall Q, \int_{F_{L,Q}} [\tilde v_h] = 0 \ \forall F_{L,Q}\}. \tag{4.23}$$

We point out that we have different spaces V_h for $\mathcal{T}_h$ and $\mathcal{Q}_h$ but in both cases the same space $\tilde{V}_h$. In analogy to (4.11) we define for an arbitrary but fixed $v_h \in V_h$ a function $\tilde{v}_h \in \tilde{V}_h$ such that

$$\int_{F_{L,Q}} v_h = \int_{F_{L,Q}} \tilde{v}_h \quad \forall F_{L,Q}. \tag{4.24}$$

An equality like (4.12) can only be shown for pentahedral meshes. It does not hold in the tetrahedral case since the derivative $\partial_i v_h$, $i \in \{S1, S2\}$, is only piecewise constant in Q. However, it turns out to be sufficient to have the following lemma.

Lemma 4.4 *For any pentahedron $Q \in \mathcal{Q}_h$ which can be but needs not to be divided into three tetrahedra K, the equation*

$$\int_Q \partial_i (v_h - \tilde{v}_h) = 0, \quad i \in \{S1, S2\}, \tag{4.25}$$

is valid.

Proof If v_h is defined with respect to $\mathcal{Q}_h$ then we simply have by Green's formula and (4.24)

$$\int_Q \partial_i (v_h - \tilde{v}_h) = \sum_{F_{L,Q} \subset \partial Q} n_i \int_{F_{L,Q}} (v_h - \tilde{v}_h) = 0, \quad i \in \{S1, S2\},$$

where n_i is the component of the outward unit normal n in direction of the x_i-axis.

In the tetrahedral case we have intermediately more terms,

$$\begin{aligned} \int_Q \partial_i (v_h - \tilde{v}_h) &= \sum_{K \subset Q} \int_K \partial_i (v_h - \tilde{v}_h) = \sum_{K \subset Q} \int_{\partial K} (v_h - \tilde{v}_h)\, n_i \\ &= \sum_{F \subset \mathrm{int} Q} n_i \int_F ([v_h] - [\tilde{v}_h]) + \sum_{F_{L,Q} \subset \partial Q} n_i \int_{F_{L,Q}} (v_h - \tilde{v}_h), \end{aligned}$$

but also these terms vanish due to the definition of V_h and $\tilde{V}_h$. □

Since equality (4.12) was used to prove (4.15) we have to modify this estimate in the tetrahedral case.

Lemma 4.5 *For any pentahedron $Q \in \mathcal{Q}_h$ which is divided into three tetrahedra K, the estimates*

$$\|v_h - \tilde{v}_h; L^q(Q)\| \lesssim (\mathrm{meas}_3 Q)^{1/q - 1/p} \sum_{K \subset Q} \sum_{i \in \{L, S1, S2\}} h_{i,Q} \|\partial_i v_h; L^p(K)\|, \tag{4.26}$$

$$\sum_{K \subset Q} \|\partial_i (v_h - \tilde{v}_h); L^q(K)\| \lesssim (\mathrm{meas}_3 Q)^{1/q - 1/p} \sum_{K \subset Q} |v_h; W^{1,p}(K)|, \quad i \in \{L, S1, S2\}, \tag{4.27}$$

are valid for any $p, q \in [1, \infty]$.

Proof Consider the reference element $\hat{Q} := \{(\hat{x}_{S1}, \hat{x}_{S2}, \hat{x}_L) \in I\!R^3 : 0 < \hat{x}_{S1} < 1, 0 < \hat{x}_{S2} < 1 - \hat{x}_{S1}, 0 < \hat{x}_L < 1\}$ with three rectangular faces $F_{L,\hat{Q}} \subset \partial\hat{Q}$. We have for any $\hat{w}_h \in V_h|_{\hat{Q}}$

$$\tilde{\hat{w}}_h = \sum_{F_{L,\hat{Q}}} (\text{meas}_2 F_{L,\hat{Q}})^{-1} \left(\int_{F_{L,\hat{Q}}} \hat{w}_h \right) \hat{\varphi}_{F_{L,\hat{Q}}} \tag{4.28}$$

where $\hat{\varphi}_{F_{L,\hat{Q}}} \in \text{span}\,\{1, \hat{x}_{S1}, \hat{x}_{S2}\}$ is the polynomial which is equal to one on $F_{L,\hat{Q}}$ and vanishes at the midpoints of the other two rectangular faces. Hence

$$\|\tilde{\hat{w}}_h; L^q(\hat{Q})\| \lesssim \|\hat{w}_h; L^1(\hat{Q})\|. \tag{4.29}$$

We prove now that $|||\cdot|||$,

$$|||\, \hat{w}_h \,||| := \sum_{\hat{K} \subset \hat{Q}} |\hat{w}_h; W^{1,p}(\hat{K})| + \left| \int_{\hat{Q}} \hat{w}_h \right|, \tag{4.30}$$

is a norm in $V_h|_{\hat{Q}}$. It is simple to see that $|||\cdot|||$ is a seminorm. Assume now that $|||\, \hat{v}_h \,||| = 0$ for some $\hat{v}_h \in V_h|_{\hat{Q}}$. Consequently $|\hat{v}_h; W^{1,p}(\hat{K})| = 0$ for any $\hat{K} \subset \hat{Q}$, this means that $\hat{v}_h$ is piecewise constant. Since by definition (2.7) $\int_{\hat{F}} [\hat{v}_h] = 0$ on the interior faces, $\hat{v}_h$ is even constant in $\hat{Q}$. Since $\int_{\hat{Q}} \hat{v}_h = 0$ we obtain $\hat{v}_h \equiv 0$. Hence $|||\cdot|||$ is a norm.

Since all norms in finite spaces ($V_h|_{\hat{Q}}$ is ten-dimensional) are equivalent, we conclude from (4.29), (4.30)

$$\begin{aligned} \|\hat{w}_h - \tilde{\hat{w}}_h; L^q(\hat{Q})\| &\lesssim \|\hat{w}_h; L^q(\hat{Q})\| + \|\hat{w}_h; L^1(\hat{Q})\| \\ &\lesssim \sum_{\hat{K} \subset \hat{Q}} |\hat{w}_h; W^{1,p}(\hat{K})| + \left| \int_{\hat{Q}} \hat{w}_h \right|. \end{aligned}$$

Set $\hat{w}_h = v_h - M_{\hat{Q}} \hat{v}_h$ and note that $\hat{w}_h - \tilde{\hat{w}}_h = \hat{v}_h - \tilde{\hat{v}}_h$ by (4.28). Hence

$$\|\hat{v}_h - \tilde{\hat{v}}_h; L^q(\hat{Q})\| \lesssim \sum_{\hat{K} \subset \hat{Q}} |\hat{v}_h; W^{1,p}(\hat{K})|.$$

The affine transformation from $\hat{Q}$ to Q leads to the estimate (4.26).

Estimate (4.27) is trivial for $i = L$ since $\partial_L \tilde{v}_h = 0$. For $i \in \{S1, S2\}$ we use the equivalence of norms and Lemma 4.4 on the reference element,

$$\|\partial_i \tilde{\hat{v}}_h; L^q(\hat{Q})\| \sim \left| \int_{\hat{Q}} \partial_i \tilde{\hat{v}}_h \right| = \left| \sum_{\hat{K} \subset \hat{Q}} \int_{\hat{K}} \partial_i \hat{v}_h \right| \lesssim \sum_{\hat{K} \subset \hat{Q}} \|\partial_i \hat{v}_h; L^p(\hat{K})\|.$$

Consequently

$$\sum_{\hat{K} \subset \hat{Q}} \|\partial_i (\hat{v}_h - \tilde{\hat{v}}_h); L^q(\hat{K})\| \lesssim \sum_{\hat{K} \subset \hat{Q}} \|\partial_i \hat{v}_h; L^p(\hat{K})\|.$$

By transformation from $\hat{Q}$ to Q we conclude estimate (4.27). □

We are now prepared to prove the consistency error estimate.

Lemma 4.6 *For pentahedral and tetrahedral meshes the estimate*

$$\sup_{v_h\in V_h}\frac{|(\eta,\nabla v_h)_h-(f,v_h)|}{\|v_h\|_{1,h}}\lesssim$$
$$\left(\sum_Q\sum_{i,j\in\{L,S1,S2\}}h_{S,K}^{-2\beta_{i,j,K}}h_{i,Q}^2\|r^{\beta_{i,j,K}}\partial_i\eta_j;L^2(Q)\|^2\right)^{1/2}+\left(\sum_Q h_{L,Q}^2\|f+\partial_L\eta_L;L^2(Q)\|^2\right)^{1/2}$$

holds provided that η and f satisfy (4.1), (4.19) and (4.20), and $\beta_{i,j,K}\in[0,1)$ for all K and for all $j\in\{L,S1,S2\}$.

We prove the lemma for the case of tetrahedral meshes. In the other case the proof is analogous; some simplifications could be made.

Proof We introduce $\tilde v_h\in\tilde V_h$ by (4.23), (4.24) and modify the proof of Lemma 4.2 by using (4.25) instead of (4.12). Let us first write

$$\begin{aligned}(\eta,\nabla v_h)_h-(f,v_h) &= \sum_K\int_K[\eta_L\partial_L v_h+\eta_{S1}\partial_{S1}\tilde v_h+\eta_{S2}\partial_{S2}\tilde v_h-fv_h]+\\ &\quad\sum_K\int_K[\eta_{S1}\partial_{S1}(v_h-\tilde v_h)+\eta_{S2}\partial_{S2}(v_h-\tilde v_h)].\end{aligned}\tag{4.31}$$

The first term is known from the proof of Lemma 4.2 and will be estimated similarly, only taking into account the weaker assumption (4.19) instead of (4.13). By using Green's formula and being careful about (4.21) we have

$$\begin{aligned}&\sum_K\int_K[\eta_L\partial_L v_h+\eta_{S1}\partial_{S1}\tilde v_h+\eta_{S2}\partial_{S2}\tilde v_h-fv_h]\\ &=\sum_K\int_K[\eta\cdot\nabla\tilde v_h+\eta_L\partial_L(v_h-\tilde v_h)-fv_h]\\ &=\sum_K\int_K[-(\nabla\cdot\eta)\tilde v_h-\partial_L\eta_L(v_h-\tilde v_h)-fv_h]+\sum_K\int_{\partial K}((\eta\cdot n)\tilde v_h+\eta_L(v_h-\tilde v_h)n_L)\\ &=-\sum_Q\int_Q(f+\partial_L\eta_L)(v_h-\tilde v_h)+\sum_K\int_{\partial K}(\eta_L v_h n_L+\eta_{S1}\tilde v_h n_{S1}+\eta_{S2}\tilde v_h n_{S2})\end{aligned}\tag{4.32}$$

The right hand side of (4.32) is analogous to that of (4.14). So we can proceed as in the proof of Lemma 4.2. We have only to use Lemma 4.5 instead of estimate (4.15), Lemma 4.3 instead of Lemma 4.1, and the equality

$$\sum_K\int_{\partial K}(\eta_{S1}n_{S1}+\eta_{S2}n_{S2})\tilde v_h=\sum_Q\int_{\partial Q}(\eta_{S1}n_{S1}+\eta_{S2}n_{S2})\tilde v_h.$$

It remains to estimate the second term of (4.31). Using Lemma 4.4 and the operator $M_Q : L^1(Q) \to \mathcal{P}_0$, $M_Q w := (\text{meas}_3 Q)^{-1} \int_Q w$, we get

$$\begin{aligned}
&\left| \sum_{j\in\{S1,S2\}} \sum_Q \sum_{K\subset Q} \int_K \eta_j \partial_j (v_h - \tilde{v}_h) \right| \\
&= \left| \sum_{j\in\{S1,S2\}} \sum_Q \sum_{K\subset Q} \int_K (\eta_j - M_Q\eta_j) \partial_j (v_h - \tilde{v}_h) \right| \\
&\le \sum_{j\in\{S1,S2\}} \sum_Q \|\eta_j - M_Q\eta_j; L^1(Q)\| \left(\sum_{K\subset Q} \|\partial_j (v_h - \tilde{v}_h); L^\infty(K)\| \right). \qquad (4.33)
\end{aligned}$$

As in the proof of Lemma 4.3 we use the Poincaré inequality and the Hölder inequality to get

$$\begin{aligned}
\|\eta_j - M_Q\eta_j; L^1(Q)\| &\lesssim \sum_{i\in\{L,S1,S2\}} h_{i,Q} \|\partial_i \eta_j; L^1(Q)\| \\
&\lesssim (\text{meas}_3 Q)^{1/2} \sum_{i\in\{L,S1,S2\}} h_{S,Q}^{-\beta_{i,j,K}} h_{i,Q} \|r^{\beta_{i,j,K}} \partial_i \eta_j; L^2(Q)\|. \qquad (4.34)
\end{aligned}$$

Combining (4.33), (4.34) and using Lemma 4.5 we conclude

$$\begin{aligned}
&\left| \sum_{j\in\{S1,S2\}} \sum_Q \sum_{K\subset Q} \int_K \eta_j \partial_j (v_h - \tilde{v}_h) \right| \\
&\lesssim \sum_{j\in\{S1,S2\}} \sum_Q \left(\sum_{i\in\{L,S1,S2\}} h_{S,Q}^{-\beta_{i,j,K}} h_{i,Q} \|r^{\beta_{i,j,K}} \partial_i \eta_j; L^2(Q)\| \right) \left(\sum_{K\subset Q} |v_h; W^{1,2}(K)| \right)
\end{aligned}$$

By using the discrete Cauchy-Schwarz inequality we finish the proof. □

5 Error estimates for the model problem

In view of the second Lemma of Strang, estimate (2.10), we have to bound the global interpolation error and the consistency error for the family of meshes defined by (2.6). The properties of u were stated in Lemma 2.1.

Theorem 5.1 *Let u be a function satisfying (2.4), (2.5). Then the estimate*

$$\|u - \mathrm{I}_h u\|_{1,h} \lesssim h \, \|f; L^2(\Omega)\|$$

holds if $\mu < \pi/\omega$.

Proof We prove the lemma for the case of tetrahedral meshes, pentahedral meshes can be treated in the same way. The estimation of the global error is reduced to the evaluation of the local errors where we distinguish between the elements far from the singular edge, $r_K > 0$, and the elements touching the edge, $r_K = 0$.

For all elements K with $r_K > 0$ we can apply Lemma 3.3 with $p = q = 2$, and use that $r^{-\beta} < r_K^{-\beta}$ in K,

$$\begin{aligned} |u - \mathrm{I}_h u; W^{1,2}(K)| &\lesssim \sum_{i=1}^{3} h_{i,K} |\partial_i u; W^{1,2}(K)| \\ &\lesssim \sum_{i\in\{S1,S2\}} h_{i,K} r_K^{-\beta} \left|\partial_i u; V_\beta^{1,2}(K)\right| + h_{L,K} \left|\partial_L u; V_0^{1,2}(K)\right| \quad (5.1) \end{aligned}$$

for any $\beta > 1 - \pi/\omega$. We apply now the assumption (2.6) and obtain for $r_K \le R$ and $\beta = 1 - \mu$ the relation $h_{i,K} r_K^{-\beta} \sim h r_K^{1-\mu-\beta} = h$ ($i \in \{S1, S2\}$). The choice $\beta = 1 - \mu$ is admissible due to the refinement condition $\mu < \pi/\omega$. In the case $r_K > R$ we have $h_{i,K} r_K^{-\beta} \lesssim h R^{-\beta} \sim h$. Combining this with (5.1) we obtain

$$|u - \mathrm{I}_h u; W^{1,2}(K)| \lesssim h \sum_{i\in\{S1,S2\}} \left|\partial_i u; V_\beta^{1,2}(K)\right| + h \left|\partial_L u; V_0^{1,2}(K)\right|. \quad (5.2)$$

Consider now the elements K with $r_K = 0$. We use Lemma 3.4 with $p = q = 2$, $\beta_{S1,K} = \beta_{S2,K} = \beta = 1 - \mu \in (1 - \pi/\omega, 1)$, $\beta_{L,K} = 0$,

$$\begin{aligned} |u - \mathrm{I}_h u; W^{1,2}(K)| &\lesssim \sum_{i\in\{S1,S2\}} h_{S,K}^{1-\beta} \|\partial_j u; V_\beta^{1,2}(K)\| + h_{L,K} \|\partial_L u; V_0^{1,2}(K)\| \\ &\lesssim h \sum_{i\in\{S1,S2\}} \left\|\partial_i u; V_\beta^{1,2}(K)\right\| + h \left\|\partial_L u; V_0^{1,2}(K)\right\|. \quad (5.3) \end{aligned}$$

We also used that $h_{S,K}^{1-\beta} \sim h^{(1-\beta)/\mu} = h$ for $\beta = 1 - \mu$.

Summing up the square of the estimates (5.2), (5.3) over all elements we obtain

$$|u - \mathrm{I}_h u; W^{1,2}(\Omega)| \lesssim h \sum_{i=1}^{2} \left\|\partial_i u; V_\beta^{1,2}(\Omega)\right\| + h \left\|\partial_L u; V_0^{1,2}(\Omega)\right\|.$$

By applying Lemma 2.1 the theorem is proved. □

Theorem 5.2 *Let u be the solution of (2.1), (2.2). Then the estimate*

$$\sup_{v_h \in V_h} \frac{|(\nabla u, \nabla v_h)_h - (f, v_h)|}{\|v_h\|_{1,h}} \lesssim h \,\|f; L^2(\Omega)\|$$

holds if $\mu < \pi/\omega$.

Proof In view of Lemma 4.6 it remains to prove

$$\left(\sum_Q \sum_{i,j\in\{L,S1,S2\}} h_{S,Q}^{-2\beta_{i,j,Q}} h_{i,Q}^2 \|r^{\beta_{i,j,Q}}\partial_i\partial_j u; L^2(Q)\|^2\right)^{1/2} \lesssim h\,\|f; L^2(\Omega)\|, \tag{5.4}$$

$$\left(\sum_Q h_{L,Q}^2 \|f + \partial_L^2 u; L^2(Q)\|^2\right)^{1/2} \lesssim h\,\|f; L^2(\Omega)\|. \tag{5.5}$$

The second estimate is trivial since $h_{L,Q} = h$ for all Q and $\|f + \partial_L^2 u; L^2(\Omega)\| \lesssim \|f; L^2(\Omega)\|$ due to (2.5).

In the left hand side of (5.4) we set $\beta_{i,j,Q} = \beta$ if $r_Q = 0$ and $i, j \in \{S1, S2\}$, and $\beta_{i,j,Q} = 0$ otherwise. Then we insert the definition (2.6) of $h_{i,Q}$ and proceed similarly to the proof of Theorem 5.1, namely

$$\begin{aligned}
&\left(\sum_Q \sum_{i,j\in\{L,S1,S2\}} h_{S,Q}^{-2\beta_{i,j,Q}} h_{i,Q}^2 \|r^{\beta_{i,j,Q}}\partial_i\partial_j u; L^2(Q)\|^2\right)^{1/2} \\
&= \Bigg(\sum_{Q:r_Q=0} \sum_{i,j\in\{S1,S2\}} h_{S,Q}^{2(1-\beta)} \|r^\beta \partial_i\partial_j u; L^2(Q)\|^2 + \sum_{Q:r_Q=0} \sum_{i\in\{L,S1,S2\}} h_{L,Q}^2 \|\partial_i\partial_L u; L^2(Q)\|^2 + \\
&\qquad + \sum_{Q:r_Q>0} \sum_{i,j\in\{L,S1,S2\}} h_{i,Q}^2 \|\partial_i\partial_j u; L^2(Q)\|^2\Bigg)^{1/2} \\
&\lesssim \Bigg(h^{2(1-\beta)/\mu} \sum_{Q:r_Q=0} \sum_{i\in\{S1,S2\}} |\partial_i u; V_\beta^{1,2}(Q)|^2 + h^2 \sum_{Q:r_Q=0} |\partial_L u; V_0^{1,2}(Q)|^2 + \\
&\qquad + h^2 \sum_{Q:r_Q>0} \left(\sum_{i\in\{S1,S2\}} r_Q^{2(1-\mu-\beta)} |\partial_i u; V_\beta^{1,2}(Q)|^2 + |\partial_3 u; V_0^{1,2}(Q)|^2\right)\Bigg)^{1/2}.
\end{aligned}$$

With $\beta = 1 - \mu > 1 - \pi/\omega$, by using the Cauchy-Schwarz inequality, and by applying Lemma 2.1 we get the desired estimate (5.4). This finishes the proof. □

Corollary 5.3 *Let u be the solution of (2.1), (2.2) and let u_h be the finite element solution defined by (2.9). Assume that the mesh is refined according to $\mu < \pi/\omega$. Then the finite element error can be estimated by*

$$\begin{aligned}
\|u - u_h\|_{1,h} &\lesssim h\,\|f; L^2(\Omega)\|, \\
\|u - u_h; L^2(\Omega)\| &\lesssim h^2\,\|f; L^2(\Omega)\|.
\end{aligned}$$

Proof The first estimate follows from Theorems 5.1 and 5.2 via (2.10). The second estimate can be proved in the standard way by using the first estimate, see, for example, [16, §III.1]. □

By analogy one can prove for $\pi/\omega < \mu \le 1$ that

$$\begin{aligned} |u - u_h; W^{1,2}(\Omega)| &\lesssim h^{\pi/(\mu\omega)-\varepsilon} \, \|f; L^2(\Omega)\| \\ \|u - u_h; L^2(\Omega)\| &\lesssim h^{2\pi/(\mu\omega)-2\varepsilon} \, \|f; L^2(\Omega)\| \end{aligned}$$

for arbitrary small $\varepsilon > 0$, compare [7]. That means that we get for the unrefined mesh ($\mu = 1$) only an approximation order $\pi/\omega - \varepsilon$. We conjecture that the ε can be omitted but this needs another way of proof.

6 Numerical test

Consider the Laplace equation with Dirichlet boundary conditions,

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega,$$

in the three-dimensional domain

$$\Omega = \{(x_1, x_2, x_3) = (r\cos\phi, r\sin\phi, z) \in I\!R^3 : r < 1,\ 0 < \phi < 3\pi/2,\ 0 < z < 1\}.$$

The right hand side g is taken such that

$$u = (10 + z)\, r^{2/3} \sin \tfrac{2}{3}\phi$$

is the exact solution of the problem. It has the typical singular behaviour at the edge.

We constructed tetrahedral meshes as described in Section 2, with $\mu = 1$ (quasi-uniform) and $\mu = 0.5$ (anisotropically refined) and with different numbers of elements. The numerical solution was computed by using conforming and non-conforming $\mathcal{P}_1$ elements. From these numerical solutions and the known exact solution, the energy norm $\|u - u_h\|_{1,h}$ and the L^2-norm $\|u - u_h; L^2(\Omega)\|$ of the finite element error was computed in the four cases. Figures 6.1 and 6.2 show the plot of these norms against the number N of unknowns and the number N_{el} of elements, respectively. A double logarithmic scale was used such that the slope of the curves corresponds to the approximation order. The example verifies the theoretically predicted convergence orders.

Comparing the conforming with the non-conforming strategy we see that the conforming strategy is superior when the number of unknowns is considered whereas the non-conforming strategy is superior when the number of elements is taken into consideration. A good criterion for a comparison is computing time. The amount of computational work is proportional to the number of elements in the assembling step and whereas it is proportional to the number of unknowns in one iteration of the solver. The latter statement is, however, only partially convincing since the amount of work depends also on the number

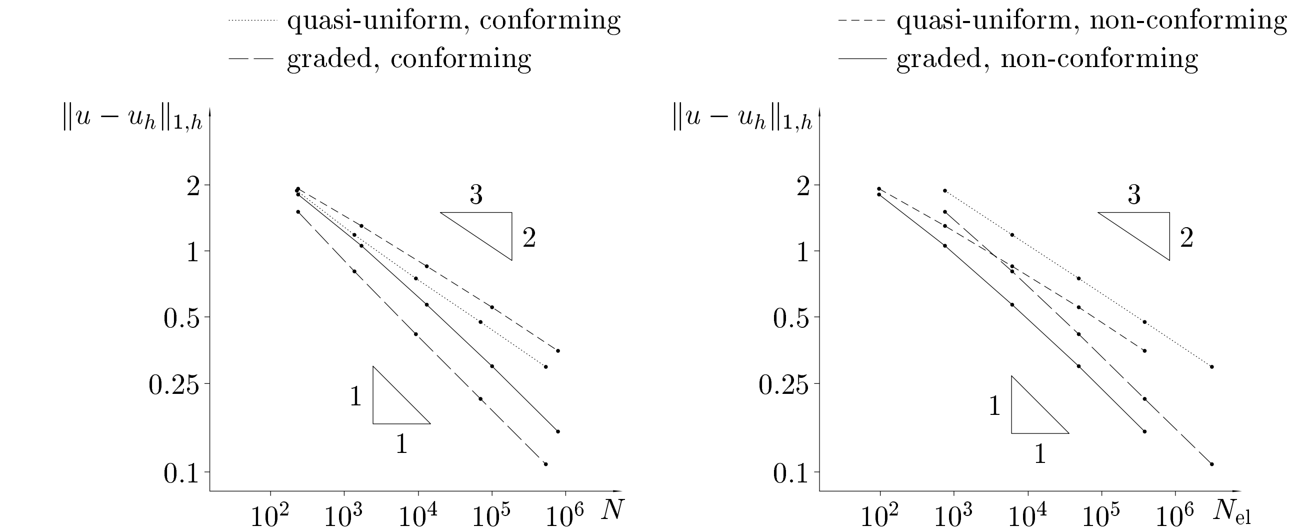

Figure 6.1: Comparison of uniform vs. graded meshes and conforming vs. non-conforming methods: energy norm of the error against number of nodes (left), energy norm of the error against number of elements (right).

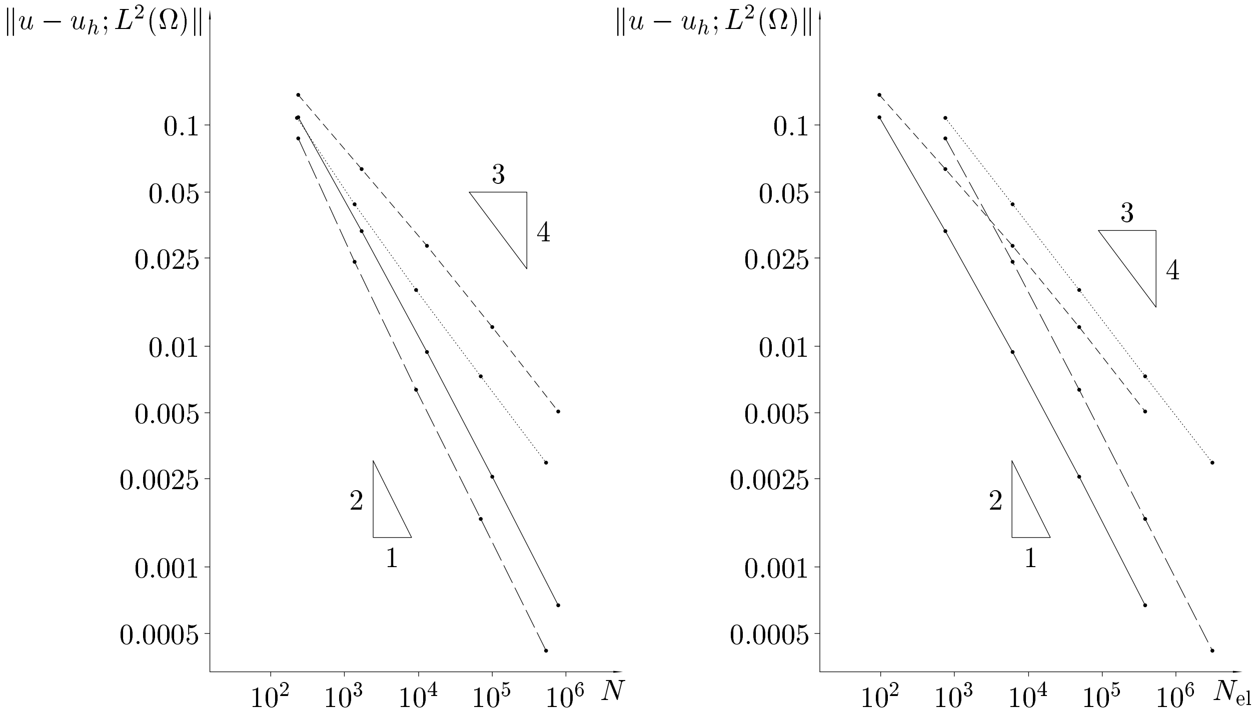

Figure 6.2: Comparison of uniform vs. graded meshes and conforming vs. non-conforming methods: $L^2(\Omega)$-norm of the error against number of nodes (left), $L^2(\Omega)$-norm of the error against number of elements (right).

of non-zero entries in the matrix and these numbers are different multiples of the number of unknowns for the conforming and the non-conforming strategy. Moreover, the number of iterations can hardly be compared since optimal preconditioners for graded meshes near edges are not available now.

Finally we like to remark that we did tests also with non-tensor product meshes as they were described for the treatment of general polyhedral domains in [8]. The same convergence rates were verified so that we expect that the anisotropic non-conforming strategy could also be proved for classes of more general meshes than we assumed in this paper. This is a task for future work.

Acknowledgement The work of the third author is supported by the Austrian Science Fund 'Fonds zur Förderung der wissenschaftlichen Forschung", Spezialforschungsbereich F013. The visit of the first author in Valenciennes was financed by the Université de Valenciennes et du Hainaut Cambrésis. The visit of the third author in Chemnitz was financed by the DFG (German Research Foundation), Sonderforschungsbereich 393.

References

[1] G. Acosta and R. G. Durán. The maximum angle condition for mixed and non-conforming elements. application to the stokes equations. Impresiones previas 108, Departamento de Matemática, Facultad de Ciencias Exactas, Universidad de Buenos Aires.

[2] Th. Apel. Interpolation of non-smooth functions on anisotropic finite element meshes. Preprint SFB393/97-6, TU Chemnitz-Zwickau, 1997. To appear in Math. Modeling Numer. Anal.

[3] Th. Apel. *Anisotropic finite elements: Local estimates and applications.* Advances in Numerical Mathematics. Teubner, Stuttgart, 1999. To appear.

[4] Th. Apel and M. Dobrowolski. Anisotropic interpolation with applications to the finite element method. *Computing*, 47:277–293, 1992.

[5] Th. Apel and G. Lube. Anisotropic mesh refinement in stabilized Galerkin methods. *Numer. Math.*, 74:261–282, 1996.

[6] Th. Apel and G. Lube. Anisotropic mesh refinement for a singularly perturbed reaction diffusion model problem. *Appl. Numer. Math.*, 26:415–433, 1998.

[7] Th. Apel and S. Nicaise. Elliptic problems in domains with edges: anisotropic regularity and anisotropic finite element meshes. In J. Cea, D. Chenais, G. Geymonat, and J. L. Lions, editors, *Partial Differential Equations and Functional Analysis (In Memory of Pierre Grisvard)*, pages 18–34. Birkhäuser, Boston, 1996. Shortened version of Preprint SPC94_16, TU Chemnitz-Zwickau, 1994.

[8] Th. Apel and S. Nicaise. The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges. *Math. Methods Appl. Sci.*, 21:519–549, 1998.

[9] Th. Apel, S. Nicaise, and J. Schöberl. A non-conforming finite element method with anisotropic mesh grading for the Stokes problem in domains with edges. In preparation.

[10] Th. Apel and J. Schöberl. Anisotropic meshes for the treatment of boundary layers in Mindlin-Reissner plates. In preparation.

[11] I. Babuška and A. K. Aziz. On the angle condition in the finite element method. *SIAM J. Numer. Anal.*, 13:214–226, 1976.

[12] E. Bänsch. Anisotropic interpolation estimates. Preprint 373, SFB 256, Rheinische Friedrich-Wilhelms-Universität Bonn, 1994.

[13] R. E. Barnhill and J. A. Gregory. Interpolation remainder theory from Taylor expansions on triangles. *Numer. Math.*, 25:401–408, 1976.

[14] R. Becker. *An adaptive finite element method for the incompressible Navier–Stokes equations on time-dependent domains.* PhD thesis, Ruprecht-Karls-Universität Heidelberg, 1995.

[15] R. Becker and R. Rannacher. Finite element solution of the incompressible navier-stokes equations on anisotropically refined meshes. In *Fast solvers for flow problems*, volume 49 of *Notes on Numerical Fluid Mechanics*, pages 52–62, Wiesbaden, 1995. Vieweg.

[16] D. Braess. *Finite Elemente.* Springer, Berlin, 1997.

[17] P. G. Ciarlet. *The finite element method for elliptic problems.* North-Holland, Amsterdam, 1978.

[18] M. Crouzeix and P. A. Raviart. Conforming and non-conforming finite elements for solving the stationary Stokes equations. *R.A.I.R.O. Anal. Numér.*, 7:33–76, 1973.

[19] E. F. D'Azevedo and R. B. Simpson. On optimal triangular meshes for minimizing the gradient error. *Numer. Math.*, 59:321–348, 1991.

[20] M. Dobrowolski and H.-G. Roos. A priori estimates for the solution of convection-diffusion problems and interpolation on Shishkin meshes. *Z. Anal. Anwend.*, 16:1001–1012, 1997.

[21] P. Jamet. Estimations d'erreur pour des éléments finis droits presque dégénérés. *R.A.I.R.O. Anal. Numér.*, 10:43–61, 1976.

[22] M. Křížek. On semiregular families of triangulations and linear interpolation. *Appl. Math.*, 36:223–232, 1991.

[23] M. Křížek. On the maximum angle condition for linear tetrahedral elements. *SIAM J. Numer. Anal.*, 29:513–520, 1992.

[24] L. A. Oganesyan and L. A. Rukhovets. *Variational-difference methods for the solution of elliptic equations.* Izd. Akad. Nauk Armyanskoi SSR, Jerevan, 1979. In Russian.

[25] T. von Petersdorff. *Randwertprobleme der Elastizitätstheorie für Polyeder — Singularitäten und Approximationen mit Randelementmethoden.* PhD thesis, TH Darmstadt, 1989.

[26] W. Rachowicz. An anisotropic h-type mesh refinement strategy. *Comput. Methods Appl. Mech. Engrg.*, 109:169–181, 1993.

[27] R. Rannacher and St. Turek. Simple nonconforming quadrilateral stokes element. *Numer. Meth. Partial Differ. Equations*, 8:97–111, 1992.

[28] G. Raugel. Résolution numérique par une méthode d'éléments finis du problème Dirichlet pour le Laplacien dans un polygone. *C. R. Acad. Sci. Paris, Sér. A*, 286(18):A791–A794, 1978.

[29] S. Rippa. Long and thin triangles can be good for linear interpolation. *SIAM J. Numer. Anal.*, 29:257–270, 1992.

[30] N. A. Shenk. Uniform error estimates for certain narrow Lagrangian finite elements. *Math. Comp.*, 63:105–119, 1994.

[31] R. B. Simpson. Anisotropic mesh transformation and optimal error control. *Applied Numerical Mathematics*, 14:183–198, 1994.

[32] E. P. Stephan and J. R. Whiteman. Singularities of the Laplacian at corners and edges of three-dimensional domains and their treatment with finite element methods. *Math. Methods Appl. Sci.*, 10(3):339–350, 1988.

[33] J. L. Synge. *The hypercircle in mathematical physics.* Cambridge University Press, Cambridge, 1957.

[34] H. Walden and R. B. Kellogg. Numerical determination of the fundamental eigenvalue for the Laplace operator on a spherical domain. *J. Eng. Math.*, 11:299–318, 1977.

[35] A. Ženíšek and M. Vanmaele. The interpolation theorem for narrow quadrilateral isoparametric finite elements. *Numer. Math.*, 72:123–141, 1995.

Other titles in the SFB393 series:

99-01 P. Kunkel, V. Mehrmann, W. Rath. Analysis and numerical solution of control problems in descriptor form. January 1999.

99-02 A. Meyer. Hierarchical preconditioners for higher order elements and applications in computational mechanics. January 1999.

99-03 T. Apel. Anisotropic finite elements: local estimates and applications (Habilitationsschrift). January 1999.

99-04 C. Villagonzalo, R. A. Römer, M. Schreiber. Thermoelectric transport properties in disordered systems near the Anderson transition. February 1999.

99-05 D. Michael. Notizen zu einer geometrisch motivierten Plastizitätstheorie. Februar 1999.

99-06 T. Apel, U. Reichel. SPC-PM Po 3D V 3.3, User's Manual. February 1999.

99-07 F. Tröltzsch, A. Unger. Fast solution of optimal control problems in the selective cooling of steel. March 1999.

99-08 W. Rehm, T. Ungerer (Eds.). Ausgewählte Beiträge zum 2. Workshop Cluster-Computing 25./26. März 1999, Universität Karlsruhe. März 1999.

99-09 M. Arav, D. Hershkowitz, V. Mehrmann, H. Schneider. The recursive inverse eigenvalue problem. March 1999.

The complete list of current and former preprints is available via
`http://www.tu-chemnitz.de/sfb393/preprints.html`.