QUASI-UNIFORMITY OF BDDC PRECONDITIONERS FOR THE MITC REISSNER-MINDLIN PROBLEM

L. BEIRÃO DA VEIGA*, C. CHINOSI†, C. LOVADINA‡, L. F. PAVARINO§, AND J. SCHÖBERL ¶

Abstract. We consider the BDDC preconditioner for Reissner-Mindlin plate problems, discretized with the MITC element, introduced and analyzed in [11]. In that contribution the authors prove that the condition number of the ensuing linear system is independent of the plate thickness and scalable with respect to the mesh. We here prove, in addition, that the BDDC preconditioner of [11] is also quasi-optimal, a property which was shown only numerically in [11].

1. Introduction. The Reissner-Mindlin theory is widely used to describe the bending behavior of an elastic plate loaded by a transverse force. Despite its simple formulation, the discretization by means of finite elements is not straightforward, since standard low-order schemes exhibit a severe lack of convergence whenever the thickness is too small with respect to the other characteristic dimensions of the plate. This undesirable phenomenon, known as *shear locking*, is nowadays well understood: the most popular way to overcome the shear locking phenomenon is to reduce the influence of the shear energy by considering a mixed formulation. A vast engineering and mathematical literature is devoted to the design and analysis of plate elements, see e.g. the works [12, 3, 4, 5, 8, 13, 14, 15, 19, 21, 25, 26, 27, 28, 33, 34, 9]. However, a limited number of domain decomposition works are available for the efficient iterative solution of the resulting discrete plate problems; a list of references is given in [11].

The goal of this paper is to improve a result already shown in [11]. In that paper, the authors present and analyze a Balancing Domain Decomposition Method by Constraints (BDDC) for the Reissner-Mindlin plate bending problem discretized with the well known MITC elements. The BDDC preconditioner in [11] is based on selecting the plate rotations and deflection degrees of freedom at the subdomain vertices as primal continuity constraints. Introduced by Dohrmann [16] and analyzed by Mandel, Dohrmann and Tezaur [29, 30], BDDC methods have evolved from previous domain decomposition work on Balancing Neumann-Neumann methods. BDDC algorithm have been extended in recent years from scalar elliptic problems to almost incompressible elasticity (Dohrmann [17, 18]), the Stokes system (Li and Widlund [24]), flow in porous media (Tu [36]), spectral element discretizations (Pavarino [31], Klawonn et al. [23]).

In [11] the authors prove that the condition number of the preconditioned system is independent of the plate thickness t and scalable with respect to the mesh, which are very desirable properties in plate analysis. This means a bound on the condition

* Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italy. Email: lourenco.beirao@unimi.it, URL: http://mat.unimi.it/∼beirao/

†Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale, Via Bellini 25/G, I-15100 Alessandria, ITALY. Email: chinosi@ mfn.unipmn.it, URL: http://www.mfn.unipmn.it/∼chinosi/

‡Dipartimento di Matematica, Università di Pavia, Via Ferrata 1, 27100 Pavia, ITALY. Email: carlo.lovadina@unipv.it, URL: http://www-dimat.unipv.it/lovadina/

§Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, ITALY. Email: luca.pavarino@unimi.it, URL: http://mat.unimi.it/∼pavarino/

¶Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstrasse 8-10, 1040 Wien, Austria. Email: joachim.schoeber@tuwien.ac.at URL: http://www.asc.tuwien.ac.at/∼schoeberl/wiki/index.php/Joachim_Schöberl

number K as

$$K \leq C(H/h) \ ,$$

with C a constant independent of t, h, H and where t, h, H represent, respectively, the plate thickness, the mesh size and the coarse problem mesh size. The bound above must be compared with the condition number of the non preconditioned MITC plate bending problem K_{no}, which is

$$K_{no} \sim Ch^{-2p}t^{-2}$$

with p the polynomial degree adopted.

Although the extensive numerical tests in [11] clearly show also a quasi-uniformity property, such property does not appear among the theoretical results of that paper. In the present contribution we prove the improved result

$$K \leq C(1 + \log^3{(H/h)}) \ ,$$

which shows that the BDDC preconditioner of [11] is also quasi-uniform. This result is a non negligible improvement, since it gives a growth rate in h/H which is logarithmic instead of linear. On the other hand it must be noted that the ideas used here are very similar to, and strongly inherit from, those already present in [11].

The present paper is organized as follows. In Section 2, we present the Reissner-Mindlin plate bending problem and its discretization with MITC finite elements. The basic substructuring procedure is introduced in Section 3, while the BDDC algorithm of [11] is recalled in Section 4. The theoretical analysis of the BDDC quasi-uniformity is finally developed in Section 5.

2. The MITC Reissner-Mindlin plate bending problem. Let Ω be a polygonal domain in $\mathbb{R}^2$ representing the midsurface of the plate. For simplicity of exposition, we assume that the plate is clamped on the whole boundary $\partial\Omega$, although what follows extends identically to more general cases. Following the Reissner-Mindlin model, see for instance [6], the plate bending problem requires to solve

$$\begin{cases} \text{Find } \boldsymbol{\theta}^{ex} \in [H^1_0(\Omega)]^2, u^{ex} \in H^1_0(\Omega) \text{ such that} \\ a(\boldsymbol{\theta}^{ex}, \boldsymbol{\eta}) + \mu k t^{-2}(\boldsymbol{\theta}^{ex} - \nabla u^{ex}, \boldsymbol{\eta} - \nabla v) = (f, v) \quad \forall \boldsymbol{\eta} \in [H^1_0(\Omega)]^2, v \in H^1_0(\Omega) \ , \end{cases} \tag{2.1}$$

where μ is the shear modulus and k is the so-called shear correction factor. Above, t represents the plate thickness, u^{ex} the deflection, $\boldsymbol{\theta}^{ex}$ the rotation of the normal fibers and f the applied scaled normal load. Moreover, $(\cdot, \cdot)$ stands for the standard scalar product in $L^2(\Omega)$ and the bilinear form $a(\cdot, \cdot)$ is defined by

$$a(\boldsymbol{\theta}^{ex}, \boldsymbol{\eta}) = (\mathbb{C}\varepsilon(\boldsymbol{\theta}^{ex}), \varepsilon(\boldsymbol{\eta})),$$

with $\mathbb{C}$ the positive definite tensor of bending moduli and $\varepsilon(\cdot)$ the symmetric gradient operator. Introducing the scaled shear stresses $\boldsymbol{\gamma}^{ex} = \mu k t^{-2}(\boldsymbol{\theta}^{ex} - \nabla u^{ex})$, Problem (2.1) can be written in terms of the following mixed variational formulation:

$$\begin{cases} \text{Find } \boldsymbol{\theta}^{ex} \in [H^1_0(\Omega)]^2, u^{ex} \in H^1_0(\Omega), \boldsymbol{\gamma}^{ex} \in [L^2(\Omega)]^2 \text{ such that} \\ a(\boldsymbol{\theta}^{ex}, \boldsymbol{\eta}) + (\boldsymbol{\gamma}^{ex}, \boldsymbol{\eta} - \nabla v) = (f, v) \quad \forall \boldsymbol{\eta} \in [H^1_0(\Omega)]^2, v \in H^1_0(\Omega) \\ (\boldsymbol{\theta}^{ex} - \nabla u^{ex}, \boldsymbol{s}) - \dfrac{t^2}{\mu k}(\boldsymbol{\gamma}, \boldsymbol{s}) = 0 \quad \forall \boldsymbol{s} \in [L^2(\Omega)]^2 \ . \end{cases} \tag{2.2}$$

To simplify notation, and without any loss of generality, we will assume $\mu k = 1$ in the analysis that follows.

2.1. Discretization of the problem with the MITC elements. We will now present the discretization of the problem following the MITC (Mixed Interpolation of Tensorial Components) elements. Since the MITC is a large family of elements, we will keep a general standpoint without detailing the particular description of the discrete spaces and operators, which depend on the particular MITC element chosen; a list of elements can be found for instance in [6, 7, 13].

Let τ_h denote a triangular or quadrilateral conforming finite element mesh on Ω, of characteristic mesh size h. Let $\mathbf{\Theta}$, U and $\mathbf{\Gamma}$ represent the discrete spaces for rotations, deflections and shear stresses, respectively. In the sequel, we will set the compact notation $\mathbf{X} = \mathbf{\Theta} \times U$. Then the Reissner-Mindlin plate bending problem (2.2) discretized with MITC elements reads

$$\begin{cases} \text{Find } (\boldsymbol{\theta}, u) \in \mathbf{X},\ \boldsymbol{\gamma} \in \mathbf{\Gamma} \text{ such that} \\ a(\boldsymbol{\theta}, \boldsymbol{\eta}) + (\boldsymbol{\gamma}, \Pi\, \boldsymbol{\eta} - \nabla v) = (f, v) \quad \forall (\boldsymbol{\eta}, v) \in \mathbf{X} \\ (\Pi\, \boldsymbol{\theta} - \nabla u, \boldsymbol{s}) - t^2 (\boldsymbol{\gamma}, \boldsymbol{s}) = 0 \quad \forall \boldsymbol{s} \in \mathbf{\Gamma}\ , \end{cases} \tag{2.3}$$

where $\Pi\ :\ ([H^1(\Omega)]^2 + \mathbf{\Gamma}) \longrightarrow \mathbf{\Gamma}$ is the MITC reduction operator. Using the second equation of (2.3), shear stresses can be eliminated to obtain the following positive definite discrete formulation:

$$\begin{cases} \text{Find } (\boldsymbol{\theta}, u) \in \mathbf{X} \text{ such that} \\ b((\boldsymbol{\theta}, u), (\boldsymbol{\eta}, v)) = (f, v) \quad \forall (\boldsymbol{\eta}, v) \in \mathbf{X}\ , \end{cases} \tag{2.4}$$

where we have introduced the compact notation

$$b((\boldsymbol{\theta}, u), (\boldsymbol{\eta}, v)) := a(\boldsymbol{\theta}, \boldsymbol{\eta}) + t^{-2} (\Pi\, \boldsymbol{\theta} - \nabla u, \Pi\, \boldsymbol{\eta} - \nabla v)\ . \tag{2.5}$$

The MITC spaces and the associated operator Π are well known to satisfy the following five properties, which will play a key role in the sequel. In what follows, $Q \subset L^2(\Omega)$ represents an ad hoc discrete auxiliary space. For the proofs of these properties, see [7, 13].

P1. $\nabla U \subset \mathbf{\Gamma}$.
P2. $\operatorname{curl} \mathbf{\Gamma} \subset Q$.
P3. $\operatorname{curl} \Pi\, \boldsymbol{\eta} = P \operatorname{curl} \boldsymbol{\eta}$, $\boldsymbol{\eta} \in [H^1_0]^2$, where $P : L^2 \to Q$ denotes the L^2 projection.
P4. $\{\boldsymbol{\gamma} \in \mathbf{\Gamma}\ :\ \operatorname{curl} \boldsymbol{\gamma} = 0\} = \nabla U$.
P5. $(\mathbf{\Theta}, Q)$ is a stable pair of spaces for the Stokes problem.

We will also require the following additional property to hold, see [8].

P6. For every edge l of the mesh, denote with $\boldsymbol{\tau}$ its tangent vector. We assume that the combined operator $(\Pi \boldsymbol{\theta}|_l) \cdot \boldsymbol{\tau}$ depends only on $\boldsymbol{\theta}|_l \cdot \boldsymbol{\tau}$ for all $\boldsymbol{\theta} \in \mathbf{\Theta}$. Therefore the above combined operator is well defined also when applied to functions living only on edges.

Note that in this paper we address directly the positive definite Problem (2.4), in the spirit of [10, 11], instead of the mixed formulation. A vast literature exists on the convergence analysis of the MITC elements, see for instance [7, 13, 19, 32] and also [9, 20]. The MITC elements perform optimally with respect to the polynomial degree and regularity of the solution, and their rate of convergence is independent of the thickness parameter t.

In the sequel we will need also the following mild assumptions, which are included separately in a Remark since those conditions where not requested in [11], but are needed here in order to derive the quasi-uniformity condition. We start by the following definition of edge bubble functions. Let Ω_i, for $i = 1, .., N$, be the subdomains in which we decompose Ω, which are defined more in detail in Section 3, and let Γ_i be the boundary of each Ω_i. Let e be an edge of the mesh laying on Γ_i, and K_l represent the element inside Ω_i with l as an edge. Then an edge bubble function B_l^i is a quadratic (or biquadratic in the quadrilateral case) function living on K_l which is equal to 1 at the center of the edge l and null on the two (respectively 3) other edges of K_l. In the quadrilateral case, we moreover enforce that B_l^i is null at the baricenter of K_e, in order to make it uniquely defined. Then, we introduce the following additional conditions on the mesh and method.

REMARK 2.1. *We assume the following. Given any edge l on any Γ_i, $i = 1, 2, .., N$, the function $B_l^i \boldsymbol{\tau}$ is contained into the restriction of $\boldsymbol{\Theta}$ to the element K_l. Moreover it holds*

$$\int_l \Pi \left(\boldsymbol{\theta}_i \cdot \boldsymbol{\tau}\right) = \int_l \boldsymbol{\theta}_i \cdot \boldsymbol{\tau} \quad \forall l \in \Gamma_i, \ \forall \boldsymbol{\theta}_i \in \boldsymbol{\Theta}_i.$$

Furthermore, it exists a constant C' such that $0 < t \leq C'h$ for all meshes in the family.

The first two conditions above are quite easy to fullfil, it is for example sufficient that the rotation space is at least of second (polynomial) order and that a standard definition of Π is adopted. But it is for instance satisfied also by the Duran-Liberman element [19], even if this element does not contain all quadratic functions. The condition above is satisfied by almost all the plate MITC elements in the literature. Regarding the last condition, this is also quite natural and in general satisfied. Indeed, when $h \leq t$, we can expect that the error in the model becomes larger than the finite element error, thus making useless any further mesh refinement.

3. Iterative substructuring. We decompose the domain Ω into N open, nonoverlapping subdomains Ω_i of characteristic size H forming a shape-regular finite element mesh τ_H. This coarse triangulation τ_H is further refined into a finer triangulation τ_h of characteristic size h; both meshes will typically be composed of triangles or quadrilaterals. In the sequel, we assume that the material tensor $\mathbb{C}$ is constant through the whole domain; see Remark 5.1.

As it is standard in iterative substructuring methods, we first reduce the problem to the interface

$$\Gamma = \left(\bigcup_{i=1}^{N} \partial\Omega_i \right) \setminus \partial\Omega,$$

by implicitly eliminating the interior degrees of freedom, a process also known as static condensation. In variational form, this process consists in a suitable decomposition of the discrete space $\mathbf{X} = \boldsymbol{\Theta} \times U$. More precisely, let us define $\boldsymbol{W} = \mathbf{X}_{|\Gamma}$, i.e. the space of the traces of functions in $\mathbf{X}$, as well as the local spaces

$$\mathbf{X}_i = \mathbf{X} \cap [H_0^1(\Omega_i)]^3 \ . \tag{3.1}$$

The space $\mathbf{X}$ can be decomposed as

$$\mathbf{X} = \oplus_{i=1}^{N} \mathbf{X}_i \ \oplus \ \overline{\mathcal{H}}(\boldsymbol{W}) \ .$$

Here $\overline{\mathcal{H}} : \boldsymbol{W} \longrightarrow \mathbf{X}$ is the discrete "plate-harmonic" extension operator defined by solving the problem

$$\begin{cases} \text{Find } \overline{\mathcal{H}}(\boldsymbol{w}_\Gamma) \in \mathbf{X} \ \text{ such that:} \\ b(\overline{\mathcal{H}}(\boldsymbol{w}_\Gamma), \boldsymbol{v}_I) = 0 \qquad \forall\, \boldsymbol{v}_I \in \mathbf{X}_i \quad i = 1, 2, ..., N \\ \overline{\mathcal{H}}(\boldsymbol{w}_\Gamma)|_\Gamma = \boldsymbol{w}_\Gamma \ . \end{cases} \tag{3.2}$$

Defining the Schur complement bilinear form

$$s(\boldsymbol{w}_\Gamma, \boldsymbol{v}_\Gamma) = b(\overline{\mathcal{H}}(\boldsymbol{w}_\Gamma), \overline{\mathcal{H}}(\boldsymbol{v}_\Gamma)), \tag{3.3}$$

it follows that the interface component of the discrete solution satisfies the reduced system

$$s(\boldsymbol{u}_\Gamma, \boldsymbol{v}_\Gamma) =< \tilde{\boldsymbol{f}}, \boldsymbol{v}_\Gamma > \qquad \forall\, \boldsymbol{v}_\Gamma \in \boldsymbol{W}, \tag{3.4}$$

for a suitable right-hand side $\tilde{\boldsymbol{f}}$. In order to simplify the notation, in the sequel we will drop the index Γ for functions in W if there is no risk of confusion. Moreover, in the rest of the contribution C will indicate a general scalar constant, independent of H and h, which may change on different occurrences.

4. A Balancing Domain Decomposition method by Constraints. The BDDC preconditioner, introduced by Dohrmann [16] and analyzed by Mandel and Dohrmann [29], applies to the classical Schur complement system and can be regarded as an evolution of the Balancing Neumann-Neumann preconditioner. In this section, we introduce the BDDC preconditioner of Ref. [16, 29], formulated with the notation of Ref. [35, 11]. We need a set of preliminary definitions.

In the sequel, in order to shorten the notation, we indicate with

$$\Gamma_i := \partial\Omega_i \ ,$$

while $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$, $i, j \in \{1, 2, .., N\}$, will represent the common edge between two adjacent subdomains Ω_i and Ω_j.

We introduce the local spaces $\overline{\boldsymbol{W}}_i$ as the spaces of discrete functions defined by $\overline{\boldsymbol{W}}_i = \boldsymbol{W}_{|\Gamma_i}$, $i = 1, 2, .., N$. Let $\overline{\mathcal{H}}_i : \overline{\boldsymbol{W}}_i \longrightarrow \mathbf{X}|_{\Omega_i}$, $i = 1, 2, .., N$, represent the restriction of the operator $\overline{\mathcal{H}}$ to the domain Ω_i

$$\begin{cases} \text{Find } \overline{\mathcal{H}}_i(\boldsymbol{w}_i) \in \mathbf{X}|_{\Omega_i} \text{ such that:} \\ b_i(\overline{\mathcal{H}}_i(\boldsymbol{w}_i), \boldsymbol{v}_i) = 0 \qquad \forall\, \boldsymbol{v}_i \in \mathbf{X}_i \\ \overline{\mathcal{H}}(\boldsymbol{w}_i)|_{\Gamma_i} = \boldsymbol{w}_i \ . \end{cases} \tag{4.1}$$

where the $b_i(\cdot, \cdot)$ are given by restricting the integrals in $b(\cdot, \cdot)$ to the domain Ω_i, $i = 1, 2, .., N$.

We then define the local bilinear forms on the space $\overline{\boldsymbol{W}}_i$

$$s_i(\boldsymbol{w}_i, \boldsymbol{v}_i) = b_i(\overline{\mathcal{H}}_i\boldsymbol{w}_i, \overline{\mathcal{H}}_i\boldsymbol{v}_i) \ , \qquad \forall \boldsymbol{w}_i, \boldsymbol{v}_i \in \overline{\boldsymbol{W}}_i \ , \tag{4.2}$$

For simplicity, we adopt the shortened notation

$$s_i(\boldsymbol{w}_i) := s_i(\boldsymbol{w}_i, \boldsymbol{w}_i) \quad \forall \boldsymbol{w}_i \in \overline{\boldsymbol{W}}_i \ ,$$

and the same for all the other bilinear forms appearing in the paper.

Furthermore, let the prolongation operators R_i^T, $i = 1, 2, .., N$ be maps which extend any function of $\overline{\boldsymbol{W}}_i$ to the function of $\boldsymbol{W}$ which is zero at all the nodes not in Γ_i. Conversely, we call R_i, $i = 1, 2, .., N$, the restriction operators $\boldsymbol{W} \to \overline{\boldsymbol{W}}_i$ that leave the function unchanged on Γ_i. Note that, by definition of the s_i, it holds

$$\sum_{i=1}^{N} s_i(R_i \boldsymbol{w}, R_i \boldsymbol{v}) = s(\boldsymbol{w}, \boldsymbol{v}) \qquad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{W} \ . \tag{4.3}$$

We also need the following definition (see for example Section 6.2.1 of Ref. [35]). Given any node $x \in \Gamma$, define $N_x = \# \{ j \in \mathbb{N} \mid x \in \partial\Omega_j \}$. Then, the weighted counting operators $\delta_i : \overline{\boldsymbol{W}}_i \longrightarrow \overline{\boldsymbol{W}}_i$ (and their inverse operators $\delta_i^\dagger$) are defined by

$$\begin{aligned} \delta_i \boldsymbol{v}_i \, (x) &= N_x \boldsymbol{v}_i(x) \quad \forall x \text{ node of } \Gamma_i \cap \Gamma \\ \delta_i^\dagger \boldsymbol{v}_i \, (x) &= N_x^{-1} \boldsymbol{v}_i(x) \quad \forall x \text{ node of } \Gamma_i \cap \Gamma \ . \end{aligned} \tag{4.4}$$

Let the local constraint operators $C_i : \overline{\boldsymbol{W}}_i \to \mathbb{R}^{3cc_i}$ be the operators that read the function values at the corners of the subdomain Ω_i, with cc_i the number of corners of the subdomain. Then the local constrained spaces are

$$\boldsymbol{W}_i = \{ \boldsymbol{w}_i \in \overline{\boldsymbol{W}}_i \mid C_i \boldsymbol{w}_i = \boldsymbol{0} \} \ . \tag{4.5}$$

We will moreover introduce a global coarse space $\boldsymbol{W}_0 \subset \boldsymbol{W}$, whose degrees of freedom are the function values at the subdomain corner nodes. Given the number m of such subdomain corners, let $w_c \in \mathbb{R}^{3m}$ be a vector representing the respective nodal values. Then the space $\boldsymbol{W}_0$ is defined by

$$\boldsymbol{W}_0 = \{ \sum_{i=1}^{N} R_i^T \delta_i^\dagger \boldsymbol{w}_{0,i} \mid C_i \boldsymbol{w}_{0,i} = R_i^C w_c, w_c \in \mathbb{R}^{3m}, s_i(\boldsymbol{w}_{0,i}, \boldsymbol{w}_{0,i}) \to \min \}, \tag{4.6}$$

where R_i^C is the operator that extracts the corner node values for the subdomain Ω_i from the global vector w_c of all the subdomain corner node values.

Any element $\boldsymbol{w} \in \boldsymbol{W}$ can be uniquely decomposed as

$$\boldsymbol{w} = \boldsymbol{w}_0 + \sum_{i=1}^{N} \boldsymbol{w}_i \ , \quad \boldsymbol{w}_0 \in \boldsymbol{W}_0 \ , \ \boldsymbol{w}_i \in \boldsymbol{W}_i \text{ for } i = 1, .., N \ . \tag{4.7}$$

Let the inexact bilinear forms, see (4.6), be defined by

$$\begin{aligned} \tilde{s}_0(\boldsymbol{w}_0, \boldsymbol{v}_0) &= \sum_{i=1}^{N} s_i(\boldsymbol{w}_{0,i}, \boldsymbol{v}_{0,i}) \quad \forall \boldsymbol{w}_0, \boldsymbol{v}_0 \in \boldsymbol{W}_0 \ , \\ \tilde{s}_i(\boldsymbol{w}_i, \boldsymbol{v}_i) &= s_i(\delta_i \boldsymbol{w}_i, \delta_i \boldsymbol{v}_i) \quad \forall \boldsymbol{w}_i, \boldsymbol{v}_i \in \boldsymbol{W}_i, \ i = 1, 2, .., N \ . \end{aligned} \tag{4.8}$$

Finally, we define the coarse operator $P_0 : \boldsymbol{W} \longrightarrow \boldsymbol{W}_0$ by

$$\tilde{s}_0(P_0 \boldsymbol{u}, \boldsymbol{v}_0) = s(\boldsymbol{u}, \boldsymbol{v}_0) \qquad \forall \, \boldsymbol{v}_0 \in \boldsymbol{W}_0, \tag{4.9}$$

and the local operators $P_i = R_i^T \tilde{P}_i : \boldsymbol{W} \longrightarrow R_i^T \boldsymbol{W}_i$ by

$$\tilde{s}_i(\tilde{P}_i \boldsymbol{u}, \boldsymbol{v}_i) = s(\boldsymbol{u}, R_i^T \boldsymbol{v}_i) \qquad \forall \, \boldsymbol{v}_i \in \boldsymbol{W}_i. \tag{4.10}$$

Then, our BDDC method is defined by the preconditioned operator

$$P = \sum_{i=0}^{N} P_i \ . \tag{4.11}$$

5. Quasi-uniformity of the BDDC preconditioned operator. In this section, we bound the condition number of the preconditioned BDDC operator P introduced in [11] and here described in (4.11). The results shown here are an improvement of those presented in [11], where a scalability result was proven, but not a quasi-uniformity result. Indeed, the bound shown in [11] exhibits a bound of the type $\sim C\,H/h$, while here we can show $\sim C(1+\log^3(h/H))$. Nevertheless, it must be noted that the ideas used here are very similar to, and strongly those, from those already present in [11].

We start by briefly re-stating the general results for BDDC preconditioners of Ref. [29] stated in the same setting and notations of Ref. [11, 35]. The main part lays in Section 5.1, where we will address the following fundamental assumption for the MITC plate bending elements.

Assumption 5.1. *Given any Γ_i, $i = 1, 2, .., N$, let $\mathcal{E}_i$ represent the set of the edges of Γ_i. Then, we assume that there exist two positive constants k_*, k^* and a boundary seminorm $|\cdot|_{\tau(\Gamma_i)}$ on $\overline{\boldsymbol{W}}_i$, $i = 1, 2, .., N$, such that*

$$|\boldsymbol{w}_i|^2_{\tau(\Gamma_i)} \leq k^* s_i(\boldsymbol{w}_i) \quad \forall \boldsymbol{w}_i \in \overline{\boldsymbol{W}}_i \ ; \tag{5.1}$$

$$|\boldsymbol{w}_i|^2_{\tau(\Gamma_i)} \geq k_* s_i(\boldsymbol{w}_i) \quad \forall \boldsymbol{w}_i \in \boldsymbol{W}_i \ ; \tag{5.2}$$

$$|\boldsymbol{w}_i|^2_{\tau(\Gamma_i)} = \sum_{e \in \mathcal{E}_i} |\boldsymbol{w}_i|^2_{\tau(e)} \quad \forall \boldsymbol{w}_i \in \overline{\boldsymbol{W}}_i \ , \tag{5.3}$$

where $|\cdot|_{\tau(e)}$ is a given seminorm on the edge e.

We notice that we cannot adopt the obvious choice $|\boldsymbol{w}_i|_{\tau(\Gamma_i)} = s_i(\boldsymbol{w}_i)$, since it can be shown that it does not satisfy (5.3), not even with a bound up to a uniform constant. We have the following results.

Theorem 5.1. *If Assumption 5.1 holds, then the condition number of the preconditioned operator P in (4.11) satisfies the bound*

$$Cond(P) \leq C(1 + 5k_*^{-1}k^*).$$

The proof of the above result can be found in [29] but also, in a different form and with a notation more consistent to that of the present paper, in [11].

Theorem 5.2. *The constants k_* and k^* of Assumption 5.1 are bounded by*

$$k^* \leq C_1, \qquad k_*^{-1} \leq C_2(1 + \log^3(H/h)),$$

with constants C_1, C_2 depending only on the material constants and mesh regularity and not on the plate thickness t. Therefore, we have the convergence rate bound

$$Cond(P) \leq C\left(1 + \log^3(H/h)\right),$$

with the constant C depending only on the material constants and mesh regularity, and not on the plate thickness t.

The proof of this result is given in the following Section 5.1.

Remark 5.1. *An extended set of numerical tests, also including jump in the coefficients, which are in complete accordance with Theorem 5.2 can be found in [11].*

5.1. Proof of Assumption 5.1. In this section we prove that Assumption 5.1 holds for the MITC plate bending problem (2.4), and show the respective bounds for the constants k_*, k^*.

The local spaces $\overline{\boldsymbol{W}}_i$, $i = 1, 2, .., N$, are composed by rotation and deflection parts, which we indicate in the sequel as

$$\overline{\boldsymbol{W}}_i = \overline{\boldsymbol{\Theta}}_i \times \overline{U}_i.$$

Accordingly, we indicate the rotation and deflection parts of the constrained space by

$$\boldsymbol{W}_i = \boldsymbol{\Theta}_i \times U_i \ ,$$

where the functions of $\boldsymbol{\Theta}_i$ and U_i vanish at the subdomain corner nodes. In the sequel, given any $\boldsymbol{w}_i = (u_i, \boldsymbol{\theta}_i) \in \overline{\boldsymbol{W}}_i$, we will indicate with $\overline{\mathcal{H}}_i\boldsymbol{\theta}_i$ the rotation part of its energy-harmonic extension $\overline{\mathcal{H}}_i\boldsymbol{w}_i$ defined in (4.1). Similarly, $\overline{\mathcal{H}}_i u_i$ will represent the deflection part.

Proof of the upper bound (5.1). The proof of this bound is identic to that of [11] and is included here only for completeness. We start by defining the following edge seminorm on the rotation part

$$|\boldsymbol{\theta}_i|_{\gamma(e)} := \inf_{\boldsymbol{\psi}\in[H^1(\Omega_i)]^2, \boldsymbol{\psi}|_e=\boldsymbol{\theta}_i|_e} ||\boldsymbol{\varepsilon}(\boldsymbol{\psi})||_{L^2(\Omega_i)} \tag{5.4}$$

for all $e \in \mathcal{E}_i$. Note that, simply restricting the choice in the infimum and since the number of edges of each subdomain is finite, it holds

$$\sum_{e\in\mathcal{E}_i} |\boldsymbol{\theta}_i|^2_{\gamma(e)} \leq C \inf_{\boldsymbol{\psi}\in[H^1(\Omega_i)]^2, \boldsymbol{\psi}|_{\Gamma_i}=\boldsymbol{\theta}_i} ||\boldsymbol{\varepsilon}(\boldsymbol{\psi})||^2_{L^2(\Omega_i)} \ . \tag{5.5}$$

We can now introduce the following seminorm on the space $\overline{\boldsymbol{W}}_i$:

$$\begin{aligned} |\boldsymbol{w}_i|^2_{\tau(\Gamma_i)} &= \sum_{e\in\mathcal{E}_i} |\boldsymbol{w}_i|^2_{\tau(e)} \quad \forall \boldsymbol{w}_i = (\boldsymbol{\theta}_i, u_i) \in \overline{\boldsymbol{W}}_i \ , \\ |\boldsymbol{w}_i|^2_{\tau(e)} &= |\boldsymbol{\theta}_i|^2_{\gamma(e)} + ht^{-2}||\Pi\, \boldsymbol{\theta}_i \cdot \boldsymbol{\tau} - u_i'||^2_{L^2(e)} \ , \end{aligned} \tag{5.6}$$

where $\boldsymbol{\tau}$ is the tangent unit vector at the boundary and the apex indicates as usual the derivative, in the direction of $\boldsymbol{\tau}$, for functions defined on the (one dimensional) boundary. Note that due to property $(P6)$, the operator Π is well defined also when restricted on boundary edges. Norm (5.6) clearly satisfies (5.3) by definition. We will now show the remaining two properties. Consider $\boldsymbol{w}_i = (\boldsymbol{\theta}_i, u_i) \in \overline{\boldsymbol{W}}_i$. Using bound (5.5) with the choice $\boldsymbol{\psi} = \overline{\mathcal{H}}_i\boldsymbol{\theta}_i$, it follows

$$\sum_{e\in\mathcal{E}_i} |\boldsymbol{\theta}_i|^2_{\gamma(e)} \leq C||\boldsymbol{\varepsilon}(\overline{\mathcal{H}}_i\boldsymbol{\theta}_i)||^2_{L^2(\Omega_i)} \leq 4C\alpha^{-1}\, a_i(\overline{\mathcal{H}}_i\boldsymbol{\theta}_i) \ , \tag{5.7}$$

where the local bilinear forms $a_i(\cdot,\cdot) = a|_{\Omega_i}(\cdot,\cdot)$ and $\alpha = \alpha(E,\nu) > 0$ is the coercivity constant for the elastic moduli $\mathbb{C}$.

First recalling $(P6)$ and the definition of gradient, then using the Agmon inequality (see [1, 2]) and an inverse estimate, we get

$$\begin{aligned} &\sum_{e\in\mathcal{E}_i} ht^{-2}||\Pi\, \boldsymbol{\theta}_i \cdot \boldsymbol{\tau} - u_i'||^2_{L^2(e)} = \sum_{e\in\mathcal{E}_i} ht^{-2}||\, (\Pi\, \overline{\mathcal{H}}_i\boldsymbol{\theta}_i - \boldsymbol{\nabla}\overline{\mathcal{H}}_i u_i)|_e \cdot \boldsymbol{\tau}||^2_{L^2(e)} \\ &\leq Ct^{-2}||\Pi\, \overline{\mathcal{H}}_i\boldsymbol{\theta}_i - \boldsymbol{\nabla}\overline{\mathcal{H}}_i u_i||^2_{L^2(\Omega_i)} \ , \end{aligned} \tag{5.8}$$

where the constant C depends only on the mesh shape regularity. Finally, combining (5.7) and (5.8) with definition (5.6) we get

$$\begin{aligned} |\boldsymbol{w}_i|^2_{\tau(\Gamma_i)} &\leq 4C\alpha^{-1}\, a_i(\overline{\mathcal{H}}_i\boldsymbol{\theta}_i) + Ct^{-2}||\Pi\, \overline{\mathcal{H}}_i\boldsymbol{\theta}_i - \boldsymbol{\nabla}\overline{\mathcal{H}}_i u_i||^2_{L^2(\Omega_i)} \\ &\leq k^* b_i(\overline{\mathcal{H}}_i\boldsymbol{w}_i) = k^* s_i(\boldsymbol{w}_i) \ , \end{aligned} \tag{5.9}$$

where k^* is a new constant depending only on the mesh regularity and material parameters. Assumption (5.1) is proved.

Proof of the lower bound (5.2). Assumption (5.2) is without doubt the more involved; this is where the present contribution constitutes an improvement with respect to [11].

We start recalling that the definition of edge bubble functions was given at the end of Section 2.1. Consider now $\boldsymbol{w}_i = (\boldsymbol{\theta}_i, u_i) \in \boldsymbol{W}_i$. We introduce the following splitting of the rotation variable. Let

$$\boldsymbol{\theta}_i^2 \in \boldsymbol{\Theta}_i^2 := \text{span}\{B_l^i \boldsymbol{\tau}\}_{l \in \Gamma_i},$$

be defined by

$$\int_l \boldsymbol{\theta}_i^2 \cdot \boldsymbol{\tau} = \int_l \boldsymbol{\theta}_i \cdot \boldsymbol{\tau} - u_i' \quad \forall l \in \Gamma_i,$$

and let $\boldsymbol{\theta}_i^1 = \boldsymbol{\theta}_i - \boldsymbol{\theta}_i^2$ so that $\boldsymbol{\theta}_i = \boldsymbol{\theta}_i^1 + \boldsymbol{\theta}_i^2$. Note that, by construction, it holds

$$\int_l u_i' - \boldsymbol{\theta}_i^1 \cdot \boldsymbol{\tau} = 0 \quad \forall e \in \Gamma_i. \tag{5.10}$$

We introduce also the related splitting of $\boldsymbol{w}_i$

$$\boldsymbol{w}_i = \boldsymbol{w}_i^1 + \boldsymbol{w}_i^2 \,, \qquad \boldsymbol{w}_i^1 = (u_i, \boldsymbol{\theta}_i^1),\ \boldsymbol{w}_i^2 = (0, \boldsymbol{\theta}_i^2).$$

We have the following lemma.

LEMMA 5.3. *It exists a constant $C > 0$ independent of h such that for all edges e of all subdomains Ω_i*

$$||\boldsymbol{w}_i||_{\tau(e)} = ||(u_i, \boldsymbol{\theta}_i)||_{\tau(e)} \geq C\big(||(u_i, \boldsymbol{\theta}_i^1)||_{\tau(e)} + ||(0, \boldsymbol{\theta}_i^2)||_{\tau(e)}\big).$$

Proof. It is clearly sufficient to show

$$||(0, \boldsymbol{\theta}_i^2)||_{\tau(e)} \leq C||\boldsymbol{w}_i||_{\tau(e)} \tag{5.11}$$

for some (possibly different) constant C. From the definition of $\boldsymbol{\theta}_i^2$ and Remark 2.1 it follows immediately

$$\int_l \boldsymbol{\theta}_i^2 \cdot \boldsymbol{\tau} = \int_l \Pi\, \boldsymbol{\theta}_i \cdot \boldsymbol{\tau} - u_i' \quad \forall l \in \Gamma_i. \tag{5.12}$$

For every face $l \in \Gamma_i$, since $\boldsymbol{\theta}_i^2|_l = c\, B_l^i|_l \boldsymbol{\tau}$ for some $c \in \mathbb{R}$, a scaling argument and bound (5.12) easily give

$$\begin{aligned} ||\boldsymbol{\theta}_i^2||_{L^2(l)} &\leq C||\frac{1}{|l|}\int_l \boldsymbol{\theta}_i^2 \cdot \boldsymbol{\tau}||_{L^2(l)} = C||\frac{1}{|l|}\int_l u_i' - \Pi\, \boldsymbol{\theta}_i \cdot \boldsymbol{\tau}||_{L^2(l)} \\ &\leq C||u_i' - \Pi\, \boldsymbol{\theta}_i \cdot \boldsymbol{\tau}||_{L^2(l)}. \end{aligned} \tag{5.13}$$

By an inverse inequality and recalling Remark 2.1 we obtain

$$\begin{aligned} |(0, \boldsymbol{\theta}_i^2)|^2_{\tau(e)} &= |\boldsymbol{\theta}_i^2|^2_{\gamma(e)} + ht^{-2}||\Pi\, \boldsymbol{\theta}_i^2 \cdot \boldsymbol{\tau}||^2_{L^2(e)} \leq C\big(h^{-1}||\boldsymbol{\theta}_i^2||^2_{L^2(e)} + ht^{-2}||\boldsymbol{\theta}_i^2||^2_{L^2(e)}\big) \\ &\leq Cht^{-2}||\boldsymbol{\theta}_i^2||^2_{L^2(e)}. \end{aligned} \tag{5.14}$$

Bound (5.11) follows combining equations (5.13) and (5.14). ☐

We have also the following result.

LEMMA 5.4. *It exists a constant $C > 0$ independent of h such that*

$$s_i(\boldsymbol{w}_i^2) \leq C\, |\boldsymbol{w}_i^2|^2_{\tau(\Gamma_i)}.$$

Proof. Let $\widehat{\boldsymbol{\theta}}$ be the extension of $\boldsymbol{\theta}_i^2$ inside Ω_i obtained by setting all the internal degrees of freedom to zero. Then, the couple $(\widehat{\boldsymbol{\theta}}, 0)$, restricted on the boundary Γ_i is equal to $\boldsymbol{w}_i^2 = (\boldsymbol{\theta}_i^2, 0)$. As a consequence, by definition of energy-harmonic extension, it holds

$$\begin{aligned} s_i(\boldsymbol{w}_i^2) &= b_i(\overline{\mathcal{H}}\boldsymbol{w}_i^2, \overline{\mathcal{H}}\boldsymbol{w}_i^2) \leq b_i(\, (\widehat{\boldsymbol{\theta}}, 0), (\widehat{\boldsymbol{\theta}}, 0)\,) \\ &= ||\boldsymbol{\varepsilon}(\widehat{\boldsymbol{\theta}})||^2_{L^2(\Omega_i)} + t^{-2}||\Pi\, \widehat{\boldsymbol{\theta}}||^2_{L^2(\Omega_i)}. \end{aligned} \tag{5.15}$$

Since $\widehat{\boldsymbol{\theta}}$ is obtained from $\boldsymbol{\theta}_i^2$ setting all the internal degrees of freedom to zero, with a scaling argument applied to (5.15) it is easy to check

$$s_i(\boldsymbol{w}_i^2) \leq C\big(h^{-1}||\boldsymbol{\theta}_i^2||^2_{L^2(\Gamma_i)} + ht^{-2}||\boldsymbol{\theta}_i^2||^2_{L^2(\Gamma_i)}\big). \tag{5.16}$$

Recalling Remark 2.1 and with an argument similar to that used in (5.13), we get from (5.16)

$$\begin{aligned} s_i(\boldsymbol{w}_i^2) &\leq Cht^{-2}||\boldsymbol{\theta}_i^2||^2_{L^2(\Gamma_i)} = Cht^{-2}||\boldsymbol{\theta}_i^2 \cdot \boldsymbol{\tau}||^2_{L^2(\Gamma_i)} \\ &\leq Cht^{-2}||\Pi\, \boldsymbol{\theta}_i^2 \cdot \boldsymbol{\tau}||^2_{L^2(\Gamma_i)} \leq C|(0, \boldsymbol{\theta}_i^2)|^2_{\tau(\Gamma_i)} = |\boldsymbol{w}_i^2|^2_{\tau(\Gamma_i)}. \end{aligned} \tag{5.17}$$

☐

We now show the following result.

PROPOSITION 5.5. *It exists a constant $C > 0$ independent of h such that*

$$s_i(\boldsymbol{w}_i^1) \leq C\,(1 + \log^3(1 + h/H))\, |\boldsymbol{w}_i^1|^2_{\tau(\Gamma_i)}.$$

Proof. In the sequel, Q_i will indicate the restriction of the auxiliary space Q, introduced in (P5), to the domain Ω_i. We start by solving the following rotated Stokes problem

$$\begin{cases} \text{Find } \tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}|_{\Omega_i}, p \in Q_i/\mathbb{R} \text{ s.t.} \\ (\boldsymbol{\nabla}\tilde{\boldsymbol{\theta}}, \boldsymbol{\nabla}\tilde{\boldsymbol{\eta}}) + (p, \operatorname{curl}\boldsymbol{\eta}) = 0 \qquad \forall \boldsymbol{\eta} \in \boldsymbol{\Theta}|_{\Omega_i} \cap [H_0^1(\Omega_i)]^2 \\ (\operatorname{curl}\tilde{\boldsymbol{\theta}}, q) = 0 \qquad \forall q \in Q_i/\mathbb{R} \\ \tilde{\boldsymbol{\theta}} = \boldsymbol{\theta}_i^1 \qquad \text{on } \Gamma_i\ . \end{cases} \tag{5.18}$$

Due to the stability property ($P5$), problem (5.18) has a unique solution and, using standard techniques, it can be shown that

$$|\tilde{\boldsymbol{\theta}}|_{H^1(\Omega_i)} \leq C|\boldsymbol{\theta}_i^1|_{H^{1/2}(\Gamma_i)}\ . \tag{5.19}$$

Using an integration by parts, (5.18) and recalling the definition of $\boldsymbol{\theta}_i^1$ yields

$$\int_{\Omega_i} \operatorname{curl}\tilde{\boldsymbol{\theta}} = \int_{\Gamma_i} \boldsymbol{\theta}_i^1 \cdot \boldsymbol{\tau} = \int_{\Gamma_i} u_i' = 0\ . \tag{5.20}$$

As a consequence of the second identity in problem (5.18) and due to (5.20) one has

$$(\mathrm{curl}\,\tilde{\boldsymbol{\theta}}, q) = 0 \quad \forall q \in Q_i, \tag{5.21}$$

i.e.

$$P\,\mathrm{curl}\,\tilde{\boldsymbol{\theta}} = 0\ , \tag{5.22}$$

where the projection operator P was defined in property $(P3)$.

Combining (5.22) with property $(P3)$ we get $\mathrm{curl}\,\Pi\tilde{\boldsymbol{\theta}} = 0$, which, due to (P4), gives the existence of a function $\Psi \in U|_{\Omega_i}$ such that

$$\Pi(\tilde{\boldsymbol{\theta}}) = \nabla\Psi\ . \tag{5.23}$$

We now introduce the additional problem

$$\begin{cases} \text{Find } \tilde{u} \in U|_{\Omega_i} \text{ s.t.} \\ (\nabla\tilde{u} - \nabla\Psi, \nabla v) = 0 \qquad \forall v \in U|_{\Omega_i} \cap H_0^1(\Omega_i) \\ \tilde{u} = u_i \qquad \text{on } \Gamma_i\ . \end{cases} \tag{5.24}$$

Using identity (5.23) we obtain

$$||\Pi\tilde{\boldsymbol{\theta}} - \nabla\tilde{u}||_{L^2(\Omega_i)} = ||\nabla\Psi - \nabla\tilde{u}||_{L^2(\Omega_i)} = |\Psi - \tilde{u}|_{H^1(\Omega_i)}\ . \tag{5.25}$$

Note that by definition (5.24) it holds

$$\tilde{u} - \Psi = \mathcal{H}_i(u_i - \Psi|_{\Gamma_i})\ , \tag{5.26}$$

with $\mathcal{H}_i$ the standard harmonic extension in the discrete space $\overline{U}_i$.

First using (5.26) and well known properties of the discrete Harmonic extension (see for instance [35]), then using Lemma 7.1 in [11] and an inverse inequality, we get

$$\begin{aligned} |\Psi - \tilde{u}|^2_{H^1(\Omega_i)} &\le C|(\Psi|_{\Gamma_i}) - u_i|^2_{H^{1/2}(\Gamma_i)} \le C\big(1 + \log^2(H/h)\big) \sum_{e\in\mathcal{E}_i} |(\Psi|_{\Gamma_i}) - u_i|^2_{H^{1/2}(e)} \\ &\le C\big(1 + \log^2(H/h)\big)\ h^{-1} \sum_{e\in\mathcal{E}_i} ||(\Psi|_{\Gamma_i}) - u_i||^2_{L^2(e)}\ . \end{aligned} \tag{5.27}$$

We now observe that, for all edges l in Γ_i, due to the definition of $\boldsymbol{\theta}_i^1$ and Remark 2.1 it holds

$$\int_l (\Psi|_{\Gamma_i} - u_i)' = \int_l \nabla\Psi|_{\Gamma_i} \cdot \boldsymbol{\tau} - u_i' = \int_l \Pi\,\tilde{\boldsymbol{\theta}}\,|_{\Gamma_i} \cdot \boldsymbol{\tau} - u_i' = \int_l \boldsymbol{\theta}_i^1 \cdot \boldsymbol{\tau} - u_i' = 0.$$

As a consequence, it is immediate to check that $\Psi(\nu) = u_i(\nu)$ for all points ν in Γ_i which are vertexes of the mesh τ_h. Therefore the continuous and piecewise linear nodal interpolant (living on Γ_i) to the function $\Psi|_{\Gamma_i} - u_i$ is null. A standard interpolation result then gives, for all $e \in \mathcal{E}_i$,

$$\begin{aligned} h^{-1}||(\Psi|_{\Gamma_i}) - u_i||^2_{L^2(e)} &\le Ch|(\Psi|_{\Gamma_i}) - u_i|^2_{H^1(e)} = Ch||(\Psi|_{\Gamma_i} - u_i)'||^2_{L^2(e)} \\ &= Ch||\Pi\,\boldsymbol{\theta}_i^1 \cdot \boldsymbol{\tau} - u_i'||^2_{L^2(e)}\ . \end{aligned} \tag{5.28}$$

Combining (5.25) with (5.27) and (5.28) yields

$$||\Pi\tilde{\boldsymbol{\theta}} - \nabla\tilde{u}||^2_{L^2(\Omega_i)} \le C\big(1 + \log^2(H/h)\big)\ h \sum_{e\in\mathcal{E}_i} ||\Pi\,\boldsymbol{\theta}_i^1 \cdot \boldsymbol{\tau} - u_i'||^2_{L^2(e)}, \tag{5.29}$$

which in turn finally gives

$$\begin{aligned} t^{-2}||\Pi\tilde{\boldsymbol{\theta}} - \nabla\tilde{u}||^2_{L^2(\Omega_i)} &\le C\big(1+\log^2(H/h)\big)\ ht^{-2}\sum_{e\in\mathcal{E}_i}||\Pi\,\boldsymbol{\theta}_i^1\cdot\boldsymbol{\tau} - u_i'||^2_{L^2(e)} \\ &= C\big(1+\log^2(H/h)\big)\ ht^{-2}||\Pi\,\boldsymbol{\theta}_i^1\cdot\boldsymbol{\tau} - u_i'||^2_{L^2(\Gamma_i)}. \end{aligned} \tag{5.30}$$

We are now ready to bound $s_i(\boldsymbol{w}_i^1)$. By definition, the local energy harmonic extension $\overline{\mathcal{H}}_i\boldsymbol{w}_i^1 = (\overline{\mathcal{H}}_i\boldsymbol{\theta}_i^1, \overline{\mathcal{H}}_i u_i)$ is given by

$$\begin{cases} \text{Find } \overline{\mathcal{H}}_i(\boldsymbol{w}_i^1)\in\overline{\boldsymbol{W}}_i \text{ such that:} \\ b_i(\overline{\mathcal{H}}_i\boldsymbol{w}_i^1, \boldsymbol{v}) = 0 \qquad \forall\,\boldsymbol{v}\in\mathbf{X}_i \\ \overline{\mathcal{H}}_i\boldsymbol{w}_i^1|_{\Gamma_i} = \boldsymbol{w}_i^1\ , \end{cases} \tag{5.31}$$

where $\mathbf{X}_i$ is defined in (3.1). Let in the sequel $\tilde{\boldsymbol{w}}\in\mathbf{X}|_{\Omega_i}$ be given by $\tilde{\boldsymbol{w}} = (\tilde{\boldsymbol{\theta}}, \tilde{u})$ with $\tilde{\boldsymbol{\theta}}$ defined in (5.18) and $\tilde{u}$ defined in (5.24). Note that, due to the definitions (5.18), (5.24) and (5.31) we have

$$\overline{\mathcal{H}}_i\boldsymbol{w}_i^1 - \tilde{\boldsymbol{w}} = 0 \quad \text{on } \Gamma_i\ .$$

Therefore $(\overline{\mathcal{H}}_i\boldsymbol{w}_i^1 - \tilde{\boldsymbol{w}})\in\mathbf{X}_i$ and, due to (5.31), it satisfies

$$b_i(\overline{\mathcal{H}}_i\boldsymbol{w}_i^1 - \tilde{\boldsymbol{w}}, \boldsymbol{v}) = -b_i(\tilde{\boldsymbol{w}}, \boldsymbol{v}) \qquad \forall\,\boldsymbol{v}\in\mathbf{X}_i\ . \tag{5.32}$$

As a consequence of (5.32) it easily follows

$$b_i(\overline{\mathcal{H}}_i\boldsymbol{w}_i^1 - \tilde{\boldsymbol{w}}) \le b_i(\tilde{\boldsymbol{w}})\ , \tag{5.33}$$

which, recalling the definition of s_i, gives

$$s_i(\boldsymbol{w}_i^1) = b_i(\overline{\mathcal{H}}_i\boldsymbol{w}_i^1) \le 4b_i(\tilde{\boldsymbol{w}})\ . \tag{5.34}$$

Therefore, we need to bound

$$\begin{aligned} b_i(\tilde{\boldsymbol{w}}) &= a_i(\tilde{\boldsymbol{\theta}}) + t^{-2}||\Pi\tilde{\boldsymbol{\theta}} - \nabla\tilde{u}||^2_{L^2(\Omega_i)} \\ &\le C||\boldsymbol{\varepsilon}(\tilde{\boldsymbol{\theta}})||^2_{L^2(\Omega_i)} + t^{-2}||\Pi\tilde{\boldsymbol{\theta}} - \nabla\tilde{u}||^2_{L^2(\Omega_i)}\ . \end{aligned} \tag{5.35}$$

From (5.19) we get

$$||\boldsymbol{\varepsilon}(\tilde{\boldsymbol{\theta}})||^2_{L^2(\Omega_i)} \le |\tilde{\boldsymbol{\theta}}|^2_{H^1(\Omega_i)} \le C|\boldsymbol{\theta}_i^1|_{H^{1/2}(\Gamma_i)}\ . \tag{5.36}$$

Recalling that $\boldsymbol{\theta}_i^1$ vanishes at the subdomain corner nodes, we can apply Lemma 7.1 in [11] and get from (5.36)

$$||\boldsymbol{\varepsilon}(\tilde{\boldsymbol{\theta}})||^2_{L^2(\Omega_i)} \le C(1+\log^2 H/h)\sum_{e\in\mathcal{E}_i}|\boldsymbol{\theta}_i^1|^2_{H^{1/2}(e)}\ . \tag{5.37}$$

Furthermore, using again that $\boldsymbol{\theta}_i^1$ vanishes at the subdomain corner nodes, we combine Lemma 7.2 in [11] and (5.37) in order to obtain

$$||\boldsymbol{\varepsilon}(\tilde{\boldsymbol{\theta}})||^2_{L^2(\Omega_i)} \le C(1+\log^3 H/h)\sum_{e\in\mathcal{E}_i}|\boldsymbol{\theta}_i^1|^2_{\gamma(e)}\ . \tag{5.38}$$

Finally, joining (5.34), (5.35), (5.38), (5.30) and recalling definition (5.6) it follows

$$s_i(\boldsymbol{w}_i^1) \le C\Big(\big(1+\log^3(H/h)\big)\sum_{e\in\mathcal{E}_i}|\boldsymbol{\theta}_i^1|^2_{\gamma(e)} + \big(1+\log^2(H/h)\big)ht^{-2}||\Pi\boldsymbol{\theta}_i^1 - u_i'||^2_{L^2(\Gamma_i)}\Big)$$
$$\le C\,\big(1+\log^3(H/h)\big)\,|\boldsymbol{w}_i^1|^2_{\tau(\Gamma_i)}\ . \qquad (5.39)$$

□

The upper bound now follows combining the three previous results. Indeed, first recalling the splitting $\boldsymbol{w}_i = \boldsymbol{w}_i^1 + \boldsymbol{w}_i^2$ and using a triangle inequality, then applying Lemma 5.4 and Proposition 5.5, finally using Lemma 5.3 yields

$$s_i(\boldsymbol{w}_i) \le 2\Big(s_i(\boldsymbol{w}_i^1) + s_i(\boldsymbol{w}_i^2)\Big) \le C\Big((1+\log^3(1+h/H))\,|\boldsymbol{w}_i^1|^2_{\tau(\Gamma_i)} + |\boldsymbol{w}_i^2|^2_{\tau(\Gamma_i)}\Big)$$
$$\le C(1+\log^3(1+h/H))|\boldsymbol{w}_i|^2_{\tau(\Gamma_i)}.$$

Bound (5.2) is therefore proved with

$$k_*^{-1} = C\,(1+\log^3 H/h),$$

with the constant C depending only on the material constants and mesh regularity.

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