Exercises in Numerics of Differential Equations

5th / 7th June 2019

Exercise 1. Consider an $m$-stage Runge–Kutta method with Butcher tableau $\begin{pmatrix} c \\ A \\ b \end{pmatrix}$. In the lecture, we have formulated the integrator via increments $k_j$, i.e.,

$$y_{\ell+1} = y_{\ell} + h \sum_{j=1}^{m} b_j k_j,$$

where $k_i = f(t_\ell + c_i h, y_{\ell} + h \sum_{j=1}^{m} A_{ij} k_j)$ for all $i = 1, \ldots, m$.

Equivalently, one can formulate Runge–Kutta methods with stages $Y_j$, i.e.,

$$y_{\ell+1} = y_{\ell} + h \sum_{j=1}^{m} b_j f(t_\ell + c_j h, Y_j),$$

where $Y_i = y_{\ell} + h \sum_{j=1}^{m} A_{ij} f(t_\ell + c_j h, Y_j)$. (1)

Show that both approaches lead to the same method (under the usual assumptions on $f$).

Exercise 2. Consider an $m$-stage Runge–Kutta method with Butcher tableau $\begin{pmatrix} c \\ A \\ b \end{pmatrix}$ such that the coefficients satisfy

$$b_i A_{ij} + b_j A_{ji} = b_i b_j$$

for all $1 \leq i, j \leq m$.

Show that the integrator preserves quadratic invariants of an autonomous ODE.

Hint. It suffices to consider invariants of the form $I(y) = y^T C y$. Furthermore, use the form (1) for the RK-method.

Exercise 3. Consider the Hamiltonian $H(q, p) = \frac{1}{2} q^2 + \frac{1}{2} p^2$ and the corresponding system

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \partial_p H(q, p) \\ -\partial_q H(q, p) \end{pmatrix}$$

in $[0, T]$, \quad $\begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (2)

Argue that the solution is unique and compute the exact solution of (2). Moreover, consider the explicit Euler method, the implicit Euler method, and the implicit midpoint method, i.e.,

$$y_{\ell+1} = y_\ell + h f(y_\ell), \quad y_{\ell+1} = y_\ell + h f(y_{\ell+1}), \quad y_{\ell+1} = y_\ell + h f\left(\frac{y_\ell + y_{\ell+1}}{2}\right).$$

(3)

Show that the first two methods do not conserve the discrete energy, whereas the latter does.

To this end, show that for the discrete energy there holds

$$H(q_\ell, p_\ell) - H(q_0, p_0) \begin{cases} \geq 0 & \text{for the explicit Euler method,} \\ \leq 0 & \text{for the implicit Euler method,} \\ = 0 & \text{for the implicit midpoint method.} \end{cases}$$

Hint. For $a, b \in \mathbb{R}$, use the identity

$$\frac{1}{2} (a^2 - b^2) - \frac{1}{2} (a - b)^2 = (a - b)b.$$
Exercise 4. Use the methods from (3) to solve the system (2) numerically. For $h = 0.01$ and $T = 10$, plot the values $q_\ell, p_\ell$ on the $q$-$p$ plane. What do you expect? What do you observe? Furthermore, for varying step-sizes $h$, plot the energy differences and errors

$$\left| H(q_\ell, p_\ell) - H(q(T), p(T)) \right|, \quad \|(q_\ell, p_\ell)^T - (q(T), p(T))^T\|$$

at time $T = 10$ over the step-size $h$. What do you expect? What do you observe?