Exercises in Numerics of Differential Equations

22\textsuperscript{th} / 17\textsuperscript{th} May 2019

Exercise 1. On the time-interval \([t_0, T]\) consider a general \(m\)-step method of the form
\[
\sum_{j=0}^{m} \alpha_j y_{\ell+j} = h \Phi(t_{\ell}, y_{\ell}, \ldots, y_{\ell+m}, h).
\]
(1)

Suppose that \(\Phi\) is Lipschitz continuous in \(y\), i.e., for all \(t_{\ell} \in [t_0, T], h > 0\) and \(y_{\ell+j}, \tilde{y}_{\ell+j} \in \mathbb{R}^n\) it holds that
\[
\|\Phi(t_{\ell}, y_{\ell}, \ldots, y_{\ell+m}, h) - \Phi(t_{\ell}, \tilde{y}_{\ell}, \ldots, \tilde{y}_{\ell+m}, h)\| \leq L \sum_{j=0}^{m} \|y_{\ell+j} - \tilde{y}_{\ell+j}\|.
\]
Show that for sufficiently small \(h > 0\) and arbitrary \(y_{\ell}, \ldots, y_{\ell+m-1} \in \mathbb{R}^n\), the equation (1) has a unique solution \(y_{\ell+m} \in \mathbb{R}^n\).

Exercise 2. For a general \(m\)-step method as given in (1) we define the polynomial
\[
q(\lambda) := \sum_{j=0}^{m} \alpha_j \lambda^j.
\]
Show that the Adams–Bashforth and Adams–Moulton methods from the lecture satisfy the root condition from the convergence theorem, i.e., all solutions \(\lambda_i\) of \(q(\lambda) = 0\) satisfy \(|\lambda_i| \leq 1\) and all \(\lambda_i\) with \(|\lambda_i| = 1\) are simple roots.
Furthermore, show that \(\lambda = 1\) is a root of \(q(\lambda)\), if the \(m\)-step method has consistency order \(p \geq 1\).

Exercise 3. Consider the so-called backwards differentiation formulas (BDF-methods) for the approximate solution of \(y'(t) = f(t, y(t))\): Let \(m \in \mathbb{N}\). Suppose we already have computed approximations \(y_{\ell} \approx y(t_{\ell})\) at points \(t_{\ell} \in [t_0, \ldots, T]\) for \(\ell = 0, \ldots, m-1\). We construct a polynomial \(q(t) \in \mathbb{P}_m\) approximating the exact solution by Lagrange interpolation in the \(m+1\) points \((t_j, y_j)\) for \(j = 0, \ldots, m\), i.e.
\[
q(t) = \sum_{j=0}^{m} y_j L_j(t),
\]
where the \(L_j\) are the Lagrange basis functions. The value \(y_m\) is the sought approximation. To find this value, we additionally ask for
\[
q'(t_m) = f(t_m, y_m).
\]
Show that this defines a linear \(m\)-step method is given, i.e., provide formulas for the coefficients \(\alpha_j\) and the incremental function \(\Phi\). Compute the coefficients of the BDF-methods for \(m = 1, 2, 3\).
Exercise 4. Consider a general explicit linear two-step method of the form
\[ y_{\ell+2} + \alpha_1 y_{\ell+1} + \alpha_0 y_\ell = h [\beta_1 f(t_{\ell+1}, y_{\ell+1}) + \beta_0 f(t_\ell, y_\ell)]. \] (2)
Choose the parameters \( \alpha_0, \alpha_1 \) and \( \beta_0, \beta_1 \) such that the consistency order of (2) is maximal. Implement method (2) to approximate the solution of
\[ y' = -y \quad \text{on } [0, 1], \quad y(0) = 1, \]
where you can set \( y_1 = y(h) = \exp(-h) \). Vary the step-size \( h = 2^{-1}, 2^{-2}, \ldots \). Does this method converge?