Exercises in Numerics of Differential Equations

3rd / 5th April 2019

Exercise 1. Consider an implicit $m$-stage Runge–Kutta method with Butcher tableaux and increment function

$$\phi(t, y, h) = \sum_{j=1}^{m} b_j k_j,$$

respectively. Let further be $f(t, y)$ Lipschitz continuous in the second argument and $H > 0$ sufficiently small. Under these assumptions, show that the RK-method is stable, i.e., there exists a constant $C > 0$ such that for all $t \in [0, T]$, $h \in (0, H)$ and $y, \tilde{y} \in \mathbb{R}^n$ there holds

$$\|\phi(t, y, h) - \phi(t, \tilde{y}, h)\| \leq C \|y - \tilde{y}\|.$$ 

Exercise 2. Consider a Runge–Kutta method of consistency order $p \geq 1$ on a mesh $\Delta = (t_0, \ldots, t_N)$. Under the assumptions of the convergence theorem (in particular, $f$ is Lipschitz continuous in the second argument), we obtain a vector of approximations $y_\ell \approx y(t_\ell)$ satisfying

$$\max_{\ell=0,\ldots,N} \|y(t_\ell) - y_\ell\| = \mathcal{O}(h_\Delta^p).$$

a) Let $y_h$ be the interpolating linear spline (i.e., $y_h$ is a polynomial of degree 1 for every interval $[t_\ell, t_{\ell+1}]$) given by $y_h(t_\ell) = y_\ell$ for all $\ell = 0, \ldots, N$. Show that

$$\|y - y_h\|_\infty = \mathcal{O}(h_\Delta^{\min\{2, p\}}).$$

b) Let $y_h$ be the interpolating cubic spline (i.e., $y_h$ is a polynomial of degree 3 for every interval $[t_\ell, t_{\ell+1}]$) given by piecewise Hermite-Interpolation

$$y_h(t_\ell) = y_\ell, \quad y'_h(t_\ell) = f(t_\ell, y_\ell)$$

for all $\ell = 0, \ldots, N$. Which convergence order do you expect for $\|y - y_h\|_\infty$?

Hint. Use the $\Delta$-piecewise Lagrange, or Hermite interpolant $\tilde{y}(t)$, which approximates the exact solution $y(t)$, in a suitable manner. For b) show further, $\|q\| := |q(t_\ell)| + |q(t_{\ell+1})| + |q'(t_\ell)| + |q'(t_{\ell+1})|$ is a norm on the space of polynomials of degree $\leq 3$ on the interval $[t_\ell, t_{\ell+1}]$ and equivalent to $\|\cdot\|_\infty$. 
Exercise 3. Consider a linear initial value Problem

\[ y'(t) = My(t), \quad y(0) = y_0 \]  

with a matrix \( M \in \mathbb{R}^{n \times n} \). Implement a general solver for this kind of problem, based on implicit Runge–Kutta methods as given in (1). To this end, write a function `linearImplicitRK` that takes as input the Matrix \( M \), a discretization \( \Delta = (t_0, \ldots, t_N) \) of the interval \([0, T]\), the initial value \( y_0 \), and the Butcher tableaux of the implicit RK-method. Your function should return the corresponding vector of approximations \( y_\ell \approx y(t_\ell) \).

To validate your implementation, you might want to consider \( M = \text{diag}(\lambda_1, \ldots, \lambda_n) \), \( y_0 = (d_1, \ldots, d_n)^\top \), and \( y(t) = (d_1 \exp(\lambda_1 t), \ldots, d_n \exp(\lambda_n t))^\top \).

**Hint.** To get an explicit representation of the stages \( k_i \), write them as \( K := (k_1^\top, \ldots, k_m^\top)^\top \in \mathbb{R}^{nm} \). Now formulate the implicit formula for the stages,

\[ k_i = f\left( t + c_i h_\ell, y_\ell + h_\ell \sum_{j=1}^m A_{ij} k_j \right), \]

as implicit equation for the vector \( K \) using matrix-vector multiplication with suitable matrices in \( \mathbb{R}^{nm \times nm} \).

Exercise 4. Implement the embedded Runge–Kutta method of Bogacki and Shampine. This is a scheme for adaptive time-stepping as given in the lecture (see lecture notes chapter 2, page 20) with two Runge–Kutta methods of order 2 and 3, respectively. They have the Butcher tableaux

\[
\begin{array}{c|ccc}
0 & 1/2 & 1/2 \\
1/2 & 0 & 3/4 \\
3/4 & 2/9 & 1/3 & 4/9 \\
1 & 7/24 & 1/4 & 1/3 & 1/8 \\
\end{array}
\]

where the first \( b \)-row gives the method of order 3 and the second the method of order 2.

With this method solve the initial value problem

\[ y'(t) = -200ty^2(t), \quad y(0) = 1, \quad y(t) = \frac{1}{1 + 100t^2}. \]

For different tolerances \( \tau \), plot the solution and the vector of used step-sizes. Finally, plot the error at \( t = 1 \) over the tolerance \( \tau \).