HABILITATIONSSCHRIFT

EFFECTIVE NUMERICAL TREATMENT OF THE LANDAU-LIFSHITZ EQUATION IN RELAXED MICROMAGNETICS

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von

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In der vorliegenden Schrift sind drei wissenschaftliche Arbeiten zusammengefasst:


Diese Schrift liefert Beiträge zur Finite-Elemente-Methode für ein Variationsproblem mit Nebenbedingungen im Bereich des Mikromagnetismus.


ANALYSIS OF THE OPERATOR $\Delta^{-1}\text{div}$ ARISING IN MAGNETIC MODELS

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Abstract. In the context of micromagnetics the partial differential equation

$$\text{div}(-\nabla u + m) = 0 \text{ in } \mathbb{R}^d$$

has to be solved in the entire space for a given magnetization $m : \Omega \to \mathbb{R}^d$ and $\Omega \subseteq \mathbb{R}^d$. For an $L^p$ function $m$ we show that the solution might fail to be in the classical Sobolev space $W^{1,p}(\mathbb{R}^d)$ but has to be in a Beppo-Levi class $W^{1,p}_b(\mathbb{R}^d)$. We prove unique solvability in $W^{1,p}_b(\mathbb{R}^d)$ and provide a direct ansatz to obtain $u$ via a non-local integral operator $L_p$ related to the Newtonian potential. A possible discretization to compute $\nabla (L_2 m)$ is mentioned and it is shown how recently established matrix compression techniques using hierarchical matrices can be applied to the full matrix obtained from the discrete operator.

1. Motivation & Introduction

Let $\Omega \subseteq \mathbb{R}^d$, for $d = 2, 3$, denote the spatial domain of a ferromagnetic body. In the classical model for stationary micromagnetics due to Weiss, Landau, and Lifshitz [3, 11], an energy functional $E$ has to be minimized over an admissible set $A$ of magnetizations $m : \Omega \to \mathbb{R}^d$. The functional $E$ comprises four terms, which are known as exchange energy, anisotropic energy, exterior energy, and stray-field (or magnetostatic) energy,

$$E(m) = \frac{1}{2} \int_\Omega |\nabla m|^2 \, dx + \int_\Omega \phi(m) \, dx - \int_\Omega f \cdot m \, dx + \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx.$$

Here, the exchange parameter $\alpha > 0$ and the anisotropy density $\phi \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ are given. Frequently, $\Omega$ is supposed to be a large body so that the exchange contribution can be neglected, i.e. $\alpha = 0$ in (1.1) [7]. The magnetic potential $u : \mathbb{R}^d \to \mathbb{R}$ and the magnetization $m$ are linked through Maxwell’s equations which imply the partial differential equation

$$\Delta u = \text{div} m$$

in the entire space $\mathbb{R}^d$ [where $m$ is extended by zero outside of $\Omega$]. As usual, Equation (1.2) is treated in the sense of distributions. By definition, we are looking for a weakly differentiable function $u$ which satisfies

$$\langle \nabla u ; \nabla v \rangle = \langle m ; \nabla v \rangle \quad \text{for all } v \in D(\mathbb{R}^d),$$

where $\langle \cdot ; \cdot \rangle$ denotes the scalar product of $L^2(\mathbb{R}^d; \mathbb{R}^d)$ and $D(\mathbb{R}^d)$ the vector space of all $C^\infty$-functions with compact support.

The length $|m|$ of the vector field depends only on the temperature and is therefore usually assumed to be constant. In particular, one has $m \in L^\infty(\Omega; \mathbb{R}^d)$. Thus, it seems to be interesting to investigate the solvability of (1.3) for $m \in L^p(\Omega; \mathbb{R}^d)$.

Key words and phrases. Laplace equation, integral representation, Calderón-Zygmund kernel, micromagnetics, magnetic potential, panel clustering, hierarchical matrices.
For bounded $\Omega$ and $d = 3$ it is well known that, given $m \in L^2(\Omega; \mathbb{R}^d)$, there is a unique solution $u \in H^1(\Omega; \mathbb{R}^d)$ which solves (1.3) [12, 14]. But, for unbounded $\Omega$ or $d = 2$, the solution $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ fails, in general, to be in $L^2(\mathbb{R}^d; \mathbb{R}^d)$. In particular, we show that, for $d = 2$, this is related to the fundamental solution of the Laplacian.

The paper is organized as follows: Section 2 recalls the necessary definitions and classical results applied in the following sections. In Section 3 the Banach spaces $W^p_1(\mathbb{R}^d)$ are introduced and it is shown that for a magnetization $m \in L^2(\mathbb{R}^d, \mathbb{R}^d)$ the Hilbert space $W^p_1(\mathbb{R}^d)$ is the right space to be considered; there is a unique solution $u \in W^p_1(\mathbb{R}^d)$ of (1.3). Section 4 recalls the definition of Calderón-Zygmund kernels and states the main theorem on Calderón-Zygmund convolutions which is applied in Section 5. We show that for $m \in L^p(\mathbb{R}^d; \mathbb{R}^d)$ and $1 < p < \infty$ the potential equation (1.3) has a unique solution $u = \mathcal{L}_p m \in W^p_1(\mathbb{R}^d)$. At least for a magnetization $m \in L^1(\mathbb{R}^d; \mathbb{R}^d) \cap L^p(\mathbb{R}^d; \mathbb{R}^d)$, the potential $\mathcal{L}_p m$ is given as a classical convolution

$$
\mathcal{L}_p m = \sum_{j=1}^d \frac{\partial G}{\partial x_j} * m_j,
$$

where $G$ denotes the Newtonian kernel and $m_j$ is the $j$-th component of $m$. The extension of $\mathcal{L}_p$ defines a continuous linear operator from $L^p(\mathbb{R}^d; \mathbb{R}^d)$ to $W^p_1(\mathbb{R}^d)$. Finally, Section 6 gives the application of the provided results for a Galerkin discretization with piecewise constant ansatz functions in the context of computational micromagnetics. We show how the theory of $H^2$-matrices can be applied to the Galerkin elements to decrease computational cost down to (almost) linear.

2. Preliminaries

For functions $u, v : \mathbb{R}^d \to \mathbb{R}$ we define the convolution $u * v$ of $u$ and $v$ by

$$(u * v)(x) := \int_{\mathbb{R}^d} u(x - y)v(y) \, dy \quad \text{for all } x \in \mathbb{R}^d,$$

whenever the integral exists. As usual in the context of convolutions, functions $w : \Omega \to \mathbb{R}$, for $\Omega \subseteq \mathbb{R}^d$, are identified with their trivial extension $w : \mathbb{R}^d \to \mathbb{R}$, i.e. $w(x) := 0$ for $x \in \mathbb{R}^d \setminus \Omega$. We summarize some well-known facts about convolutions [17, 20, 21, 22].

**Proposition 2.1.** (i) For $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$, the convolution $u * v$ of $u \in L^p(\mathbb{R}^d)$ and $v \in L^q(\mathbb{R}^d)$ satisfies

$$
\|u * v\|_{L^r(\mathbb{R}^d)} \leq \|u\|_{L^p(\mathbb{R}^d)} \|v\|_{L^q(\mathbb{R}^d)}.
$$

(ii) For $q = p/(p - 1) =: p'$, the convolution $u * v \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is uniformly continuous. Further, $1 < p < \infty$ and $q = p'$ imply $u * v \in C^0(\mathbb{R}^d)$, i.e. $u * v$ vanishes at infinity.

(iii) For $k \in \mathbb{N}$, $u \in L^p_{\text{loc}}(\mathbb{R}^d)$, $v \in C^k_c(\mathbb{R}^d)$, and multi-indices $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq k$, we have

$$
\frac{\partial^\alpha (u * v)}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} = u \ast (\partial^\alpha v).
$$

For $d \geq 2$, we define the Newtonian kernel $G : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ by

$$
G(x) := \begin{cases}
\frac{1}{|x|^d} \log |x| & \text{for } d = 2, \\
\frac{1}{(2 - d)\gamma_d} |x|^{2-d} & \text{for } d > 2,
\end{cases}
$$

(2.1)
where $\gamma_d = |\partial B(0, 1)|$ denotes the surface measure of the unit sphere, in particular $\gamma_2 = 2\pi$, $\gamma_3 = 4\pi$. The Newtonian kernel is the fundamental solution of the Laplacian, i.e. we have the following well-known proposition [8].

**Proposition 2.2.** For any test function $f \in D(\mathbb{R}^d)$ the Newtonian potential $w := G*f$ satisfies $w \in C^\infty(\mathbb{R}^d)$ with $\Delta w = f$ in $\mathbb{R}^d$. Moreover, for the partial derivatives of $w$ it holds that $\partial w/\partial x_j = (\partial G/\partial x_j) * f = G * (\partial f/\partial x_j)$. \hfill \square

**Corollary 2.3.** For any smooth magnetization $m = (m_1, \ldots, m_d) \in D(\mathbb{R}^d; \mathbb{R}^d)$, a solution of Equation (1.2) is given by $u := G * (\text{div } m) = \sum_{j=1}^d (\partial G/\partial x_j) * m_j$. \hfill \square

For later use, we need the standard notation of Sobolev and Lebesgue spaces. For $1 \leq p < \infty$ and $\omega \subseteq \mathbb{R}^d$ an open set, we denote with $L^p(\omega)$ the Banach space of all measurable functions whose absolute value to the power $p$ is integrable. The inner product of the Hilbert space $L^2(\omega)$ is given by

\begin{equation}
\langle u ; v \rangle := \int_\omega u(x)v(x) \, dx \quad \text{for all } u, v \in L^2(\omega).
\end{equation}

For $p = \infty$, $L^\infty(\omega)$ denotes the Banach space of all measurable functions which are essentially bounded. For $n \in \mathbb{N}_0$, the classical Sobolev space $W^{n,p}(\omega)$ consists of all functions $u : \omega \to \mathbb{R}$ which are $n$ times weakly differentiable and whose (weak) derivatives of order $|\alpha| \leq n$ satisfy $\partial^\alpha u \in L^p(\omega)$. The norm on $W^{n,p}(\omega)$ is given by

\begin{equation}
\|u\|_{W^{n,p}(\omega)} := \left( \sum_{|\alpha| \leq n} \|\partial^\alpha u\|^p_{L^p(\omega)} \right)^{1/p}, \quad \|u\|_{W^{n,\infty}(\omega)} := \max_{|\alpha| \leq n} \|\partial^\alpha u\|_{L^\infty(\omega)}.
\end{equation}

In this sense we have $L^p(\omega) = W^{0,p}(\omega)$ and it is quite common to denote the Hilbert space $W^{n,2}(\omega)$ by $H^n(\omega)$. The subspace $W^{0,p}_0(\omega)$ is the completion of the test functions $D(\omega)$ with respect to $\|\cdot\|_{W^{0,p}(\omega)}$. The subscript $\text{loc}$, e.g. in $W^{n,p}_\text{loc}(\omega)$, indicates that $u \in W^{n,p}_\text{loc}(\omega)$ satisfies $u \in W^{n,p}(K)$ for all compact subsets $K \subseteq \omega$. We write $W^{n,p}(\omega; \mathbb{R}^d)$ whenever we are dealing with vector valued functions. Finally, we point out that, for any $1 \leq p \leq \infty$, the conjugate index is denoted with $p' := p/(p-1) \in [1, \infty]$. Further, $B(x, \varepsilon)$ denotes the closed ball with radius $\varepsilon > 0$ and center $x \in \mathbb{R}^d$. As usual, $|\cdot|$ denotes both the absolute value of a scalar $\lambda \in \mathbb{R}$ and the (Euclidean) norm of a vector $x \in \mathbb{R}^d$, respectively. The scalar product of two vectors $x, y \in \mathbb{R}^d$ is written as $x \cdot y$.

3. **The Banach Spaces $W^{p}_0(\mathbb{R}^d)$**

For $1 \leq p \leq \infty$, we define the vector space [15, 13]

\begin{equation}
L^p_\text{loc}(\mathbb{R}^d; \mathbb{R}^d) := \{ f \in L^p(\mathbb{R}^d; \mathbb{R}^d) \mid \exists u \in W^{1,p}_\text{loc}(\mathbb{R}^d), \nabla u = f \}
\end{equation}

of all $L^p$ functions which are weak gradients.

**Lemma 3.1.** $L^p_\text{loc}(\mathbb{R}^d; \mathbb{R}^d)$ is a closed subspace of $L^p(\mathbb{R}^d; \mathbb{R}^d)$, whence a Banach space. Further, $L^p_\text{loc}(\mathbb{R}^d; \mathbb{R}^d)$ is reflexive for $1 < p < \infty$ and a Hilbert space for $p = 2$.

**Proof.** Let $(f_n)$ be a Cauchy sequence in $L^p_\text{loc}(\mathbb{R}^d; \mathbb{R}^d)$ with limit $f \in L^p(\mathbb{R}^d; \mathbb{R}^d)$. Further, let $(u_n)$ be a sequence in $W^{1,p}_\text{loc}(\mathbb{R}^d)$ with $f_n = \nabla u_n$. For $k \in \mathbb{N}$, define $v_n^{(k)} := u_n - \bar{f}_B(k) \, u_n \, dx$, where $\bar{f}_B \, dx := \int_\omega \, dx/|B|$ denotes the integral mean over $B \subseteq \mathbb{R}^d$. Since $\nabla v_n^{(k)} = f_n$, a Poincaré inequality on $B(0, k)$ yields that $(v_n^{(k)})_{n \in \mathbb{N}}$ converges to a function
Proposition 3.4. Fourier transform.

Lemma 3.3. and $D$ defines a bounded linear functional

$$\text{Remark}$$

With respect to Lemma 3.3, is $\text{Remark}$ (1.3) is $\text{Remark}$ with $\text{Remark} \neq 0$. Thus, $k \to \infty$ gives a function $u \in W^{1,p}(\mathbb{R}^d)$ with $\nabla u = f$. The remaining claims follow from principles of functional analysis [20].

Now, we consider the vector space

$$\tilde{W}_1^p(\mathbb{R}^d) := \{ u \in W_0^{1,p}(\mathbb{R}^d) \mid \nabla u \in L^p(\mathbb{R}^d;\mathbb{R}^d) \},$$

Note that the natural definition

$$\|u\|_{W_1^p(\mathbb{R}^d)} := \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}$$

only induces a seminorm on this space. Two functions $u, v \in \tilde{W}_1^p(\mathbb{R}^d)$ have the same gradient, i.e. $\nabla u = \nabla v$, if and only if $u = v + c$ for a constant $c \in \mathbb{R}$. Factored out the piecewise constants from $\tilde{W}_1^p(\mathbb{R}^d)$, i.e

$$W_1^p(\mathbb{R}^d) := \tilde{W}_1^p(\mathbb{R}^d)/\mathbb{R},$$

the quotient space equipped with $\| \cdot \|_{W_1^p(\mathbb{R}^d)}$ obviously satisfies the following lemma.

**Lemma 3.2.** $W_1^p(\mathbb{R}^d)$ is a Banach space which is reflexive for $1 < p < \infty$, and $W_1^2(\mathbb{R}^d)$ is a Hilbert space. Moreover, the gradient $\nabla : W_1^p(\mathbb{R}^d) \to L_2^p(\mathbb{R}^d;\mathbb{R}^d)$ is an isometric isomorphism.

**Remark 1.** The inclusion $i_p : W_1^p(\mathbb{R}^d) \hookrightarrow W_1^2(\mathbb{R}^d)$, that maps a function $u \in W_1^p(\mathbb{R}^d)$ to the corresponding equivalence class in $W_1^2(\mathbb{R}^d)$, is continuous and injective. Thus, $W_1^p(\mathbb{R}^d)$ and $D(\mathbb{R}^d)$ can be treated as subspaces of $W_1^2(\mathbb{R}^d)$.

The following result can be found in [13, Appendix A] or easily be verified by use of the Fourier transform.

**Lemma 3.3.** The test functions $D(\mathbb{R}^d)$ are dense within $W_1^2(\mathbb{R}^d)$.

**Proposition 3.4.** For $m \in L^2(\mathbb{R}^d;\mathbb{R}^d)$, there is a unique $u = u_m \in W_1^2(\mathbb{R}^d)$ which satisfies (1.3). The operator $L : L^2(\mathbb{R}^d;\mathbb{R}^d) \to W_1^2(\mathbb{R}^d), m \mapsto u_m$ is linear and bounded with operator norm $\|L\| = 1$. The composition

$$P := \nabla \circ L \in L(L^2(\mathbb{R}^d;\mathbb{R}^d); L^2(\mathbb{R}^d;\mathbb{R}^d))$$

is the $L^2$-orthogonal projection onto $L_v^2(\mathbb{R}^d;\mathbb{R}^d)$, and we have

$$L_v^2(\mathbb{R}^d;\mathbb{R}^d) = \{ m \in H(\text{div};\mathbb{R}^d) \mid \text{div} m = 0 \}.$$

**Proof.** According to the Cauchy inequality and $\|v\|_{W_1^2(\mathbb{R}^d)} = \|\nabla v\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$,

$$\Phi(v) := \langle m ; \nabla v \rangle$$

defines a bounded linear functional $\Phi \in W_1^2(\mathbb{R}^d)^*$ with norm $\|\Phi\| \leq \|m\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$. Now, (1.3) reads

$$\langle u ; v \rangle_{W_1^2(\mathbb{R}^d)} = \Phi(v) \quad \text{for all } v \in D(\mathbb{R}^d).$$

With respect to Lemma 3.3, $D(\mathbb{R}^d)$ in (3.7) can be replaced by $W_1^2(\mathbb{R}^d)$ and Riesz’ theorem yields the existence of a unique $u \in W_1^2(\mathbb{R}^d)$ satisfying the equality. The estimation of the norm again follows from the Cauchy inequality,

$$\|u\|_{W_1^2(\mathbb{R}^d)} = \langle \nabla u ; \nabla u \rangle = \langle m ; \nabla u \rangle \leq \|m\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)} \|u\|_{W_1^2(\mathbb{R}^d)},$$
i.e. \( \|Lm\|_{W_2^1(\mathbb{R}^d)} \leq \|m\|_{L^2(\mathbb{R}^d;\mathbb{R}^e)} \). For \( m \in L^2_\nu(\mathbb{R}^d;\mathbb{R}^d) \), we have \( m = \nabla(Lm) \), whence \( \|L\| = 1 \) and \( \mathcal{P} \) is a projection onto \( L_\nu^2(\mathbb{R}^d;\mathbb{R}^d) \). From \( \|\mathcal{P}\| = \|L\| = 1 \), we derive that \( \mathcal{P} \) is orthogonal [20]. Equation (3.6) follows directly from Lemma 3.3.

\[ \square \]

**Remark 2.** Since the embedding \( i_2 : H^1(\mathbb{R}^d) \hookrightarrow W_2^1(\mathbb{R}^d) \) is injective and (1.3) has a unique solution in \( W_2^1(\mathbb{R}^d) \), there is at most one solution in \( H^1(\mathbb{R}^d) \). Later we will investigate in which cases the unique solution \( u \in W_2^1(\mathbb{R}^d) \) is represented by a function in \( H^1(\mathbb{R}^d) \).

**Remark 3.** For the numerical treatment of \( \mathcal{P} \) the latter proposition is meaningless. But in Section 5 an analytical representation of \( \mathcal{L} \) is introduced which carries over to the case \( 1 < p < \infty \) instead of \( p = 2 \).

### 4. The Analytical Main Result

The theorem we want to prove requires some preliminaries on the Calderón-Zygmund kernels defined below. For any kernel \( h : \mathbb{R}^d \setminus \{0\} \to \mathbb{R} \) we make the convention to write \( \tilde{h} \) and \( h_\varepsilon \) for \( h(x) := h(-x) \) and \( h_\varepsilon := h_{R,\varepsilon}(\mathbb{R}^d) \) with arbitrary \( \varepsilon > 0 \) respectively, where \( \chi_{\mathbb{R}^d \setminus B(0,\varepsilon)} \) denotes the characteristic function.

**Definition 1.** A measurable function \( \kappa : \mathbb{R}^d \setminus \{0\} \to \mathbb{R} \) is called Calderón-Zygmund kernel if there is a constant \( c_1 > 0 \) such that for any \( x \neq 0 \) and \( 0 < r < R < \infty \) there holds

\[
\begin{align*}
(4.1) \quad & |\kappa(x)| \leq c_1|x|^{-d}, \\
(4.2) \quad & \int_{|y| > 2|x|} |\kappa(y - x) - \kappa(y)| \, dy \leq c_1, \\
(4.3) \quad & \int_{r < |y| < R} \kappa(y) \, dy = 0.
\end{align*}
\]

**Theorem 4.1 (Calderón-Zygmund [22]).** For a Calderón-Zygmund kernel \( \kappa \), \( 1 < p < \infty \), \( \varepsilon > 0 \) and \( f \in L^p(\mathbb{R}^d) \), the convolution of \( \kappa_\varepsilon \) and \( f \) satisfies

\[
(4.4) \quad \kappa_\varepsilon * f \in L^p(\mathbb{R}^d) \quad \text{with} \quad \|\kappa_\varepsilon * f\|_{L^p(\mathbb{R}^d)} \leq c_p \|f\|_{L^p(\mathbb{R}^d)},
\]

where the constant \( c_p > 0 \) depends only on \( p \) and \( \kappa \) but neither on \( \varepsilon \) nor \( f \). Further \( \kappa_\varepsilon * f \) converges in \( L^p(\mathbb{R}^d) \) for \( \varepsilon \to 0 \). Consequently,

\[
(4.5) \quad S_pf := \lim_{\varepsilon \to 0}(\kappa_\varepsilon * f) \in L^p(\mathbb{R}^d)
\]

defines a bounded operator \( S_p \in L(L^p(\mathbb{R}^d); L^p(\mathbb{R}^d)) \) with norm \( \|S_p\| \leq c_p \). Since \( S_p \) extends the convolution, we shall write \( \kappa f := S_pf \).

\[ \square \]

**Remark 4.** The notation \( \kappa f \) is independent of \( p \) in the following sense: For \( 1 < p, q < \infty \) and \( f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \), one has \( S_pf = S_qf \) since \( L^p \) convergence for \( \varepsilon \to 0 \) implies pointwise convergence almost everywhere [21].

The partial derivatives of the Newtonian kernel \( G \),

\[
\frac{\partial G}{\partial x_j}(x) = \frac{1}{\gamma_d} \frac{x_j}{|x|^2},
\]
give rise to the following definition.

**Definition 2.** A kernel \( h : \mathbb{R}^d \setminus \{0\} \to \mathbb{R} \) is homogeneous of degree \( \alpha \in \mathbb{R} \) if \( h(\lambda x) = \lambda^\alpha h(x) \) for any \( \lambda > 0 \) and \( x \neq 0 \).
To give first examples of Calderón-Zygmund kernels and to see that the second order
partial derivatives of the Newtonian kernel \( \kappa := \partial^2 G / (\partial x_j \partial x_k) \) are of Calderón-Zygmund
type, we cite the following lemma from [1].

**Lemma 4.2.** For \( h \in C^2(\mathbb{R}^d \setminus \{0\}) \) homogeneous of degree \( 1 - d \), any partial derivative
\( \kappa := \partial h / \partial x_j \) is a Calderón-Zygmund kernel. \( \square \)

**Theorem 4.3.** Let \( 1 < p < \infty \) and \( h \in C^1(\mathbb{R}^d \setminus \{0\}) \) be homogeneous of degree \( 1 - d \) such
that the first order partial derivatives of \( h \) are Calderón-Zygmund kernels. Then, there is a
unique bounded operator \( T_p \in L(L^p(\mathbb{R}^d); W^p_1(\mathbb{R}^d)) \) with
\[
T_p f = h \ast f \quad \text{for all } f \in D(\mathbb{R}^d).
\]

For \( f \in L^p(\mathbb{R}^d) \), the weak derivative of \( T_p f \) is given by
\[
\frac{\partial}{\partial x_j} (T_p f) = \kappa_j \ast f + \lambda_j f,
\]
where \( \kappa_j := \partial h / \partial x_j \) and \( \lambda_j := \int_{\partial B(0,1)} h(x) x_j \, ds_x \). The operator \( T_p \) has the following mapping properties:

(a) \( T_p f = h \ast f \) for \( f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \),
(b) \( T_p f = h \ast f \) for \( f \in L^q(\mathbb{R}^d) \cap L^r(\mathbb{R}^d) \) with \( 1 \leq q < d < r \leq \infty \) and \( q \leq p \leq r \),
(c) \( T_p f = h \ast f \in W^{1,p}(\mathbb{R}^d) \) for \( f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \) and \( d > p' \) (i.e. \( pd > p + d \)).

Further, if \( \Omega \) is bounded, the restriction to \( L^p(\Omega) \) satisfies
\( (d) T_p \in L(L^p(\Omega); W^{1,p}(\mathbb{R}^d)) \) for \( d > p' \).

Finally, for \( 1 < p, q < \infty \) we have
\( (e) T_p f = T_q f \) for all \( f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \).

**Remark 5.** The equalities in Theorem 4.3, e.g. (4.6), have to be understood in the sense
that \( h \ast f \in W^{1,p}_0(\mathbb{R}^d) \) (resp. \( h \ast f \in W^{1,p}(\mathbb{R}^d) \)) exists and is a representer of the equivalence
class \( T_p f \). Since the operator \( T_p \) extends the convolution with \( h \), we write \( h \ast f := T_p f \) for
\( f \in L^p(\mathbb{R}^d) \). Due to (e) this notation is independent of \( p \).

**Remark 6.** For the first order partial derivatives \( h := \partial G / \partial x_k \) of the Newtonian kernel \( G \),
\( \lambda_j \) from (4.7) can be computed,
\[
\lambda_j = \begin{cases} 
0 & \text{for } j \neq k, \\
1/d & \text{for } j = k.
\end{cases}
\]

With the unit sphere \( S := \partial B(0,1) \) this follows from \( \lambda_j = 1 / |S| \int_S x_j x_k \, ds_x \) for \( j \neq k \) by
symmetry and from \( |S| = \sum_{j=1}^d \int_S x_j^2 \, ds_x = d\int_S x_j^2 \, ds_x \) for \( j = k \).

The proof of the theorem needs the following elementary lemma which can be derived
directly from Proposition 2.1.

**Lemma 4.4.** For a measurable function \( h : \mathbb{R}^d \setminus \{0\} \to \mathbb{R} \) which is of degree \( 1 - d \) and
\( h_1 := h \chi_{\mathbb{R}^d \setminus B(0,1)} \), the following holds:

(i) \( h_1 \in L^1(\mathbb{R}^d) \), \( h - h_1 \in L^s(\mathbb{R}^d) \) for \( 1 \leq s < d' < t \leq \infty \),
(ii) \( h \ast f \in L^p_{loc}(\mathbb{R}^d) \) for \( f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \), and \( 1 \leq q < d \),
(iii) \( h \ast f \in L^\infty(\mathbb{R}^d) \) for \( f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \) and \( 1 \leq q < d < p \leq \infty \),
(iv) \( \langle h * f ; g \rangle = \langle f ; \tilde{h} * g \rangle \) for \( f \) as in (iii), \( g \in L^1(\mathbb{R}^d) \), and \( \tilde{h}(x) := h(-x) \).

\[ \square \]

**Proof of Theorem 4.3 (main part).** For \( f \in \mathcal{D}(\mathbb{R}^d) \), we have \( h * f \in L^\infty(\mathbb{R}^d) \). To verify \( h * f \in W^p_1(\mathbb{R}^d) \) it remains to show that

\[ \langle h * f ; \partial \phi / \partial x_j \rangle = -\langle \kappa_j \tilde{h} * f + \lambda_j f ; \phi \rangle \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}^d). \]

For the notation, we use the conventions introduced above. Let \( \phi \in \mathcal{D}(\mathbb{R}^d) \) and choose \( r > 0 \) with \( \text{supp}(f) \cup \text{supp}(\phi) \subseteq B(0, r) \) and note

\[ \langle h * f ; \partial \phi / \partial x_j \rangle = \langle f ; \tilde{h} * (\partial \phi / \partial x_j) \rangle = \int_{\text{supp}(f)} f(y) \lim_{\epsilon \to 0} \int_{B(0, r) \setminus B(y, \epsilon)} h(x - y) \frac{\partial \phi}{\partial x_j}(x) \, dx \, dy. \]

For fixed \( y \in \text{supp}(f) \) and small \( \epsilon > 0 \), the inner integral reads with partial integration

\[ \int_{B(0, r) \setminus B(y, \epsilon)} h(x - y) \frac{\partial \phi}{\partial x_j}(x) \, dx = -\int_{B(0, r) \setminus B(y, \epsilon)} \kappa_j(x - y) \phi(x) \, dx + \int_{\partial B(0, r) \setminus B(y, \epsilon)} h(x - y) \phi(x) n_j(x) \, ds_x \]

\[ = -\langle \kappa_j \tilde{h} * \phi(y) \rangle - \int_{\partial B(y, \epsilon)} h(x - y) \phi(x) \frac{x_j - y_j}{\epsilon} \, ds_x, \]

where the Calderón-Zygmund kernel \( \kappa_j \) is defined by \( \kappa_j(x) = \kappa_j(-x) \). Recall that according to Theorem 4.1 \( \kappa_j \tilde{h} * \phi \) converges to \( \kappa_j \tilde{h} * \phi \) in \( L^p(\mathbb{R}^d) \) for \( \epsilon \to 0 \). (This allows us to exchange the limit and the integration with respect to \( y \).) With transformations, the surface integral reads

\[ \int_{\partial B(y, \epsilon)} h(x - y) \phi(x) \frac{x_j - y_j}{\epsilon} \, ds_x = \int_{\partial B(0, 1)} h(x)x_j \phi(y + \epsilon x) \, ds_x \]

\[ = \lambda_j \phi(y) + \int_{\partial B(0, 1)} h(x)x_j (\phi(y + \epsilon x) - \phi(y)) \, ds_x, \]

and the second term in the sum vanishes for \( \epsilon \to 0 \). Combining both equations, we end up with

\[ \langle h * f ; \partial \phi / \partial x_j \rangle = -\lim_{\epsilon \to 0} \langle f ; \kappa_j \tilde{h} * \phi \rangle - \lambda_j \phi \]

\[ = -\lim_{\epsilon \to 0} \langle \kappa_j \tilde{h} * f ; \phi \rangle - \lambda_j \phi = \langle \kappa_j \tilde{h} * f - \lambda_j f ; \phi \rangle \]

and derive (4.9). In particular, we obtain with Theorem 4.1

\[ \| \nabla (h * f) \|_{L^p(\mathbb{R}^d, \mathbb{R}^d)} \leq c_2 \| f \|_{L^p(\mathbb{R}^d)}, \]

where \( c_2 > 0 \) only depends on \( d, p, h, \) and its partial derivatives. Considering \( \mathcal{D}(\mathbb{R}^d) \) as a subspace of \( L^p(\mathbb{R}^d) \), we have shown that \( T_p f := h * f \) defines a bounded operator \( T_p \in L(\mathcal{D}(\mathbb{R}^d); W^p_1(\mathbb{R}^d)) \). Density provides a unique extension \( T_p \in L(L^p(\mathbb{R}^d); W^p_1(\mathbb{R}^d)) \). Equality (4.7) carries over from \( \mathcal{D}(\mathbb{R}^d) \) to \( L^p(\mathbb{R}^d) \) by continuity.

The remaining claims of the theorem follow from classical density arguments, which can be applied according to the additional assumptions on \( f \in L^p(\mathbb{R}^d) \).

\[ \square \]

**Proof of Theorem 4.3 (a).** We have to show that \( h * f \in L^p_{\text{loc}}(\mathbb{R}^d) \) is weakly differentiable with weak derivative \( \kappa_j \tilde{h} * f + \lambda_j f \). To this end, choose a sequence \( (f_n) \) in \( \mathcal{D}(\mathbb{R}^d) \) that converges
to \( f \) in \( L^1(\mathbb{R}^d) \) and \( L^p(\mathbb{R}^d) \) (for instance a sequence of mollifications of \( f \)). For all \( \phi \in \mathcal{D}(\mathbb{R}^d) \), a H"older inequality shows

\[
|\langle h \ast f ; \partial \phi / \partial x_j \rangle - \langle h \ast f_n ; \partial \phi / \partial x_j \rangle| \leq \|f - f_n\|_{L^1(\mathbb{R}^d)} \|\tilde{h} \ast (\partial \phi / \partial x_j)\|_{L^\infty(\mathbb{R}^d)},
\]

and the right-hand side vanishes for \( n \to \infty \). Hence, we obtain

\[
\langle h \ast f ; \partial \phi / \partial x_j \rangle = \lim_{n \to \infty} \langle h \ast f_n ; \partial \phi / \partial x_j \rangle = \lim_{n \to \infty} \langle \kappa_j \ast f_n + \lambda_j f_n ; \phi \rangle = \langle \kappa_j \ast f + \lambda_j f ; \phi \rangle,
\]

where we have used the convergence in \( L^p(\mathbb{R}^d) \).

\[\square\]

**Proof of Theorem 4.3 (b).** W.l.o.g. we may assume \( f \geq 0 \). Define \( f^n := \min(f, n) \chi_{B(0, n)} \) and note that \( f^n \) converges to \( f \) in \( L^s(\mathbb{R}^d) \) for \( s = p, q, r \). According to Proposition 2.1,

\[
\|h \ast (f - f^n)\|_{L^\infty(\mathbb{R}^d)} \leq \|h_1 \ast (f - f^n)\|_{L^\infty(\mathbb{R}^d)} + \|(h - h_1) \ast (f - f^n)\|_{L^\infty(\mathbb{R}^d)}
\]

\[
\leq \|h_1\|_{L^\nu(\mathbb{R}^d)} \|f - f^n\|_{L^r(\mathbb{R}^d)} + \|h - h_1\|_{L^\nu(\mathbb{R}^d)} \|f - f^n\|_{L^r(\mathbb{R}^d)},
\]

i.e. \( h \ast f^n \) converges to \( h \ast f \) in \( L^\infty(\mathbb{R}^d) \). The application of (a) yields

\[
\langle h \ast f ; \partial \phi / \partial x_j \rangle = \lim_{n \to \infty} \langle h \ast f_n ; \partial \phi / \partial x_j \rangle = \lim_{n \to \infty} \langle \kappa_j \ast f_n + \lambda_j f_n ; \phi \rangle = \langle \kappa_j \ast f + \lambda_j f ; \phi \rangle
\]

for any test function \( \phi \in \mathcal{D}(\mathbb{R}^d) \).

\[\square\]

**Proof of Theorem 4.3 (c).** According to (a) it remains to show that \( h \ast f \in L^p(\mathbb{R}^d) \). With Lemma 4.4 we have \( (h - h_1) \in L^1(\mathbb{R}^d) \) and \( h_1 \in L^p(\mathbb{R}^d) \) since \( p > d' \). Proposition 2.1 yields \( h_1 \ast f, (h - h_1) \ast f \in L^p(\mathbb{R}^d) \).

\[\square\]

**Proof of Theorem 4.3 (d).** Assertion (c) yields that the restriction \( T : L^p(\Omega) \to W^{1,p}(\mathbb{R}^d) \) from \( T_p \) to \( L^p(\Omega) \) is well-defined and linear. Since the inclusion \( i_p : W^{1,p}(\mathbb{R}^d) \hookrightarrow W^{1,p}_0(\mathbb{R}^d) \) and the composition \( T_p = i_p \circ T \) are continuous, Banach’s closed graph theorem implies that \( T \) is also continuous.

\[\square\]

**Proof of Theorem 4.3 (e).** The claim follows directly from (4.7) since the right-hand side is independent of \( p \) and \( q \) and defines a function in \( L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d) \) by Theorem 4.1.

\[\square\]

5. Unique Solvability of the Potential Equation (1.3)

In the subsequent section we show that also for \( 1 < p < \infty \) and \( m \in L^p(\mathbb{R}^d, \mathbb{R}^d) \) (instead of \( p = 2 \)) the potential equation (1.3) has a unique solution \( u \in W^1_p(\mathbb{R}^d) \). We provide a representation of the operator \( L \) which was introduced for \( p = 2 \) in Proposition 4.4. Recall from Corollary 2.2 that, for any arbitrary smooth magnetization \( m = (m_1, \ldots, m_d) \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d) \),

\[
(5.1) \quad u := L_0 m := \sum_{j=1}^d \frac{\partial G}{\partial x_j} * m_j
\]

is a solution of Equation (1.2). In particular, \( u \) solves the weak form (1.3).

**Proposition 5.1.** Given \( 1 \leq p \leq \infty \) and \( m \in L^p(\mathbb{R}^d, \mathbb{R}^d) \), Equation (1.3) has at most one solution \( u \in W^1_p(\mathbb{R}^d) \).
Proof. Assume that \( u_1, u_2 \in W^p_1(\mathbb{R}^d) \) solve (1.3). Then \( e := u_2 - u_1 \in W^p_1(\mathbb{R}^d) \) satisfies \( \Delta e = 0 \) in a weak sense. In particular, any derivative \( f := \partial e / \partial x_j \) satisfies \( \Delta f = 0 \) and \( f \in L^p(\mathbb{R}^d) \). For any \( \phi \in \mathcal{D}(\mathbb{R}^d) \) it follows that \( \Delta (\phi * f) = \phi * \Delta f = 0 \) and \( \phi * f \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) by Proposition 2.1. But then Liouville’s theorem implies \( \phi * f = 0 \). Since this holds for any test function \( \phi \), Lebesgue’s differentiation theorem yields \( f = 0 \), whence \( u_1 = u_2 \in W^p_1(\mathbb{R}^d) \). 

Since the kernels \( h_j := \partial G / \partial x_j \) satisfy the assumptions of Theorem 4.3, we obtain the following result which states, in particular, the unique solvability of (1.3) for \( m \in L^p(\mathbb{R}^d; \mathbb{R}^d) \) in \( W^p_1(\mathbb{R}^d) \) for \( 1 < p < \infty \).

**Theorem 5.2.** For any \( 1 < p < \infty \), there is a unique bounded operator

\[
L_p := \sum_{j=1}^d \frac{\partial G}{\partial x_j} \star m_j \quad \text{for} \quad m = (m_1, \ldots, m_d) \in L^p(\mathbb{R}^d; \mathbb{R}^d)
\]

which extends \( L_0 \) from \( \mathcal{D}(\mathbb{R}^d; \mathbb{R}^d) \) to \( L^p(\mathbb{R}^d; \mathbb{R}^d) \). For a magnetization \( m \in L^p(\mathbb{R}^d; \mathbb{R}^d) \), \( u := L_p m \) is the unique solution of (1.3). Further, \( L_p \) has the following mapping properties.

(a) \( L_p m = L_p m \) for \( m \in L^1 \cap L^p \).

(b) \( L_p m = L_p m \) for \( m \in L^q \cap L'^r \) with \( 1 \leq q < d < r \leq \infty \) and \( q \leq p \leq r \).

(c) \( L_p m = L_0 m \in W^1(\mathbb{R}^d) \) for \( m \in L^1 \cap L^p \) and \( d > p' \),

where \( L_p m = L_0 m \), in particular, states that the convolution \( L_0 m \) exists in the classical sense. Further, for a bounded open set \( \Omega \subseteq \mathbb{R}^d \), the restriction of \( L_p \) to \( L^p(\Omega; \mathbb{R}^d) \) satisfies

(d) \( L_p \in L(\mathcal{D}(\Omega; \mathbb{R}^d); W^1(\mathbb{R}^d)) \) for \( d > p' \).

Finally, for \( p = 2 \), the extended convolution operator \( L_2 \) coincides with the operator \( L \) introduced in Proposition 3.4 and we remark that (c) and (d) hold for \( d \geq 3 \).

**Proof.** According to Theorem 4.3, \( L_p \) is given by

\[
L_p m = \sum_{j=1}^d \frac{\partial G}{\partial x_j} \star m_j \quad \text{for} \quad m = (m_1, \ldots, m_d) \in L^p(\mathbb{R}^d; \mathbb{R}^d)
\]

and this extension is unique since the test functions are dense within \( L^p \). The mapping properties (a)–(d) carry over from the corresponding statements in Theorem 4.3. It remains to show that \( u = L_p m \) solves (1.3) for arbitrary \( m \in L^p(\mathbb{R}^d; \mathbb{R}^d) \). Let \( (m_k)_{k \in \mathbb{N}} \) be a sequence of test functions which converges to \( m \) in \( L^p \) and recall that (1.3) has already been shown for all \( m_k \). From definition of \( W^p_1(\mathbb{R}^d) \) we infer that \( \|L_p(m - m_k)\|_{W^p_1(\mathbb{R}^d)} \to 0 \). Combined with the Hölder inequality, this yields for all \( v \in \mathcal{D}(\mathbb{R}^d) \)

\[
|\langle -\nabla (L_p m) + m ; \nabla v \rangle| = |\langle -\nabla (L_p m) + L_p m_k ; \nabla v \rangle + \langle m - m_k ; \nabla v \rangle| 
\leq \|\nabla v\|_{L^{p'}(\mathbb{R}^d)} (\|L_p(m - m_k)\|_{W^p_1(\mathbb{R}^d)} + \|m - m_k\|_{L^p(\mathbb{R}^d)}) ,
\]

and the right-hand side vanishes for \( k \to \infty \) since \( L_p \) is continuous. 

**Remark 7.** Theorem 5.2 yields a constructive proof of Lemma 3.3: Write \( W^2_1(\mathbb{R}^d) = H \oplus H^\perp \) with \( H \) the closure of \( \mathcal{D}(\mathbb{R}^d) \) in \( W^2_1(\mathbb{R}^d) \). For \( u \in H^\perp \) we have \( \langle \nabla u ; \nabla v \rangle = 0 \) for all \( v \in \mathcal{D}(\mathbb{R}^d) \), whence \( u \) is the potential of the zero magnetization, i.e. \( u = 0 \).

6. Application to Computational Micromagnetics

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \). For a magnetization \( m \in L^2(\Omega; \mathbb{R}^d) \), let \( u := L_2 m \) denote the corresponding (unique) magnetic potential. According to (1.3) and
Proposition 3.4, the stray-field energy from (1.1) reads
\begin{equation}
\int_{\mathbb{R}^d} |\nabla u|^2 \, dx = \int_{\Omega} \mathbf{m} \cdot \nabla u \, dx.
\end{equation}
On the right-hand side, the continuous bilinear form
\begin{equation}
a : L^2(\mathbb{R}^d; \mathbb{R}^d) \times L^2(\mathbb{R}^d; \mathbb{R}^d) \to \mathbb{R}, \quad a(m, \tilde{m}) := \langle \nabla (L_2 m) ; \tilde{m} \rangle
\end{equation}
appears. For the discretization of which, let \( T = \{T_1, \ldots, T_N\} \) be a triangulation of \( \Omega \), i.e.
\begin{itemize}
  \item[(i)] every \( T \in T \) is a (bounded) Lipschitz domain which satisfies \( T \subseteq \Omega \),
  \item[(ii)] \( \overline{T} = \bigcup \{\overline{T} | T \in T\} \), where \( \overline{T} \) denotes the closure of \( T \subseteq \mathbb{R}^d \),
  \item[(iii)] for different \( T_j, T_k \in T \), we have \( T_j \cap T_k = \emptyset \).
\end{itemize}
Further, let \( S^0(T) \) denote the vector space of all \( T \)-piecewise constant functions. Then, for piecewise constant magnetizations \( m, \tilde{m} \in S^0(T)^d \), the following proposition gives a formula to compute \( a(m, \tilde{m}) \) analytically.

**Remark 8.** At least for the large body model of micromagnetics due to DE SIMONE [7] the consideration of piecewise constant functions is reasonable, cf. [5] for a discrete relaxed model and the corresponding numerical analysis.

**Proposition 6.1.** For bounded Lipschitz domains \( \omega, \overline{\omega} \subseteq \mathbb{R}^d \) and vectors \( m, \tilde{m} \in \mathbb{R}^d \), we have
\begin{equation}
a(\chi_\omega m, \chi_\overline{\omega} \tilde{m}) = -\int_{\partial \omega} \int_{\partial \overline{\omega}} G(x - y)(n(x) \cdot m)(\tilde{n}(y) \cdot \tilde{m}) \, ds_y \, ds_x,
\end{equation}
where \( \chi_\omega \) and \( \chi_\overline{\omega} \) denote the corresponding characteristic functions and \( n, \tilde{n} \) denote the outer normal vectors on \( \partial \omega \) and \( \partial \overline{\omega} \) respectively. Further, we have the symmetry properties
\begin{equation}
a(\chi_\omega m, \chi_\overline{\omega} \tilde{m}) = a(\chi_\overline{\omega} \tilde{m}, \chi_\omega m) = a(\chi_\omega \tilde{m}, \chi_\overline{\omega} m).
\end{equation}
In particular, \( B_{jk} := a(\chi_\omega e_j, \chi_\overline{\omega} e_k) \) defines a symmetric matrix \( B \in \mathbb{R}^{d \times d} \) such that
\begin{equation}
a(\chi_\omega m, \chi_\overline{\omega} \tilde{m}) = m \cdot B \tilde{m}.
\end{equation}
In the case \( \text{dist}(\omega, \overline{\omega}) > 0 \), the coefficients of \( B \) can be computed by
\begin{equation}
B_{jk} = \int_{\omega} \int_{\overline{\omega}} \frac{\partial^2 G}{\partial x_j \partial x_k}(x - y) \, dy \, dx.
\end{equation}

**Proof.** Since \( \nabla \circ L_2 \) is an orthogonal projection, the bilinear form \( a(\cdot, \cdot) \) is symmetric [20]. This shows the first equality in (6.4). To obtain the other claims of the proposition, note that the bilinearity of \( a(\cdot, \cdot) \) leads to
\begin{equation}
a(\chi_\omega m, \chi_\overline{\omega} \tilde{m}) = \sum_{j,k=1}^d m_j \tilde{m}_k a(\chi_\omega e_j, \chi_\overline{\omega} e_k) = m \cdot B \tilde{m}.
\end{equation}
Therefore only the special case \( m = e_j \) and \( \tilde{m} = e_k \) has to be treated. To abbreviate notation, we write \( h_\ell := \partial G/\partial x_\ell \) and \( \kappa_{jk} := \partial^2 G/(\partial x_j \partial x_k) \). Theorem 4.3 and Remark 6 yield
\begin{equation}
\frac{\partial (h_j * \chi_\omega)}{\partial x_k} = \kappa_{jk} \chi_\omega + \frac{\delta_{jk}}{d} \chi_\omega = \frac{\partial (h_k * \chi_\omega)}{\partial x_j}.
\end{equation}
with Kronecker’s $\delta_{jk}$. Further, we have $L_2(\chi_{\omega} e_j) = L_0(\chi_{\omega} e_j) = h_j \ast \chi_{\omega}$. With the definition of the Calderón-Zygmund convolution $\kappa_{jk} \ast \chi_{\omega}$, we obtain

$$B_{jk} = \langle \nabla \circ L(\chi_{\omega} e_j) ; \chi_{\omega} e_k \rangle = \langle \kappa_{jk} \ast \chi_{\omega} ; \chi_{\omega} \rangle + \frac{\delta_{jk}}{d} \langle \chi_{\omega} ; \chi_{\omega} \rangle$$

The symmetry $\kappa_{jk}(x) = \kappa_{kj}(-x)$ shows $\langle \kappa_{jk} \ast \chi_{\omega} ; \chi_{\omega} \rangle = \langle \chi_{\omega} ; \kappa_{jk} \ast \chi_{\omega} \rangle$ and therefore $B_{jk} = B_{kj}$, i.e. we obtain the second equality in (6.4). To prove (6.3) note that Theorem 4.3 in particular states $h_k \ast \chi_{\omega} \in W^{1,p}_{loc}(\mathbb{R}^d)$. Thus, partial integration on the bounded Lipschitz domain $\omega$ yields

$$\int_\omega \frac{\partial(h_k \ast \chi_{\omega})}{\partial x_j} \, dx = \int_{\partial \omega} (h_k \ast \chi_{\omega})(x) n_j(x) \, ds_x.$$ 

For fixed $x \in \partial \omega$ another partial integration for $G \in W^{1,1}_{loc}(\mathbb{R}^d)$ gives

$$(h_k \ast \chi_{\omega})(x) = \int_\omega \frac{\partial G}{\partial x_k} (x-y) \, dy = - \int_\omega \frac{\partial G}{\partial y_k} (x-y) \, dy$$

$$= - \int_{\partial \omega} G(x-y) n_k(y) \, ds_y.$$ 

Combining this with $L(\chi_{\omega} e_k) = h_k \ast \chi_{\omega}$ we infer

$$a(\chi_{\omega} e_k, \chi_{\omega} e_j) = \int_\omega \frac{\partial(h_k \ast \chi_{\omega})}{\partial x_j} \, dx = - \int_{\partial \omega} \int_{\partial \omega} G(x-y)n_j(x)n_k(y) \, ds_y ds_x.$$ 

Finally, (6.6) follows by simple convolution properties. We have $h_k \ast \chi_{\omega} \in C^1(\mathbb{R}^d \setminus \overline{\omega})$ with $\partial(h_k \ast \chi_{\omega}) = (\partial h_k/\partial x_j) \ast \chi_{\omega}$, whence

$$a(\chi_{\omega} e_k, \chi_{\omega} e_j) = \int_\omega \frac{\partial(h_k \ast \chi_{\omega})}{\partial x_j} \, dx = \int_\omega \kappa_{jk} \ast \chi_{\omega} \, dx = \int_\omega \int_{\partial \omega} \kappa_{jk}(x-y) \, dy dx.$$ 

This concludes the proof. \hfill $\square$

**Remark 9.** Equation (6.3) was also proved by Hackbusch and Meленк [10] for $d = 3$ by direct calculation. Although their proof does not use the result due to Calderón and Zygmund, this is what is behind when they use the Fourier transform to show the continuity of the bilinear form $a(\cdot, \cdot)$.

**Remark 10.** Obviously, the given proof of Equation (6.4) carries over to arbitrary functions $\varphi, \varphi \in L^2(\mathbb{R}^d)$, i.e. the characteristic functions $\chi_{\omega}, \chi_{\omega}$ can be replaced by $\varphi, \varphi$.

**Computing the stiffness matrix $A$ for $a(\cdot, \cdot)$.** For a Galerkin discretization of (6.1) with piecewise constant ansatz and test functions, one has to compute the matrix

$$A \in \mathbb{R}^{dN \times dN}_{sym} \text{ with } A_{jk} := a(\varphi_j, \varphi_k)$$

and a fixed basis $\{\varphi_1, \ldots, \varphi_{dN}\}$ of $S^0(T)^d$. A reasonable choice for a basis is

$$\varphi_j := \chi_{T_j} e_1, \quad \varphi_{N+j} := \chi_{T_j} e_2 \quad \text{etc. for } 1 \leq j \leq N$$

as is shown in the following: This basis gives rise to the definition of the matrices

$$A^{\alpha \beta} \in \mathbb{R}^{N \times N}_{sym} \text{ for fixed } 1 \leq \alpha, \beta \leq d, \quad A_{jk}^{\alpha \beta} := a(\chi_{T_j} e_\alpha, \chi_{T_k} e_\beta),$$
where the symmetry of \( A^{\alpha \beta} \) (i.e. an additional symmetry of \( A \)) follows from (6.4). Note that — again by Equation (6.4) — we have \( A^{\alpha \beta} = A^{\beta \alpha} \). Therefore, \( A \) is a symmetric \((d \times d)\)-block matrix with the symmetric blocks \( A^{\alpha \beta} = A^{\beta \alpha} \) of dimension \( N \times N \),

\[
A = \begin{pmatrix}
A^{11} & A^{12} \\
A^{12} & A^{22}
\end{pmatrix}
\] for \( d = 2 \) and

\[
A = \begin{pmatrix}
A^{11} & A^{12} & A^{13} \\
A^{12} & A^{22} & A^{23} \\
A^{13} & A^{23} & A^{33}
\end{pmatrix}
\] for \( d = 3 \) resp.

As a first consequence we obtain that one has only to compute and store \( d(d+1)N(N+1)/4 \) instead of \((dN)^2\) coefficients of the fully populated matrix \( A \). Provided the geometry of the elements \( T_j \in \mathcal{T} \) is simple, the entries \( A^{\alpha \beta}_{jk} \) can be computed exactly: Assume that the boundaries of \( T_j \) and \( T_k \) are finite unions of pairwise disjoint affine boundary pieces \( \Gamma_1, \ldots, \Gamma_\ell \) and \( \tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_\ell \) respectively. Then, Equation (6.3) reads for \( m = e_\alpha, \tilde{m} = e_\beta \),

\[
(6.10)
A^{\alpha \beta}_{jk} = \sum_{\mu=1}^\ell \sum_{\nu=1}^\ell (n_\alpha|\Gamma_\mu)(\bar{n}_\beta|\tilde{\Gamma}_\nu) \int_{\Gamma_\mu} \int_{\tilde{\Gamma}_\nu} G(x-y) \, ds_y \, ds_x.
\]

The double boundary integrals are well-known in the context of boundary integral methods being the Galerkin elements of Symm’s integral equation discretized by piecewise constant functions. Note that analytic formulae are known, cf. [4, 16] for \( d = 2 \) and [9, 16] for \( d = 3 \) respectively.

**Remark 11.** Equation (6.6) of Proposition 6.1 motivates panel clustering techniques to obtain an approximation \( \tilde{A} \) of \( A \) such that assembling, storage, and matrix-vector multiplication of \( \tilde{A} \) are of (almost) linear complexity although the error, for instance, in the Frobenius norm can be controlled [2]. To apply these techniques to each of the matrices \( A^{\alpha \beta} \in \mathbb{R}^{N \times N} \) note that the kernel

\[
g_{\alpha \beta}(x,y) := \frac{\partial^2 G}{\partial x_\alpha \partial x_\beta}(x-y)
\]

is asymptotically smooth and use the representation (6.6) for the entries \( A^{\alpha \beta}_{jk} \) on admissible blocks. Numerical experiments for a blockwise \( \mathcal{H}^2 \)-matrix approach will appear in [18].

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NUMERICAL ANALYSIS FOR A MACROSCOPIC MODEL IN MICROMAGNETICS

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Abstract. The macroscopic behaviour of stationary micromagnetic phenomena can be modelled by a relaxed version of the Landau-Lifshitz minimization problem. In the limit of large and soft magnets \( \Omega \), it is reasonable to exclude the exchange energy and convexify the remaining energy densities. The numerical analysis of the resulting minimization problem

\[
\min E_0^*(m) \text{ amongst } m : \Omega \to \mathbb{R}^d \text{ with } |m(x)| \leq 1 \text{ for a.e. } x \in \Omega,
\]

for \( d = 2, 3 \), faces difficulties caused by the pointwise side-constraint \( |m| \leq 1 \) and an integral over the whole space \( \mathbb{R}^d \) for the stray field energy. This paper involves a penalty method to model the side-constraint and reformulates the exterior Maxwell equation via a nonlocal integral operator \( \mathcal{P} \) acting on functions exclusively defined on \( \Omega \). The discretization with piecewise constant discrete magnetizations leads to edge-oriented boundary integrals. The implementation of which and related numerical quadrature is discussed as well as adaptive algorithms for automatic mesh-refinement. A priori and a posteriori error estimates provide a thorough rigorous error control of certain quantities. Three classes of numerical experiments study the penalization, empirical convergence rates, and the performance of the uniform and adaptive mesh-refining algorithms.

1. Introduction

Numerical simulations of stationary micromagnetic phenomena are most-frequently based on a mathematical model named after Landau and Lifshitz [2, 10]. Therein, one minimizes the energy functional

\[
E_\alpha(m) := \int_\Omega \phi(m) \, dx - \int_\Omega f \cdot m \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \alpha \int_\Omega |\nabla m|^2 \, dx
\]

over some admissible vector-valued magnetizations \( m : \Omega \to \mathbb{R}^d \) on the magnet \( \Omega \); \( m(x) := 0 \) for \( x \in \mathbb{R}^d \setminus \Omega \). Moreover, \( \phi \in \mathcal{C}^\infty(\mathbb{R}^d; \mathbb{R}_{\geq 0}) \) denotes the anisotropy density (it models material properties on a crystalline level), \( f \in L^2(\Omega; \mathbb{R}^d) \) denotes an applied exterior magnetic field, \( \alpha \geq 0 \) is the very small exchange parameter, and \( u \) is the magnetic potential related to \( m \) by Maxwell’s equation

\[
\text{div}(-\nabla u + m) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d).
\]

The model description is completed by a non-convex side-constraint given by the pointwise length condition on the magnetization vector, namely

\[
|m(x)| = 1 \quad \text{for almost every } x \in \Omega.
\]
Any of the summands in (1.1) favours another property of an energy minimizing magnetization. First, uniaxial materials such as Cobalt allow the uniaxial anisotropy energy
\begin{equation}
\phi(x) = \frac{1}{2} (1 - (x \cdot e)^2) \quad \text{for all } |x| = 1
\end{equation}
with given easy axis $e \in \mathbb{R}^d$, a fixed unit vector, which favours magnetizations $m$ aligned with $e$. Second, the exterior energy favours magnetizations $m$ aligned to the exterior field $f$. Third, the magnetic energy vanishes for divergence free magnetizations as seen in (1.2); notice that (1.2) involves a boundary condition $\frac{\partial u}{\partial n} = -m \cdot n$ for the jump $[\cdot]$ on $\partial \Omega$, where $n$ denotes the outer normal vector on $\partial \Omega$. Fourth, the exchange energy penalizes changes in the magnetization $m$ and so yields Weissian domains and rapidly changes at the Bloch walls between those.

The macroscopic material behaviour for large and soft magnets, however, is conserved in the case $\alpha = 0$. Then, the model lacks classical solutions in general [12] and hence has to be relaxed by considering measure valued solutions [16] or by convexification [6, 20]. Notice that the convexified problem is the mathematical foundation of the so-called phase theory [10, p184].

Throughout this paper, the focus is on the numerical approximation of macroscopic quantities such as the magnetic potential $u$ or the space-averages of the magnetization vector $m$. In fact, in a certain limit configuration of soft-large bodies, $\alpha \to 0$ and then $E_0$ is the correct model with generalized solutions. The well-posed macroscopic values of which are $u$ and $m$ that minimize the convexified model $E_0^{**}$. We refer to [6, 20] for justifications of this and the proof of
\begin{equation}
E_0^{**}(m) := \int_{\Omega} \phi^{**}(m) \, dx - \int_{\Omega} f \cdot m \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx
\end{equation}
with the side-constraint (1.2) and
\begin{equation}
|m(x)| \leq 1 \quad \text{for almost every } x \in \Omega.
\end{equation}

Here, $\phi^{**}$ is the convexified density defined by
\[ \phi^{**}(x) = \sup \{ \varphi(x) \mid \varphi : \mathbb{R}^d \to \mathbb{R} \text{ convex and } \varphi|_S \leq \phi \} \quad \text{for } |x| \leq 1, \]
where $S = \{ x \in \mathbb{R}^d \mid |x| = 1 \}$ denotes the unit sphere. Then, the relaxed problem (RP) reads:

Minimize $E_0^{**}(m)$ over $\mathcal{A} := \{ m \in L^\infty(\Omega; \mathbb{R}^d) \mid \|m\|_{L^\infty} \leq 1 \}$. 

In contrast to the ill-posed problem $E_0$, its convexification is well-posed. In particular, the minimum of $E_0^{**}(\mathcal{A})$ is attained in $\mathcal{A}$.

The numerical analysis of the model in [4, 19] considers $d = 2$ only, replaces the entire space $\mathbb{R}^d$ in (1.2) by a bounded Lipschitz domain $\Omega$ containing $\Omega$, and solves for $u \in H_0^1(\Omega)$. The potential $u$ is discretized by a non-conforming $u_h$ and a piecewise constant $m_h$ on $\Omega$. The choice of $u_h$ as a conforming, piecewise affine, and globally continuous finite element scheme leads to instabilities [4, 19]. In this paper, we treat (1.2) exactly via an integral representation, i.e. $u = \mathcal{L} m$ with a linear convolution operator $\mathcal{L}$ and $\mathcal{P} m := \nabla (\mathcal{L} m)$, cf. Theorem 2.1. The algorithmical realization of $\mathcal{P}$ is less obvious and discussed in Subsection 4.1. The advantage is that the resulting model requires only one discretization, e.g., by piecewise constant
approximations $m_h$. Those allow exact fulfillment of the side-constraint $|m_h| \leq 1$ involved approximately by a penalization procedure. The resulting discrete minimization problem is to minimize

\[ E_{\varepsilon,h}^{**}(m_h) := \int_{\Omega} \phi^{**}(m_h) \, dx - \int_{\Omega} f \cdot m_h \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |Pm_h|^2 \, dx + \frac{1}{2} \int_{\Omega} \varepsilon (|m_h| - 1)^2 \, dx \]

over all $T$-piecewise constant magnetizations $m_h \in \mathcal{L}^0(T)^d$, where $T$ is a partition of $\Omega$. According to the a priori error analysis, the $T$-piecewise constant penalization function $\varepsilon \in \mathcal{L}^0(T; \mathbb{R}_{>0})$ will be a power of the local mesh-size later on. It turns out that the error analysis of [4] essentially carries over to the situation presented in Section 3 and generalizes to $d = 2, 3$.

The remaining part of the paper is organized as follows: Section 2 states the Euler-Lagrange equations related to $(RP)$ and gives an alternate proof of the uniqueness of the solution of $(RP)$ in the uniaxial case. The discrete problem $(RP_{\varepsilon,h})$ is formulated and unique existence of discrete solutions is discussed. Section 3 presents the assertion and proofs of a priori and a posteriori error estimates. Section 4 displays details on a possible implementation: the computation of a discrete solution by a Newton-Raphson scheme (Subsection 4.1), an indicator-based adaptive mesh-refinement (Subsection 4.2), the implementation of the proposed refinement indicators (Subsection 4.3), and the efficient realization of the involved integral operator $P$ by an $H$-matrix approach (Subsection 4.4). Sections 5-7 report on the results of careful numerical studies. The first and second example provide a closed formula for the smooth and non-smooth exact solution with a computable error $\|m - m_h\|_{L^2(\Omega)}$. Empirical evidence supports the choice of the penalty parameter $\varepsilon = h^{3/2}$ and the superiority of adaptive mesh-refining strategies over uniform meshes. The real-life scientific computing in Section 7 with unknown solution shows, very much in surprising contrast to [4], that almost no local mesh-refinement is required.

2. Preliminaries

This section is devoted to the Euler-Lagrange equations related to $(RP)$ which characterize the minimizers and introduces the proposed discretization by a penalization strategy. For $(RP)$ and the discrete problem $(RP_{\varepsilon,h})$ we prove unique existence of solutions in the uniaxial case.

The magnetic potential is modelled via a Newton integral representation as in [14, 13]. The subsequent theorem gathers the required properties of the respective integral operator together. Proofs can be found in [18] although we expect that the result is known to the experts. The Newtonian kernel $G : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is defined by

\[ G(x) := \begin{cases} \frac{1}{\gamma_d} \log |x| & \text{for } d = 2, \\
\frac{1}{(2-d)\gamma_d}|x|^{2-d} & \text{for } d > 2 \end{cases} \quad \text{for } x \neq 0, \]

where the constant $\gamma_d := |S| > 0$ denotes the surface measure of the unit sphere.

**Theorem 2.1.** Given any $m \in L^\infty(\Omega; \mathbb{R}^d)$, there exists (up to an additive constant) a unique magnetic potential $u = Lu \in H^1_{\text{loc}}(\mathbb{R}^d)$ such that

\[ \nabla u \in L^2(\mathbb{R}^d; \mathbb{R}^d) \quad \text{and} \quad \text{div}(-\nabla u + m) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \]
The (extended) operator $\mathcal{P} : L^2(\mathbb{R}^d; \mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d; \mathbb{R}^d)$, $\mathbf{m} \mapsto \nabla(\mathcal{L}\mathbf{m})$ is an $L^2$ orthogonal projection. The potential $\mathcal{L}\mathbf{m}$ can be represented as a convolution operator

\begin{equation}
\mathcal{L}\mathbf{m} = \sum_{j=1}^{d} \frac{\partial G}{\partial x_j} \ast \mathbf{m}_j,
\end{equation}

where $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_d)$ is equivalently characterized by the corresponding Euler-Lagrange equations [6]. Thus, the potentials coincide and moreover variations. For any solutions $(\lambda, \mathbf{m})$ of (2.2) in Theorem 2.1, see Equation (2.2) in Theorem 2.1.

Since the energy functional $E_{\lambda}^{\ast\ast}$ from (1.5) is convex and (Gâteaux-) differentiable, the minima are equivalently characterized by the corresponding Euler-Lagrange equations [6]. Thus, problem $(RP)$ reads: Find $(\lambda, \mathbf{m}) \in L^2(\Omega) \times L^2(\Omega; \mathbb{R}^d)$ such that

\begin{align}
\mathcal{P}\mathbf{m} + D\phi^{\ast\ast}(\mathbf{m}) + \lambda \mathbf{m} &= \mathbf{f} \quad \text{a.e. in } \Omega, \\
\lambda &\geq 0, |\mathbf{m}| \leq 1, \lambda(1 - |\mathbf{m}|) = 0 \quad \text{a.e. in } \Omega.
\end{align}

**Remark 2.2.** For the uniaxial model case (1.4), direct calculations show $\phi^{\ast\ast}(x) = \frac{1}{2} \sum_{j=2}^{d} (x \cdot \mathbf{z}_j)^2$, where $\mathbf{e} \in \mathbb{R}^d$ is the easy axis and $\{\mathbf{e}, \mathbf{z}_2, \ldots, \mathbf{z}_d\}$ is an orthonormal basis of $\mathbb{R}^d$. Thus, $D\phi^{\ast\ast}(x) = \sum_{j=1}^{d} (x \cdot \mathbf{z}_j) \mathbf{z}_j$.

**Theorem 2.2.** Problem $(RP)$ has at least one solution $(\lambda, \mathbf{m})$. For any two solutions $(\lambda_1, \mathbf{m}_1), (\lambda_2, \mathbf{m}_2)$ of $(RP)$, the magnetic potentials coincide (modulo an additive constant), $\mathcal{L}\mathbf{m}_1 = \mathcal{L}\mathbf{m}_2$. In the uniaxial model case (1.4) the solution is unique, i.e. $(\lambda_1, \mathbf{m}_1) = (\lambda_2, \mathbf{m}_2)$ almost everywhere.

**Proof.** Existence of solutions of $(RP)$ is obtained by the direct method of the calculus of variations. For any solutions $(\lambda_j, \mathbf{m}_j)$ of $(RP)$ and $\delta = \mathbf{m}_2 - \mathbf{m}_1$ equation (2.4) yields

\begin{equation}
(\mathcal{P}\delta ; \delta)_{L^2(\Omega)} + (D\phi^{\ast\ast}(\mathbf{m}_2) - D\phi^{\ast\ast}(\mathbf{m}_1) ; \delta)_{L^2(\Omega)} + (\lambda_2 \mathbf{m}_2 - \lambda_1 \mathbf{m}_1 ; \delta)_{L^2(\Omega)} = 0.
\end{equation}

By orthogonality of $\mathcal{P}$, we have $(\mathcal{P}\delta ; \delta)_{L^2(\Omega)} = \|\mathcal{P}\delta\|_{L^2(\Omega)} \geq 0$. Further, convexity yields that the second term in (2.6) is non-negative. Direct calculation shows the same for the last term [4, Proof of Theorem 2.1]. Thus, all three terms vanish. Hence, $\mathcal{P}\delta = 0$, i.e. the potentials coincide and moreover $\delta$ is (weakly) divergence free in $\mathbb{R}^d$ by definition of $\mathcal{P}$ (and $\mathcal{L}$), see Equation (2.2) in Theorem 2.1.

In the model case we may assume that the easy axis $\mathbf{e} = \mathbf{e}_1$ is the first standard unit vector. The vanishing second term in (2.6) shows that $\delta$ vanishes in all but the $\mathbf{e}_1$ direction. Now use a standard mollification argument: For any test function $\psi \in D(\mathbb{R}^d)$, we have $\psi * \delta \in D(\mathbb{R}^d)$ with $0 = \psi * (\text{div}\delta) = \text{div}(\psi * \delta) = \partial(\psi * \delta)/\partial x_1$. Hence $\psi * \delta$ is constant in $\mathbf{e}_1$ direction and must therefore vanish. This shows $\delta = 0$. From (2.4) and (2.5) we infer that $\lambda_j$ is uniquely determined by $(\mathbf{m}_j, \mathbf{f})$. Therefore uniqueness of $\mathbf{m}_j$ implies uniqueness of $\lambda_j$.

Let $\mathcal{T} = \{T_1, \ldots, T_N\}$ be a finite family of pairwise disjoint non-empty open sets $T_j$ which satisfy $\overline{\Omega} = \bigcup_{j=1}^{N} \overline{T_j}$. The space of all $\mathcal{T}$-piecewise constant functions is denoted by $L^0(\mathcal{T})$ and
Theorem 3.1. The preceding argument applies to more general (but not for all) triangulations. This section provides an a priori and a posteriori error analysis for the proposed discrete scheme (RP_{\varepsilon,h}) with or without a further monotonicity assumption (3.4) on $\phi^{**}$ valid in the uniaxial case (1.4).

**Theorem 3.2.** Let $(\lambda, m)$ and $(\lambda_{h}, m_{h})$ solve $(RP)$ and $(RP_{\varepsilon,h})$ respectively. Then,

$$
\|\mathcal{P}m - \mathcal{P}m_{h}\|_{L^2(\Omega)}^2 + 2\langle D\phi^{**}(m) - D\phi^{**}(m_{h}) ; m - m_{h}\rangle_{L^2(\Omega)} + \|\sqrt{\varepsilon} \lambda_{h} m_{h}\|_{L^2(\Omega)}^2
$$

$$
\leq 3\|m - m_{T}\|_{L^2(\Omega)}^2 + \|D\phi^{**}(m) - (D\phi^{**}(m_{h}))_{T}\|_{L^2(\Omega)}^2 + \|\lambda m - (\lambda m)_{T}\|_{L^2(\Omega)}^2
$$

$$
+ \|\sqrt{\varepsilon} \lambda m\|_{L^2(\Omega)}^2.
$$

(Note, that according to convexity, the second term on the left-hand side is also non-negative.)
Proof. To abbreviate notation, define $\mathbf{d} := D\phi^*(\mathbf{m})$ and $\mathbf{d}_h := D\phi^*(\mathbf{m}_h)$. Using the orthogonal projection $\mathcal{P}$ and the Cauchy inequality, we infer

$$\|\mathcal{P}\mathbf{m} - \mathcal{P}\mathbf{m}_h\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|\mathcal{P}\mathbf{m} - \mathcal{P}\mathbf{m}_h\|^2_{L^2(\Omega^d)} + \frac{1}{2}\|\mathbf{m} - \mathbf{m}\|^2_{L^2(\Omega)} + \langle \mathcal{P}\mathbf{m} - \mathcal{P}\mathbf{m}_h ; \mathbf{m} - \mathbf{m}_h \rangle_{L^2(\Omega)}.$$

(3.2)

According to the Galerkin orthogonality

$$\langle \mathcal{P}\mathbf{m} - \mathcal{P}\mathbf{m}_h + \mathbf{d} - \mathbf{d}_h + \lambda \mathbf{m} - \lambda_h \mathbf{m}_h ; \nu \rangle_{L^2(\Omega)} = 0 \text{ for all } \nu \in \mathcal{L}^0(\Omega^d),$$

the last term in (3.2) may be written as

$$\langle \mathcal{P}\mathbf{m} - \mathcal{P}\mathbf{m}_h ; \mathbf{m} - \mathbf{m}_h \rangle_{L^2(\Omega)} = -\langle \mathcal{P}\mathbf{m} - \mathcal{P}\mathbf{m}_h ; \mathbf{d} - \mathbf{d}_h \rangle_{L^2(\Omega)} - \langle \mathbf{m} - \mathbf{m}_h ; \mathbf{m} - \mathbf{m}_h \rangle_{L^2(\Omega)} + \langle \mathbf{d} - \mathbf{d}_h ; \mathbf{m} - \mathbf{m}_h \rangle_{L^2(\Omega)} + \langle \lambda \mathbf{m} - \lambda_h \mathbf{m}_h ; \mathbf{m} - \mathbf{m}_h \rangle_{L^2(\Omega)}.$$

Since $(\cdot)_T$ is an orthogonal projection, $\mathbf{d}_h$ and $\lambda_h \mathbf{m}_h$ may be replaced by $\mathbf{d}_T$ and $(\lambda \mathbf{m})_T$ in the third and fourth term. Pointwise evaluation [4, Proof of Theorem 4.3] shows

$$-\langle \lambda \mathbf{m} - \lambda_h \mathbf{m}_h ; \mathbf{m} - \mathbf{m}_h \rangle_{L^2(\Omega)} \leq \frac{1}{2}\|\sqrt{\varepsilon} \lambda \mathbf{m}\|^2_{L^2(\Omega)} - \frac{1}{2}\|\sqrt{\varepsilon} \lambda_h \mathbf{m}_h\|^2_{L^2(\Omega)}.$$

Combining the last two results with two Cauchy inequalities, we conclude (3.1). \hfill \Box

Theorem 3.2. Let $(\lambda, \mathbf{m})$ and $(\lambda_h, \mathbf{m}_h)$ solve (RP) and $(RP_{\varepsilon,h})$, respectively and assume that there is a constant $c_1 > 0$ such that, for all $\mathbf{m}_1, \mathbf{m}_2 \in L^2(\Omega^d)$, there holds

$$c_1\|D\phi^*(\mathbf{m}_1) - D\phi^*(\mathbf{m}_2)\|^2_{L^2(\Omega)} \leq \langle D\phi^*(\mathbf{m}_1) - D\phi^*(\mathbf{m}_2) ; \mathbf{m}_1 - \mathbf{m}_2 \rangle_{L^2(\Omega)}.$$

(3.4)

Then, there is a constant $c_2 > 0$ which depends only on $c_1$ such that

$$\|\mathcal{P}\mathbf{m} - \mathcal{P}\mathbf{m}_h\|^2_{L^2(\Omega^d)} + \|D\phi^*(\mathbf{m}) - D\phi^*(\mathbf{m}_h)\|^2_{L^2(\Omega)} + \|\lambda \mathbf{m} - \lambda_h \mathbf{m}_h\|^2_{L^2(\Omega)}$$

$$\leq c_2 \left(1 + \|\varepsilon\|_{L^\infty(\Omega)}\right)^2 \left\{\|\mathbf{m} - \mathbf{m}_T\|^2_{L^2(\Omega)} + \|D\phi^*(\mathbf{m}) - (D\phi^*(\mathbf{m}))_T\|^2_{L^2(\Omega)} + \|\lambda \mathbf{m} - (\lambda \mathbf{m})_T\|^2_{L^2(\Omega)} + \|\varepsilon\|_{L^\infty(\Omega)}\|\sqrt{\varepsilon} \lambda \mathbf{m}\|^2_{L^2(\Omega)}\right\}.$$

(3.5)

Proof. Use notation from the proof of Theorem 3.1. Direct calculation with Galerkin orthogonality, orthogonal projections $(\cdot)_T$ and $\mathcal{P}$, and simple use of the Cauchy inequality shows

$$\|\lambda \mathbf{m} - \lambda_h \mathbf{m}_h\|^2_{L^2(\Omega)}$$

$$\leq 4\left(\|\lambda \mathbf{m} - (\lambda \mathbf{m})_T\|^2_{L^2(\Omega)} + \|\mathcal{P}\mathbf{m} - \mathcal{P}\mathbf{m}_h\|^2_{L^2(\Omega^d)} + \|\mathbf{d} - \mathbf{d}_h\|^2_{L^2(\Omega)}\right),$$

whence LHS of (3.5) $\leq 5 \times$ RHS of (3.6). Assumption (3.4) allows to dominate the last two terms by Theorem 3.1 which leads to $c_3\left(\|\sqrt{\varepsilon} \lambda \mathbf{m}\|^2_{L^2(\Omega)} - \|\sqrt{\varepsilon} \lambda_h \mathbf{m}_h\|^2_{L^2(\Omega)}\right)$ on the right-hand side with $c_3 := 5 \max\{1, c_1^{-1}\}$. Elementary calculations for scalars $a, b, c \in \mathbb{R}$ show $c(a^2 - b^2) = c(a + b)(a - b) \leq \sqrt{2}c(a^2 + b^2)^{1/2}|a - b| \leq c^2(a^2 + b^2) + |a - b|^2/2$, whence

$$c_3\left(\|\sqrt{\varepsilon} \lambda \mathbf{m}\|^2_{L^2(\Omega)} - \|\sqrt{\varepsilon} \lambda_h \mathbf{m}_h\|^2_{L^2(\Omega)}\right) \leq c_3^2\|\varepsilon \lambda \mathbf{m}\|^2_{L^2(\Omega)} + \|\varepsilon \lambda_h \mathbf{m}_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|\lambda \mathbf{m} - \lambda_h \mathbf{m}_h\|^2_{L^2(\Omega)}$$

by pointwise application and integration over $\Omega$. Finally, the second term is dominated by $\|\varepsilon \lambda_h \mathbf{m}_h\|^2_{L^2(\Omega)} \leq \|\varepsilon\|_{L^\infty(\Omega)}\|\sqrt{\varepsilon} \lambda_h \mathbf{m}_h\|^2_{L^2(\Omega)}$ and a second application of Theorem 3.1. \hfill \Box
Remark 3.1. Theorem 3.2 applies in particular for the uniaxial case, where we have equality with \( c_1 = 1 \) in the monotonicity assumption (3.4).

Remark 3.2. Assume the monotonicity assumption (3.4) and that the exact solution is sufficiently smooth. Whereas Theorem 3.1 leads to an estimate of order \( O(\varepsilon^{1/2} + h) \) for the error \( \| \mathcal{P}m \! - \! \mathcal{P}m_h \|_{L^2(\mathbb{R}^d)} + \| D\phi^{**}(m) - D\phi^{**}(m_h) \|_{L^2(\Omega)} \), Theorem 3.2 leads to \( O(\varepsilon + h) \). This favours the choice \( \varepsilon = h \) for the penalization parameter.

Remark 3.3. Note that the full \( L^2 \) convergence of \( m_h \) towards \( m \) could not be proven although it is observed in the numerical experiments. For the uniaxial case, Theorem 3.2 yields the \( L^2 \) convergence in all directions but the easy axis, cf. Remark 2.2.

Theorem 3.3. Let \((\lambda, m)\) and \((\lambda_h, m_h)\) solve \((RP)\) and \((RP_{\varepsilon,h})\) respectively and assume monotonicity as in (3.4). Then,

\[
\| \mathcal{P}m \! - \! \mathcal{P}m_h \|_{L^2(\mathbb{R}^d)}^2 + c_1 \| D\phi^{**}(m) \! - \! D\phi^{**}(m_h) \|_{L^2(\Omega)}^2 \leq (1 + 1/c_1) \|\lambda \! - \! \lambda_h \|_{L^2(\Omega)}^2 + 2\|\varepsilon |\lambda_h| m_h |(f \! - \! f_T) \! - \! (\mathcal{P}m_h \! - \! \mathcal{P}m_h)_T| \|_{L^1(\Omega)} \]

(3.7)

\[
+ 2 \langle (f - f_T) - (\mathcal{P}m_h - (\mathcal{P}m_h)_T) ; m - m_T \rangle_{L^2(\Omega)}.
\]

Remark 3.4. (a) In fact, the last term on the right-hand side of (3.7) is not an a posteriori term but can always be dominated by an application of the Hölder inequality and (1.6)

\[
\langle (f - f_T) - (\mathcal{P}m_h - (\mathcal{P}m_h)_T) ; m - m_T \rangle_{L^2(\Omega)} \leq 2 \|f - f_T\|_{L^2(\Omega)} \|\mathcal{P}m_h - (\mathcal{P}m_h)_T\|_{L^1(\Omega)},
\]

where we used the side-constraint \( \|m\|_{L^\infty(\Omega)} \leq 1 \).

(b) For \( m \in W^{1,\infty}(\Omega; \mathbb{R}^d) \) and \( C = C_P \|m\|_{W^{1,\infty}(\Omega)} \), Poincaré’s inequality yields

\[
\langle (f - f_T) - (\mathcal{P}m_h - (\mathcal{P}m_h)_T) ; m - m_T \rangle_{L^2(\Omega)} \leq C \|h \{ (f - f_T) - (\mathcal{P}m_h - (\mathcal{P}m_h)_T) \} \|_{L^1(\Omega)}.
\]

Remark 3.5. We did not succeed to derive an a posteriori bound for the a priori term \( \|\lambda m - \lambda_h m_h\|_{L^2(\Omega)} \).

Proof of Theorem 3.3. Adopt notation from the proof of Theorem 3.1. By definition of the discretization scheme, we have

\[
f - f_T = (\mathcal{P}m - \mathcal{P}m_h) + (\mathcal{P}m_h - (\mathcal{P}m_h)_T) + (d - d_h) + (\lambda m - \lambda_h m_h) \quad \text{a.e. in } \Omega.
\]

This and the elementary inequality [4, Proof of Theorem 5.2]

\[
-\langle \lambda m - \lambda_h m_h ; m - m_h \rangle_{L^2(\Omega)} \leq \int_{\Omega} \varepsilon |\lambda_h| m_h |\lambda m - \lambda_h m_h| \, dx
\]

allow to dominate the LHS of (3.7),

\[
\| \mathcal{P}m \! - \! \mathcal{P}m_h \|_{L^2(\mathbb{R}^d)}^2 + c_1 \|d - d_h\|_{L^2(\Omega)}^2 \leq \langle (f - f_T) - (\mathcal{P}m_h - (\mathcal{P}m_h)_T) ; m - m_h \rangle_{L^2(\Omega)} + \int_{\Omega} \varepsilon |\lambda_h| m_h |\lambda m - \lambda_h m_h| \, dx.
\]

In the scalar product \( m - m_h \) may be replaced my \( m - m_T \) due to orthogonality of \( (\cdot)_T \). Inserting \( \lambda m - \lambda_h m_h \) from (3.8) into the integrand and serious use of the Cauchy inequality yield the assertion. \( \square \)
4. Numerical Algorithms

This section is devoted to the implementation of \((RP_{\varepsilon,h})\) for the uniaxial case \((1.4)\) in MATLAB. The discrete problem \((RP_{\varepsilon,h})\) leads to a nonlinear systems of equations solved by a Newton-Raphson scheme. The Subsections 4.2 and 4.3 describe an adaptive mesh-refinement based on refinement indicators motivated by Theorem 3.3 and the practical computation of the refinement indicators. The computation of both, the Galerkin element and the computation of the refinement indicators, involves the integral operator \(\mathcal{P}\) and hence leads to dense matrices. Subsection 4.4 gives an outlook at their efficient approximation with an \(\mathcal{H}\)-matrix approach.

4.1. Computation of the Discrete Solution \(m_h\). Given \(T = \{T_1, \ldots, T_N\}\), the set \(\mathcal{B} := \{\chi_{T_j} e_k | 1 \leq j \leq N, 1 \leq k \leq d\}\) is a basis of \(L^0(T)^d\), where \(e_k\) denotes the \(k\)-th standard unit vector in \(\mathbb{R}^d\). The computation of a discrete solution \(m_h = \sum_{j=1}^{dN} \mu_j \varphi_j\) is done via a Newton-Raphson scheme. To abbreviate notation and to fix a numbering of the basis elements \(\varphi_\ell \in \mathcal{B}\), let

\[
\begin{cases}
[j, 1] := j & \text{for } d = 2 \\
[j, 2] := j + N & \text{for } d = 3,
\end{cases}
\]

respectively, and for all \(1 \leq j \leq N\). Further let \(\varphi_{[j,k]} := \chi_{T_j} e_k \in \mathcal{B}\). With \(x \in \mathbb{R}^{dN}\) and

\[
m_h = \sum_{j=1}^{N} \sum_{k=1}^{d} x_{[j,k]} \varphi_{[j,k]},
\]

the Equation \((2.7)\) is equivalent to the nonlinear system \(F(x) = 0\) with

\[
F : \mathbb{R}^{dN} \to \mathbb{R}^{dN}, \quad F_\ell = \langle \mathcal{P}m_h ; \varphi_\ell \rangle_{L^2(\Omega)} + \langle \lambda_h m_h + D \varphi_\ell^*(m_h) - f ; \varphi_\ell \rangle_{L^2(\Omega)}.
\]

Thus, the discrete scheme needs the computation of the matrix

\[
A \in \mathbb{R}^{dN \times dN}, \quad A_{mn} := \langle \mathcal{P} \varphi_m ; \varphi_n \rangle_{L^2(\Omega)} \quad \text{for basis functions } \varphi_m, \varphi_n \in \mathcal{B}.
\]

Provided all \(T_j \in \mathcal{T}\) are bounded Lipschitz domains, the following lemma allows for the exact computation of \((4.3)\).

Lemma 4.1 ([11, 18]). Let \(m, \tilde{m} \in \mathbb{R}^d\) and let \(\omega, \tilde{\omega} \subseteq \mathbb{R}^d\) be bounded Lipschitz domains with outer normals \(n, \tilde{n}\), respectively. Then, \(A(\chi_\omega m, \chi_\omega \tilde{m}) := \langle \mathcal{P}(\chi_\omega m) ; \chi_\omega \tilde{m} \rangle_{\mathbb{R}^d}\) satisfies

\[
A(\chi_\omega m, \chi_\omega \tilde{m}) = A(\chi_\omega \tilde{m}, \chi_\omega m) = A(\chi_\omega m, \chi_\omega m)
\]

\[
= -\int_{\partial \omega} \int_{\partial \tilde{\omega}} G(x-y)(n(x) \cdot m)(\tilde{n}(y) \cdot \tilde{m}) ds_y ds_x. \quad \square
\]

Remark 4.1. In the context of the boundary element method, boundary integrals

\[
\int_E \int_E G(x-y) ds_y ds_x
\]

occur for the computation of the Galerkin elements for Symm’s integral equation and piecewise constant ansatz functions. Analytic formula are known for \(E, \tilde{E}\) being affine boundary pieces \((d = 2) [15, 5]\) or axis oriented flat rectangles \((d = 3) [15, 9]\).
4.2. Adaptive Mesh-Refinement. Theorem 3.3 gives rise to the error estimators

\[ \mu := \left( \sum_{T \in \mathcal{T}} \mu_T^2 \right)^{1/2} \quad \text{and} \quad \eta := \left( \sum_{T \in \mathcal{T}} \eta_T^2 \right)^{1/2}, \]

where the refinement indicators \( \mu_T, \eta_T \), for \( T \in \mathcal{T} \), are defined by

\[ \ell_T := (\varepsilon \lambda_h |m_h|)|T| = (|m_h| - 1), \]

\[ \mu_T^2 := (1 + \ell_T)\|f - f_T| - (\mathcal{P}m_h - (\mathcal{P}m_h)_T)|L_1(T) + |T|\ell_T^2, \]

\[ \eta_T^2 := (\ell_T + |f - f_T| - (\mathcal{P}m_h - (\mathcal{P}m_h)_T)|L_1(T) + |T|\ell_T^2. \]

**Remark 4.2.** (a) The estimator \( \mu \) is reliable, i.e. an upper bound for the error \( \|\mathcal{P}m - \mathcal{P}m_h\|_{L^2(\mathbb{R}^d)} + \|\mathcal{D}q^* - (m_h)|L_2(\Omega) \) up to a multiplicative constant.
(b) \( \mu \) cannot be efficient, i.e. a lower bound for the error.
(c) \( \eta \) is reliable for \( m \in W^{1, \infty}(\Omega; \mathbb{R}^d) \), but not in general.
(d) Efficiency of \( \eta \) is expected but could not be proven.

**Algorithm 4.2** (Adaptive Mesh-Refinement). Let \( \mathcal{T}^{(0)} \) be the initial triangulation, \( n = 0, \alpha > 0 \), and \( 0 \leq \theta \leq 1 \).

(i) For \( T_j \in \mathcal{T}^{(n)} = \{T_1, \ldots, T_N\} \) choose a penalization parameter \( \varepsilon_j = h_j^\theta > 0 \).

(ii) Compute approximation \( m_h \) with respect to the to current triangulation \( \mathcal{T}^{(n)} \) and \( \varepsilon \in \mathcal{C}(\mathcal{T}^{(n)}) \), \( \varepsilon|_{\mathcal{T}_j} := \varepsilon_j \), by Newton-Raphson scheme.

(iii) Compute error estimators \( \mu \) and \( \eta \) from (4.5) and refinement indicators \( \eta_j := \eta_{T_j} \) and \( \mu_j := \mu_{T_j} \) from (4.6).

(iv) Mark an element \( T_j \in \mathcal{T}^{(n)} \) provided \( \eta_j \geq \theta \max_{1 \leq k \leq N} \eta_k \) for \( \eta \)-adaptive mesh-refinement and provided \( \mu_j \geq \theta \max_{1 \leq k \leq N} \mu_k \) for \( \mu \)-adaptive mesh-refinement.

(v) Refine the marked elements, update \( n \mapsto n + 1 \), and go to (i).

**Remark 4.3.** The choice \( \theta = 0 \) in Algorithm 4.2 leads to uniform mesh-refinement, whereas \( \theta \approx 1 \) leads to highly adapted meshes. In the numerical experiments, \( \theta = 0 \) or \( \theta = 1/2 \).

**Remark 4.4.** To lower the computational cost for the Newton-Raphson scheme, we used nested iterations: In step (ii) of Algorithm 4.2, the Newton-Raphson scheme was started with the prolongated discrete solution \( m_h^{(n-1)} \) for the previous grid \( \mathcal{T}^{(n-1)} \).

4.3. Implementation of the Refinement Indicators. The \( L^1 \) norm in the definition of \( \mu_T \) and \( \eta_T \), respectively, was computed by a \((2 \times 2)\)-tensor Gauss quadrature rule. The following lemma shows that the point evaluation of \( \mathcal{P}m_h \) is well-defined outside the skeleton of \( \mathcal{T} \).

**Lemma 4.3.** For \( m_h \in \mathcal{C}(\mathcal{T}; \mathbb{R}^d) \), the corresponding potential satisfies \( \mathcal{L}m_h \in \mathcal{C}(\mathbb{R}^d) \cap \mathcal{C}(\mathcal{T}, \mathcal{S}) \), where \( \mathcal{S} := \bigcup \{\partial T \mid T \in \mathcal{T}\} \) denotes the skeleton of the triangulation. Moreover, the derivative \( \mathcal{P}m_h = \nabla(\mathcal{L}m_h) \) can be computed pointwise by

\[ \mathcal{P}m_h(x) = \frac{1}{|\mathcal{S}|} \sum_{T \in \mathcal{T}} \int_{\partial T} \frac{m_h|T \cdot (x - y)}{|x - y|^d} \mathbf{n}(y) \, ds \quad \text{for} \ x \in \mathbb{R}^d \setminus \mathcal{S}, \]

where \( \mathbf{n}(y) \) denotes the outer normal with respect to \( T \in \mathcal{T} \) and \( \mathcal{S} \subseteq \mathbb{R}^d \) the unit sphere.
Lemma 4.4. and memory, by an approximation $2 \mathbf{A} \mathcal{E}cien\text{t} \text{R}ealization \text{of} \text{the Involved Integral Operator} 4.4.$

@T replaces (4.7) occur for the computation of the double layer potential for piecewise constants. If one

Remark

Computation of the

Proof.

The idea of the

4.4. Efficient Realization of the Involved Integral Operator $\mathcal{P}$. The dense matrix $\mathbf{A} \in \mathbb{R}^{dN \times dN}$ from (4.3) has certain symmetry properties. To decrease computation time and memory, $\mathcal{H}$- and $\mathcal{H}^2$-matrix approaches can be used [1, 11, 18, 17], where $\mathbf{A}$ is replaced by an approximation $\tilde{\mathbf{A}}$.

Lemma 4.4. For any bounded open sets $\omega, \tilde{\omega} \subseteq \mathbb{R}^d$ with $\text{dist}(\omega, \tilde{\omega}) > 0$, for $\alpha, \beta = 1, \ldots, d$, and the $l$-th canonical unit vector $\mathbf{e}_l \in \mathbb{R}^d$, the bilinear form $A(\cdot, \cdot)$ from Lemma 4.1 satisfies

$$A(\chi_{\omega} \mathbf{e}_\alpha, \chi_{\tilde{\omega}} \mathbf{e}_\beta) = \int_{\omega} \int_{\tilde{\omega}} \frac{\partial^2 G}{\partial x_\alpha \partial x_\beta} (x - y) \, dy \, dx.$$ 

Proof. The lemma follows from standard results on convolutions.

The idea of the $\mathcal{H}^2$-matrix approach is to approximate the kernel $g_{\alpha \beta}(x, y) := \frac{\partial^2 G}{\partial x_\alpha \partial x_\beta} (x - y)$ based on panel clustering. For certain $\sigma, \tau \subseteq \mathcal{T}$ with $\text{dist}(\cup \sigma, \cup \tau) > 0$, let vectors $x^{(\sigma)}_{m_1} \in \cup \sigma, y^{(\tau)}_{m_2} \in \cup \tau$ and polynomials $p^{(\sigma)}_{m_1}, p^{(\tau)}_{m_2}$ on $\cup \sigma$ resp. $\cup \tau$ be given and define

$$\tilde{g}_{\alpha \beta}(x, y) := \sum_{m_1 = 1}^{M_1} \sum_{m_2 = 1}^{M_2} g_{\alpha \beta}(x^{(\sigma)}_{m_1}, y^{(\tau)}_{m_2}) \, p^{(\sigma)}_{m_1}(x) \, p^{(\tau)}_{m_2}(y) \quad \text{for} \ (x, y) \in \cup \sigma \times \cup \tau.$$

For $T_j \in \sigma$ and $T_k \in \tau$ there holds the approximation

$$\int_{T_j} \int_{T_k} g_{\alpha \beta}(x, y) \, dy \, dx \approx \sum_{m_1 = 1}^{M_1} \sum_{m_2 = 1}^{M_2} \tilde{g}_{\alpha \beta}(x^{(\sigma)}_{m_1}, y^{(\tau)}_{m_2}) \left\{ \int_{T_j} p^{(\sigma)}_{m_1}(x) \, dx \right\} \left\{ \int_{T_k} p^{(\tau)}_{m_2}(y) \, dy \right\}.$$
For fixed $\alpha, \beta$, consider the matrix $B \in \mathbb{R}^{N \times N}$, $B_{jk} := A(\chi_{T_j} e_\alpha, \chi_{T_k} e_\beta)$. With the matrices $C(\alpha) \in \mathbb{R}^{|\sigma| \times M_1}$, $C(\tau) \in \mathbb{R}^{|\tau| \times M_2}$, and $D \in \mathbb{R}^{M_1 \times M_2}$ defined in (4.9), the submatrix $B|_{\sigma \times \tau}$ from $B$ satisfies
\begin{equation}
B|_{\sigma \times \tau} \approx C(\alpha) D(C(\tau))^T.
\end{equation}
The use of the latter approximation significantly reduces the computational cost for assembling the matrix $B|_{\sigma \times \tau}$ provided $\max\{M_1, M_2\} < \min\{|\sigma|, |\tau|\}$.\n
**Remark 4.7.** Notice that only the matrix $D$ in (4.9) depends on $\alpha$ and $\beta$ and the matrix $A$ can be approximated by the block-matrix $\tilde{A}$ with blocks of $\mathcal{H}^2$-matrix type. The time to assemble the matrix $B$ could be highly decreased by use of the indicated $\mathcal{H}^2$-matrix approach. However, all experiments in this paper have been made using the *exactly computed* matrix $A$, but the much cheaper $\mathcal{H}^2$-matrix approach leads to (almost) the same accuracy, in (almost) linear complexity. [Since the present implementation is in MATLAB, comparisons will appear in [17].]\n
**Remark 4.8.** The computation of the refinement indicators can also be based on an $\mathcal{H}$-matrix approach since the computation of $\mathcal{P}m_h(x)$ corresponds to a collocation method with the double layer potential, cf. Lemma 4.3.

5. **Numerical Example With Exact Solution** $m \in W^{1,\infty}(\Omega; \mathbb{R}^2)$

The unit square $\Omega = (0, 1)^2$ is filled with a uni-axial magnetic material (1.4) with easy axis $e = (-1, 1)/\sqrt{2}$, i.e. $z = (1, 1)/\sqrt{2}$ in Remark 2.2. Define
\begin{equation}
m(x) := \begin{cases} x & \text{for } |x| \leq 1, \\
x/|x| & \text{for } |x| \geq 1 \end{cases} \quad \text{and} \quad \lambda(x) := \begin{cases} 0 & \text{for } |x| < 1, \\
1 & \text{for } |x| \geq 1. \end{cases}
\end{equation}

Then, $(m, \lambda) \in W^{1,\infty}(\Omega; \mathbb{R}^2) \times L^{\infty}(\Omega)$ solves (2.4)-(2.5) with given right-hand side
\begin{equation}
f := \mathcal{P}m + (m \cdot z)z + \lambda m \in L^2(\Omega; \mathbb{R}^2).
\end{equation}

In all our numerical experiments we replaced $\mathcal{P}m$ on the right-hand side of (5.2) by $\mathcal{P}m_T$ for the elementwise integral means $m_T$ of $m$. Recall that Lemma 4.1 allows the exact integration of $\mathcal{P}m_T$. Figure 5.1 shows discrete solutions $m_h$ for the penalization parameter $\alpha = 1$.\n
**Figure 5.1.** Discrete solution $(m_h, \lambda_h)$ in Section 5 on $T_h$ (with $N = 1024$) for penalization parameter $\varepsilon = h$: $m_h$ on the left as vectors $m_h|_T$ and $|m_h|_T$ in grey scale and $\lambda_h$ in grey scale on the right.
For a given sequence of $h$-uniform meshes with $N = h^{-2}$ elements in $T^{(n)}$, the first set of experiments studies the choice of the parameter $\alpha > 0$ in the penalization $\varepsilon = h^\alpha$. Figure 5.2 displays the $L^2$ error of the magnetization vectors as a function of the mesh-size $h = N^{-2}$ for 12 values of $\alpha$. Any choice of $\alpha \geq 1$ seems to result in a linear convergence while values $\alpha < 1$ seem to result in smaller experimental convergence rates (until $\alpha = 1/4$ with almost no convergence). The length $|m_h(x)|$ for $|x| > 1$ and $\alpha = 1/2$ are about 1.1 compared to $\leq 1.01$ for $\alpha = 3/2$, cf. Figure 5.1. The value $\alpha = 3/2$ is recommended throughout all examples of this paper. Theoretical estimates concern the $z$ direction of $m_h$ exclusively. In the numerical examples, however, linear convergence is observed also for the easy axis direction $e$. Notice that $m$ is essentially smooth and hence adaptive mesh-refinements cannot improve the experimental convergence rates further.

In conclusion, the first example gives empirical support for the a priori analysis and the choice of the penalization parameter. As indicated by Theorem 3.2, the choice of $\varepsilon = h^\alpha$ with $\alpha \geq 1$ appears to be necessary for optimal experimental convergence behaviour. The lower order of convergence for a choice of $\alpha < 1$ can be explained as follows: Theorem 3.2 show the $L^2$ convergence $\lambda_h m_h \to \lambda m$ in $\Omega$, in particular, on the restricted domain $\omega := \{x \in \Omega \mid |x| \geq 1\}$, i.e. the smaller $\alpha$ the larger is the length $|m_h|$: 

$$h^{-\alpha}(|m_h| - 1)_+ = \lambda_h |m_h| \to \lambda |m| = 1 \quad \text{in } L^2(\omega).$$
6. Numerical Example With Exact Solution \( m \notin H^1(\Omega) \)

This section is devoted to the numerical approximation of a more singular magnetization
\[
(\mathbf{m}(x), \lambda(x)) := \begin{cases} 
(y(x), 0) & \text{for } x \in \omega, \\
(x_1 x_2 (1 - y_1(x))^{-1} (1 - y_2(x))^{-2} y(x), 1) & \text{for } x \in \Omega \setminus \omega
\end{cases}
\]
with a singular gradient at the three vertices \((0, 1), (1, 0), (1, 1)\) on the boundary of the
magnetic body \( \Omega = (0, 1)^2 \). Here,
\[
y(x) := \frac{(1, 1) - x}{|(1, 1) - x|} \quad \text{and} \quad \omega := \{ x \in \Omega \mid |(1, 1) - x| < 1 \}.
\]

The remaining data \( \phi, z, \) and \( f \) are as in Section 5. The magnetization vector \( \mathbf{m} \) (6.1) and
the error by the piecewise integral means are depicted in Figure 6.1. We observe a larger
elementwise \( L^2 \) error in \( \Omega \setminus \omega \) and hence expect the necessity of adaptive mesh-refining for
an effective computation. For a comparison, Figure 6.2 displays the best approximation \( \mathbf{m}_T \)
and its elementwise \( L^2 \) errors \( \| \mathbf{m} - \mathbf{m}_T \|_{L^2(T)} \) on an adapted mesh. The latter was generated
by Algorithm 4.2 with the refinement indicator \( \varrho_j := \| \mathbf{m} - \mathbf{m}_T \|_{L^2(T_j)} \), i.e. an element \( T_j \) is
marked in step (iv) if \( \varrho_j \geq 1/2 \max_{1 \leq k \leq n} \varrho_k \). The singularity at \((1, 1)\) is visible in Figure 6.1
and 6.2 as well as a refinement near the arc \( \{ x \in \Omega \mid |(1, 1) - x| = 1 \} \). There is no theoretical
support that the refinement indicator \( \varrho_j \) lead to optimal meshes; but it allows an interesting
theoretical comparison. Also, heuristically we expect optimal meshes (asymptotically) since
the mesh-refinement with respect to \( \varrho \) leads to meshes on which the best-approximation
errors are equi-distributed.

A comparison between Figure 6.1 and Figure 6.2, recall the different scalings in the right
figures, shows that adaptive meshes have the potential of improvements. Numerical evidence
for this is provided in Figure 6.3. Besides various choices of penalization parameter \( \alpha \) with
conclusions similar to those of Section 5, the sequence of uniform and \( \varrho \)-adapted mesh-
refining are compared. The adaptive meshes yield linear convergence in a reference mesh-size
\( h := N^{-2} \) even in all components of \( \| \mathbf{m} - \mathbf{m}_T \|_{L^2(\Omega)} \). The sequence of uniform meshes show
a suboptimal convergence rate.

![Figure 6.1](image-url)
Figure 6.2. Best approximation $\mathbf{m}_T$ (left) in Section 6 on an error adapted generated mesh (with $N = 1717$) and elementwise distribution of the corresponding best approximation error $\|\mathbf{m} - \mathbf{m}_T\|_{L^2(\Omega)}$ (right). For the adaptive mesh-refinement, we used Algorithm 4.2 with refinement indicators $\eta_j = \|\mathbf{m} - \mathbf{m}_T\|_{L^2(\Omega_j)}$. Notice that the grey scale displays values multiplied with $1(\text{left})$ and $10^{-4} = 1/10000(\text{right})$.

Figure 6.4 and 6.5 display the $L^2$ error $\|\mathbf{m} - \mathbf{m}_h\|_{L^2(\Omega)}$ and the error estimators $\mu$ and $\eta$ from (4.5) as functions of the number of elements $N$ for various sequences of meshes. Those are generated by Algorithm 4.2 with refinement indicators $\eta_j$ and $\mu_j$ from (4.6) and penalizations $\varepsilon = h^\alpha$ for $\alpha = 1$ and $\alpha = 3/2$. The two penalizations show similar convergence rates; the overall recommendation of $\alpha = 3/2$ is again supported in Figure 6.4 by better results. Each of the two adaptive algorithms leads to optimal convergence rates and is (asymptotically) factor 2 for $\alpha = 1$ resp. 1.3 for $\alpha = 3/2$ worse than the best approximation errors.

Figure 6.3. Error $\|\mathbf{m} - \mathbf{m}_h\|_{L^2(\Omega)}$ and dependence from penalization parameter $\varepsilon = h^\alpha$ for uniform mesh-refinement. For comparison, also the best approximation errors on uniform and adapted meshes from Figure 6.2 are shown.
Figure 6.5 illustrates the reliability-efficiency-gap [3]: What is reliable is not efficient and what is efficient is not (known to be) reliable. Theorem 3.2 and Remark 3.1 show that the error terms are bounded from above by $c_1 \mu$ and $c_2(\mathbf{m}) \eta$ and the later bound is of higher order but only valid for a smooth magnetization. The second estimate is also expected to be efficient (up to higher-order terms in the magnetization). Figure 6.5 displays $\eta$ and $\mu$ and clearly shows their different convergence rates. From Figure 6.5 there is no support that adaptive is more effective than uniform mesh-refining.

![Figure 6.4. Error $\|\mathbf{m} - \mathbf{m}_h\|_{L^2(\Omega)}$ for uniform, $\mu$-adaptive, and $\eta$-adaptive mesh-refinement with penalization parameter $\varepsilon = h^\alpha$ or $\varepsilon = h^{3/2}$. For comparison, the best approximation error is shown for uniform and optimal adapted meshes as well. Both adaptive strategies lead to optimal experimental convergence rate $1/2$ in terms of numbers of elements.](image)

7. Real-life Scientific Computing

The ferromagnetic body $\Omega = (-1/2, 1/2) \times (-5/2, 5/2)$ is loaded with a constant applied magnetic field $\mathbf{f} := (0.6, 0)$ aligned with the easy axis $\mathbf{e} = (1, 0)$. Figure 7.3 displays the magnetic potential $u_h = \mathbf{Lm}_h$ and the magnetization vectors $\mathbf{m}_h$ on an adaptively generated mesh. The exact solution $\mathbf{m}$ is unknown. First numerical computations for this example have been performed in [4]. Although there the potential equation

$$\text{div}(-\nabla u + \mathbf{m}) = 0 \quad \text{in } \mathbb{R}^2$$

is discretized and solved by a finite element scheme for a bounded domain that surrounds $\Omega$ instead of the full space, we obtain similar results.

The initial mesh $\mathcal{T}_0$ consists of 5 congruent squares with side-length 1. Figure 7.1 shows $\eta$-adaptively generated meshes $\mathcal{T}_0, \ldots, \mathcal{T}_7$ with $N = 5, \ldots, 1604$ elements. We observe some mesh-refinement towards the 4 vertices of $\Omega$ which we might expect to be caused by singularities in the stray-field. However, this refinement seems to be accomplished in $\mathcal{T}_0, \ldots, \mathcal{T}_6$ as
Figure 6.5. Illustration of Reliability-Efficiency-Gap: Experimental convergence of the error estimators $\eta$ and $\mu$ in Section 6 for uniform, $\eta$-adaptive, and $\mu$-adaptive mesh-refinement with penalization parameter $\varepsilon = h$ or $\varepsilon = h^{3/2}$. The improvement of the error by adaptive mesh-refinement strategies as shown in Figure 6.4 is not reflected by the estimators. There is (up to a multiplicative constant) no improvement of the convergence behaviour by the adaptive mesh-refinement.

Figure 7.1. $\eta$-adaptively generated meshes $T_0$ (with $N = 5$) till $T_7$ (with $N = 1604$) in Section 7 for $f = (0.6, 0)$, $e = (1, 0)$, and $\varepsilon = h^{3/2}$. The grey scale shows the length $|m_h|$ of the discrete solution.

$T_7$ and $T_8$ show a refinement of a more global zone. Figure 7.3 displays the discrete solution which follows the exterior field $f$ and develops some flowering at the tips of $\Omega$. One observes
Figure 7.2. Discrete Lagrange multiplier $\lambda_h$ on $\eta$-adaptively generated meshes $T_0$ (with $N = 5$) till $T_T$ (with $N = 1604$) in Section 7 for $f = (0.6, 0)$, $e = (1, 0)$, and $\varepsilon = h^{3/2}$. The grey scale shows the pointwise value of $\lambda_h$. In the white region we have $\lambda_h \equiv 0$, i.e. $|m_h| \leq 1$.

large curvatures of the magnetization near the top and bottom of the magnet $\Omega$ but no strong point singularity there. Furthermore, Figure 7.3 displays the corresponding magnetic potential $u_h = \mathcal{L}m_h$ computed analytically by

$$\mathcal{L}m_h(x) = \sum_{k=1}^N \int_{\partial T_k} G(x - y) m_h|_{T_k} \cdot n(y) \, ds_y \quad \text{for all } x \in \mathbb{R}^d$$

as it follows from partial integration of (2.3). In comparison with a corresponding numerical experiment in [4] we see that the potential lines of the magnetic potential are not perpendicular on the boundary of the domain displayed. This is a consequence of the correct treatment of the stray field in the full space $\mathbb{R}^2$. More important, the discretization in [4] shows a strong refinement towards the vertices; much stronger than visible in Figure 7.3 or 7.4. To monitor the asymptotic behaviour, Figure 7.4 displays the error estimators $\mu$ and $\eta$. In comparison with uniform and $\eta$- and $\mu$-adaptive mesh-refinement one deduces that, in this example, adaptivity is not important — there is a small improvement but one obtains essentially the same convergence rate for all three strategies. Our interpretation is that, very much to a surprise to us, there is no singularity in the integral-operator model at hand and so the formulation is indeed superior to that of [4].

Finally, Figure 7.2 shows the discrete Lagrange multipliers $\lambda_h$ corresponding to the triangulations from Figure 7.1. They do not indicate some particular resolution of the set \( \{x \in \Omega : \lambda_h(x) = 0\} \) (or some other level set of $\lambda_h$).

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Figure 7.3. Discrete magnetization $m_h$ (zoom on the left) on the $\eta$-adaptively generated mesh $T_4$ (with $N = 236$) and corresponding potential $u_h$ (right) for constant exterior field $f = (6, 0)$, easy axis $e = (1, 0)$, and penalization parameter $\varepsilon = h^{3/2}$. The grey scale in the zoomed magnet displays the length $|m_h|$ of the discrete magnetization. On the right, the pointwise value of $u_h$ is shown by the grey scale and some isolines have been drawn.

References

There is (up to a multiplicative constant) no improvement of the convergence behaviour by the adaptive mesh-refinement, although we obtain some local mesh-refinement towards the corners in Figure 7.1.


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APPLICATIONS OF $\mathcal{H}$-MATRIX TECHNIQUES IN MICROMAGNETICS

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ABSTRACT. The variational model by LANDAU and LIFSHITZ is frequently used in the simulation of stationary micromagnetic phenomena. We consider the limit case of large and soft magnetic bodies, treating the associated Maxwell equation exactly via an integral operator $P$. In numerical simulations of the resulting minimization problem, difficulties arise due to the imposed side-constraint and the unboundedness of the domain. We introduce a possible discretization by a penalization strategy. Here the computation of $P$ is numerically the most challenging issue, as it leads to densely populated matrices. We show how an efficient treatment of both $P$ and the corresponding bilinear form can be achieved by application of $\mathcal{H}$-matrix techniques.

1. Introduction

The simulation of stationary micromagnetic phenomena occurring in static or quasi-static processes is frequently based on a variational model named after LANDAU and LIFSHITZ. Therein, one minimizes the energy functional

$$E_\alpha(m) := \int_\Omega \phi(m) \, dx - \int_\Omega f \cdot m \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \alpha \int_\Omega |\nabla m|^2 \, dx$$

over some set of admissible vector-valued magnetizations $m : \Omega \to \mathbb{R}^d$ on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ corresponding to the magnet, with $m(x) := 0$ for $x \in \mathbb{R}^d \setminus \Omega$ and $d = 2, 3$. Here $\phi \in C^1(\mathbb{R}^d; \mathbb{R}^+)$ is the anisotropy density (depending on properties of the material on a crystalline level), $f \in L^2(\Omega; \mathbb{R}^d)$ denotes an applied exterior magnetic field, $0 \leq \alpha \ll 1$ is the exchange parameter, and $u$ is the magnetic potential related to $m$ by Maxwell’s equation

$$\text{div}(-\nabla u + m) = 0 \quad \text{in} \; D'(\mathbb{R}^d).$$

The model is completed by adding the non-convex constraint

$$|m(x)| = 1 \quad \text{for a.e.} \; x \in \Omega.$$  

For large and soft magnets, the parameter $\alpha$ in (1) vanishes. In general, the model then lacks classical solutions, see [17], and hence has to be relaxed either by considering measure valued solutions [24] or by convexification [10, 28]. In fact, for a certain limit configuration of soft-large bodies, $E_0(m)$, i.e., (1) with $\alpha \to 0$ can be justified to be the correct model, see [10]. The corresponding convexified problem $E^{**}_0$ is given by

$$E^{**}_0(m) := \int_\Omega \phi^{**}(m) \, dx - \int_\Omega f \cdot m \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx$$

subject to (2) and

$$|m(x)| \leq 1 \quad \text{for a.e.} \; x \in \Omega.$$

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Here $\phi^{**}$ is the convexified density defined by
\begin{equation}
\phi^{**}(x) = \sup \{ \varphi(x) \mid \varphi : \mathbb{R}^d \to \mathbb{R} \text{ convex and } \varphi|_S \leq \phi \} \text{ for } |x| \leq 1,
\end{equation}
where $S = \{ x \in \mathbb{R}^d \mid |x| = 1 \}$ denotes the unit sphere. Then, the relaxed problem reads:
\begin{equation}
\text{Minimize } E_0^{**} \text{ over } A := \{ m \in L^\infty(\Omega; \mathbb{R}^d) \mid \|m\|_{L^\infty(\Omega; \mathbb{R}^d)} \leq 1 \}.
\end{equation}
In contrast to the ill-posed problem $E_0$, the convexification is well-posed [10, 24, 6]. In fact, this convexified model provides the mathematical foundation of the so-called phase theory in micromagnetics, cf. [18].

**Remark 1.** For uniaxial materials such as cobalt, the anisotropy energy is given by $\phi(x) = 1/2(1 - (x \cdot e)^2)$, with $|x| = 1$ and $e \in \mathbb{R}^d$ a given fixed unit vector called the easy axis. A direct calculation shows $\phi^{**}(x) = 1/2 \sum_{j=2}^{d}(x \cdot z_j)^2$ for $|x| \leq 1$ then, where \{e, z_2, \ldots, z_d\} is an orthonormal basis of $\mathbb{R}^d$.

The numerical treatment of the minimization problem related to $E_0^{**}$ was initiated by [9] for $d = 2$, where the authors treat a simplified model obtained by replacing $\mathbb{R}^d$ in (2) by a bounded Lipschitz domain $\widehat{\Omega}$ containing $\Omega$, and solve for a potential $u \in H^1_0(\Omega)$. Here, as in [6, 7, 20, 21, 25], (2) is treated exactly via an integral representation, i.e., $u = Lm$, where $L$ is a linear convolution operator. We then set $Pm := \nabla(Lm)$, see Theorem 2.1 below, and reformulate the stray field energy contribution in (4) in terms of $P$. The advantage is that in the resulting model, only one discretization for $m$ is required, e.g., by piecewise constant functions $m_h$.

From a numerical point of view, the computation of $Pm$ for a given magnetization is the most challenging issue, since it will lead to densely populated matrices. The aim of the present work is to show how an efficient numerical treatment of both $P$ and the induced bilinear form $a(\cdot, \cdot)$ can be achieved by application of $\mathcal{H}$-matrix techniques.

**Remark 2.** The treatment of the convexified model (7) requires the explicit knowledge of the convexified anisotropy density $\phi^{**}$, which is, however, in general unknown even for simple $\phi$, cf. [10]. Both the Young measure relaxation proposed in [24] and the corresponding discretization [19] avoid the computation of $\phi^{**}$. Note that as far as the computation of the magnetic potential (2) is concerned, our ideas apply in that setting, as well. Further analysis on stabilized discrete models, as well as a comparison of the various approaches, can be found in the articles [9] and [19], as well as in the survey monograph by Prohl [26].

The remainder of this paper is organized as follows: in Section 2, we give a few preliminaries and present a possible discretization of (7); Section 3 contains some interpolation results required for the following analysis; Section 4 motivates the concept of hierarchical ($H$- resp. $H^2$-) matrices; in Section 5, we give two different approaches for a Galerkin discretization of the potential equation (2) via $H$-matrix techniques; Section 6 finally summarizes the results of our numerical experiments.

## 2. Preliminaries and Discretization

This section is devoted to the reformulation of (7) in terms of the associated Euler-Lagrange equations and introduces a possible discretization by a penalization strategy.
2.1. Preliminaries. The following Theorem 2.1 gathers some of the properties of the operator $\mathcal{P}$ required in the following. Proofs can be found in [25], although we expect the result to be known to the experts.

**Theorem 2.1** ([20, 21, 25]). Given any $\mathbf{m} \in L^\infty(\Omega; \mathbb{R}^d)$, there exists an (up to an additive constant) unique magnetic potential $u = \mathbf{Lm} \in H^1_{\text{loc}}(\mathbb{R}^d)$ such that

$$\nabla u \in L^2(\mathbb{R}^d; \mathbb{R}^d) \quad \text{and} \quad \text{div}(-\nabla u + \mathbf{m}) = 0 \quad \text{in} \, \mathcal{D}'(\mathbb{R}^d).$$

The (extended) operator $\mathcal{P} : L^2(\mathbb{R}^d; \mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d; \mathbb{R}^d)$, $\mathbf{m} \mapsto \nabla(\mathbf{Lm})$ is an $L^2$ orthogonal projection. The potential $\mathbf{Lm}$ can be represented as a convolution operator

$$\mathbf{Lm} = \sum_{j=1}^d \frac{\partial G}{\partial x_j} * \mathbf{m}_j,$$

where $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_d)$ is trivially extended [by zero] from $\Omega$ to $\mathbb{R}^d$ (so that the convolution is formally well-defined). Here $G : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$ is the Newtonian kernel

$$G(x) := \begin{cases} \frac{1}{\gamma_2} \log |x|, & d = 2, \\ \frac{1}{(2-d)\gamma_d} |x|^{2-d}, & d > 2 \end{cases}$$

for $x \neq 0$, where the constant $\gamma_d := |S| > 0$ denotes the surface measure of the unit sphere (e.g., $\gamma_2 = 2\pi$, $\gamma_3 = 4\pi$).

Since the energy functional $E_0^{\ast\ast}$ from (4) is convex and (Gâteaux) differentiable, the minima are equivalently characterized by the corresponding Euler-Lagrange equations [10]. Thus, problem $(RP)$ reads: Find $(\lambda, \mathbf{m}) \in L^2(\Omega) \times L^2(\Omega; \mathbb{R}^d)$ such that

$$\mathcal{P}\mathbf{m} + D\phi^{\ast\ast}(\mathbf{m}) + \lambda \mathbf{m} = \mathbf{f} \quad \text{a.e. in} \, \Omega,$$

$$\lambda \geq 0, \, |\mathbf{m}| \leq 1, \, \lambda(1 - |\mathbf{m}|) = 0 \quad \text{a.e. in} \, \Omega.$$

Existence results for $(RP)$ can be found in [10, 24]; in particular, in the uniaxial case the solution to $(RP)$ is unique.

2.2. The Discretized Problem. Let $\mathcal{T} = \{T_1, \ldots, T_N\}$ be a finite family of pairwise disjoint non-empty open sets $T_j$ which satisfy $\overline{\Omega} = \bigcup_{j=1}^N T_j$. The space of all $\mathcal{T}$-piecewise constant functions is denoted by $\mathcal{P}_0(\mathcal{T})$; $h \in \mathcal{P}_0(\mathcal{T})$ is the mesh-size function, $h|_T := h_T := \text{diam}(T)$. For $f \in L^2(\Omega)$, let $f_T \in \mathcal{P}_0(\mathcal{T})$ be the $\mathcal{T}$-piecewise integral mean given by

$$f_T|_T := \frac{1}{|T|} \int_T f \, dx \quad \text{for all} \, T \in \mathcal{T}.$$

The discrete problem $(RP_{\varepsilon,h})$ now reads as follows: given a penalization parameter $\varepsilon \in \mathcal{P}_0(\mathcal{T})$ with $\varepsilon > 0$, find $\mathbf{m}_h \in \mathcal{P}_0(\mathcal{T})^d$ such that

$$\langle \mathcal{P}\mathbf{m}_h + D\phi^{\ast\ast}(\mathbf{m}_h) + \lambda_h \mathbf{m}_h ; \mathbf{m}_h \rangle_{_{2^2}(\Omega)} = \langle \mathbf{f} ; \bar{m}_h \rangle_{_{2^2}(\Omega)} \quad \text{for all} \, \bar{m}_h \in \mathcal{P}_0(\mathcal{T})^d,$$

where $\lambda_h \in \mathcal{P}_0(\mathcal{T})$ is defined by

$$\lambda_h = \varepsilon^{-1} \frac{(|\mathbf{m}_h| - 1)_+}{|\mathbf{m}_h|}$$

with $(\cdot)_+ := \max\{\cdot, 0\}$. 

Remark 3. For $m_h \in P_0(T)^d$, the potential $P^h$ can be computed exactly, as the associated bilinear form

$$a(m_h, \tilde{m}_h) = \langle P m_h ; \tilde{m}_h \rangle_{L^2(\Omega)}$$

for all $m_h, \tilde{m}_h \in P_0(T)^d$

can be evaluated by a closed form formula, cf. Theorem 5.1. The evaluation of $P m_h$ is, however, computationally demanding, as it typically leads to densely populated stiffness matrices.

As for the continuous problem, we have existence of discrete solutions and uniqueness in the uniaxial model case, cf. [6]. Moreover, in that case the a priori error analysis for $(\mathcal{R}P_{\varepsilon,h})$ from [6] suggests the choice $\varepsilon = h$ for the penalization parameter, as there holds

$$\|P m - P m_h\|_{L^2(\mathbb{R}^d)} + \|D \phi^{**}(m) - D \phi^{**}(m_h)\|_{L^2(\Omega)} + \|\lambda m - \lambda_h m_h\|_{L^2(\Omega)} + \|\varepsilon \lambda m - \varepsilon \lambda m_h\|_{L^2(\Omega)} \leq C \left( \|m - m_T\|_{L^2(\Omega)} + \|\lambda m - (\lambda m)_T\|_{L^2(\Omega)} + \|\varepsilon \lambda m\|_{L^2(\Omega)} \right)$$

with a generic constant $C \geq 0$. For $(\lambda, m)$ sufficiently smooth (e.g., $m \in H^1(\Omega; \mathbb{R}^d)$, $\lambda m \in H^1(\Omega; \mathbb{R}^d)$), the above right-hand side turns out to be of order $O(\varepsilon + h)$.

The stiffness matrix $A$ induced by the bilinear form $a(\cdot, \cdot)$ from (15) will in the following be approximated by an appropriate $H^2$-matrix $\tilde{A}$. Given $(\mathcal{R}P_{\varepsilon,h})$, one then obtains an approximate discrete model $(\tilde{\mathcal{R}}P_{\varepsilon,h})$ after replacing $A$ by $\tilde{A}$ and defining the approximate bilinear form $\tilde{a}(\cdot, \cdot)$ accordingly.

3. Multidimensional Interpolation of Integral Kernels

The following section contains some results on the multidimensional interpolation of integral kernels. We restrict ourselves to one particular class of kernel functions here, known as asymptotically smooth kernels.

3.1. Asymptotically Smooth Kernels. A kernel function

$$\kappa : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}, \quad (x, y) \mapsto \kappa(x, y)$$

is said to be asymptotically smooth if there exist constants $C_{asm}$ and $c_{asm}$ such that

$$|\partial_x^\alpha \partial_y^\beta \kappa(x, y)| \leq C_{asm}(c_{asm}|x - y|)^{-|\alpha| - |\beta| - s(\alpha + \beta)!}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^d$ with $|\alpha| + |\beta| \geq 1$ and some singularity order $s \in \mathbb{R}$, where $x, y \in \mathbb{R}^d$ with $x \neq y$.

Example 3.1. For the Newtonian kernel $G$ defined in (10), $\kappa(x, y) := G(x - y)$ is asymptotically smooth for any $d \geq 2$, with $C_{asm} = \gamma_2^{-1}$ and $C_{asm} = (2 - d)\gamma_d^{-1}$ for $d = 2$ and $d \geq 3$, respectively, and $c_{asm} = 1$, see [13].

Remark 4. Note that the derivatives of an asymptotically smooth kernel $\kappa$ also are asymptotically smooth: given $\tilde{\kappa} := \partial_x^{\tilde{\alpha}} \partial_y^{\tilde{\beta}} \kappa$, (17) holds with $\tilde{s} = s + |\tilde{\alpha}| + |\tilde{\beta}|$, $\tilde{c}_{asm} = c_{asm} + \varepsilon$ for $\varepsilon > 0$ arbitrary, and some $\tilde{C}_{asm}$ depending on $C_{asm}$ and $\varepsilon$. This is a consequence of

$$((\alpha + \beta) + (\tilde{\alpha} + \tilde{\beta})! \leq \tilde{C}_{asm} \tilde{c}_{asm}^{|\alpha| + |\beta|}(\alpha + \beta)!$$
with an \(|\alpha| + |\beta|\)-dependent constant \(\tau_{asm}\). For every choice of \(\tau_{asm} > 1\), there is a \(C_{asm} > 0\) (depending on \(\tau_{asm}\)) such that the above inequality holds. One then sets \(\bar{C}_{asm} := C_{asm}C_{asm}\) and \(\bar{\tau}_{asm} := \tau_{asm}\), respectively.

### 3.2 Interpolation Operators in One Dimension.

For \(m \in \mathbb{N}_0\), let the space of \(m\)-th order polynomials in one spatial variable be denoted by \(\mathcal{P}_m\), and consider the interpolation operator

\[
\mathcal{I}_m : \mathcal{C}[-1, 1] \rightarrow \mathcal{P}_m, \quad u \mapsto \sum_{j=0}^{m} u(t_j) \mathcal{L}_j \quad \text{with} \quad \mathcal{L}_j(t) = \prod_{k=0}^{m} \frac{t - t_k}{t_j - t_k}
\]

acting on the so-called reference element \([-1, 1]\). Here \((\mathcal{L}_j(t))_{j=0}^{m}\) are the Lagrange polynomials corresponding to the interpolation points \((t_j)_{j=0}^{m}\). Note that \(\mathcal{I}_m\) is a projection, i.e., linear with \(\mathcal{I}_m^2 = \mathcal{I}_m\).

For \(m \in \mathbb{N}_0\), the Lebesgue constant \(\Lambda_m \in \mathbb{R}\) is defined as the operator norm of \(\mathcal{I}_m\),

\[
\Lambda_m := \sup_{u \in \mathcal{C}[{-1, 1}]} \frac{\|\mathcal{I}_m u\|_{\infty, [-1, 1]}}{\|u\|_{\infty, [-1, 1]}}.
\]

Clearly, we have \(\Lambda_m \geq 1\). Moreover, we assume that there are constants \(\lambda, C_\lambda \in \mathbb{R}^+\) such that

\[
\Lambda_m \leq C_\lambda (m + 1)^\lambda.
\]

For Chebyshev interpolation, where \(t_j = \cos \left((2j + 1)\pi/(2(m + 1))\right)\), this estimate holds with \(\lambda = 1 = C_\lambda\), cf. [27].

For an arbitrary compact interval \(I := [a, b] \subset \mathbb{R}\), we define the affine transformation

\[
\Phi_I : [-1, 1] \rightarrow I, \quad t \rightarrow \frac{1}{2}((a + b) + t(b - a)).
\]

The transformed interpolation operator \(\mathcal{I}_m^I\) is then given by

\[
\mathcal{I}_m^I : \mathcal{C}[a, b] \rightarrow \mathcal{P}_m, \quad u \mapsto (\mathcal{I}_m(u \circ \Phi_I)) \circ \Phi_I^{-1}.
\]

Obviously, the projection property as well as (20) now carry over from \(\mathcal{I}_m\) to \(\mathcal{I}_m^I\).

### 3.3 Tensor Interpolation Operators.

For a family of closed intervals \(I_j := [a_j, b_j] \subset \mathbb{R}, j \in \{1, \ldots, 2d\}\), define the axially parallel box \(B \subset \mathbb{R}^{2d}\) by \(B := \prod_{j=1}^{2d} I_j\). Given a family \((\mathcal{I}_m^I)_{j=0}^{2d}\) of interpolation operators on \((I_j)_{j=0}^{2d}\), the \(m\)-th order tensor product interpolation operator \(\mathcal{I}_m^B\) on \(B\) is then defined as

\[
\mathcal{I}_m^B := \mathcal{I}_m^I \otimes \cdots \otimes \mathcal{I}_m^{I_{2d}}.
\]

In analogy to \(\mathcal{I}_m^I\), \(\mathcal{I}_m^B\) is a projection from \(\mathcal{C}(B)\) to

\[
\mathcal{Q}_m := \text{span}\{ p_1 \otimes \cdots \otimes p_{2d} \mid p_j \in \mathcal{P}_m, \ j \in \{1, \ldots, 2d\}\}.
\]

We require the following result on the interpolation error of \(\mathcal{I}_m^B\) adapted from [2, 5]:

Theorem 3.2. Let \( u \in C^\infty(B) \) such that there are constants \( C_u, \gamma_u \in \mathbb{R}^+ \) satisfying
\[
\| \partial_j^n u \|_{\infty,B} \leq C_u \gamma_u^n n!
\]
for all \( j \in \{1, \ldots, 2d\} \) and \( n \in \mathbb{N}_0 \). Then, we have
\[
\| u - T_m u \|_{\infty,B} \leq 16ed C_u \Lambda^2 \left( 1 + \gamma_u \text{diam}(B) \right) \left( m + 1 \right) \left( 1 + \frac{2}{\gamma_u \text{diam}(B)} \right)^{-(m+1)}.
\]
Proof. The proof is as in [2, Theorem 3.2], with the \( d \) there replaced by \( 2d \).

3.4 Local Error Analysis for Asymptotically Smooth Kernels. We now apply interpolation to obtain an approximate degenerate kernel \( \tilde{\kappa} := T_m^B \kappa \) instead of the given asymptotically smooth integral kernel \( \kappa \). Let \( B_\sigma, B_\tau \subset \mathbb{R}^d \) be compact axially parallel boxes with positive Euclidean distance \( \text{dist}(B_\sigma, B_\tau) > 0 \):

Lemma 3.3. An asymptotically smooth kernel \( \kappa : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) satisfies (22) on \( B := B_\sigma \times B_\tau \), with constants
\[
C_\kappa = \max \left\{ \| \kappa \|_{L^\infty(B_\sigma \times B_\tau)}, \frac{C_{\text{asm}}}{(\text{asm dist}(B_\sigma, B_\tau))^s} \right\}
\quad \text{and} \quad \gamma_n = \frac{1}{c_{\text{asm dist}(B_\sigma, B_\tau)}}.
\]
Provided \( \text{diam}(B_\sigma \times B_\tau) \leq \eta \text{ dist}(B_\sigma, B_\tau) \) with \( \eta > 0 \), there holds in particular
\[
\| \kappa - T_m^{(\sigma, \tau)} \kappa \|_{L^\infty(B_\sigma \times B_\tau)} \leq c_1 c_2 C_\kappa \left( 1 + \frac{2c_{\text{asm}}}{\eta} \right)^{-(m+1)}
\]
with \( T_m^{(\sigma, \tau)} := T_m^B \), a numerical constant \( c_1 = 16ed(1 + \eta/c_{\text{asm}}) \), and a constant \( c_2 = \Lambda^2 m(m+1) \) with only polynomial increase in \( m \).

Remark 5. Note that the constant \( C_\kappa > 0 \) behaves like \( \text{dist}(B_\sigma, B_\tau)^{-s} \) for the kernels we are interested in, such as \( \kappa(x, y) = \log |x - y| \) resp. \( \kappa(x, y) = |x - y|^{-s} \).

Proof of Lemma 3.3. Direct computation shows that (22) is valid, and Theorem 3.2 yields
\[
\| \kappa - T_m^B \kappa \|_{L^\infty(B)} \leq 16ed \left( 1 + \gamma_u \text{diam}(B) \right) \Lambda^2 \left( m + 1 \right) C_\kappa \left( 1 + \frac{2}{\gamma_u \text{diam}(B)} \right)^{-(m+1)}.
\]
Combining this with \( \gamma_n \text{diam}(B) \leq \eta/c_{\text{asm}} \), we obtain (25).

The proof of Theorem 3.2 is only based on the stability constant \( \Lambda_m \) defined in (19). The advantage is that the resulting estimate can be applied to a fairly wide range of interpolation operators. If one restricts oneself to tensor Chebyshev interpolation – as we will do in the numerical experiments – one can do better by using the following error estimate for tensor Chebyshev polynomials adapted from [3]:

Lemma 3.4. Provided \( \text{diam}(B_\sigma \times B_\tau) \leq \eta \text{ dist}(B_\sigma, B_\tau) \) with \( \eta > 0 \), an asymptotically smooth kernel \( \kappa \) on \( B := B_\sigma \times B_\tau \subset \mathbb{R}^{2d} \) satisfies
\[
\| \kappa - T_m^{(\sigma, \tau)} \kappa \|_{\infty,B} \leq d C_{\text{asm}} c_{\text{asm}} \Lambda_{m-1}^{2d-1} \text{dist}(B_\sigma, B_\tau)^{-s} 4^{-m} c_{\text{asm}}^{-(m+1)} \eta^{m+1}.
\]
Proof. The proof is along the lines of [3].
Proof. Remark 6. Lemma 3.3 ensures (asymptotically) exponential convergence with respect to $m$ irrespective of the choice of $\eta > 0$. Nevertheless, the constant $c_1$ obtained in Lemma 3.3 is too pessimistic, in contrast to the reasonably good approximation results observed for small $m$, as well, cf. Section 6. From Lemma 3.4 we obtain exponential convergence provided at least $4c_{asm} > \eta$, with the highly improved constants $c_1 = dC_{asm}e^{-s} \ll 16ed(1 + \eta/c_{asm})$ and $c_2 = \Lambda_m^{2d-1} \ll \Lambda_m^{2d}(m + 1)$.

In the following, we require additional error estimates for tensor Chebyshev polynomials, in particular for the norms of the first and second derivatives of the respective interpolation errors. For the proofs, we make use of the following well-known one-dimensional error estimate, obtained in Lemma 3.3 is

\begin{equation}
\|u - I_m^I u\|_{\infty, I_j} \leq \frac{4^{-m}}{2(m + 1)!} |I_j|^{m+1} \|u^{(m+1)}\|_{\infty, I_j} \quad \text{for } u \in C^{m+1}(I_j),
\end{equation}

as well as of a result on the first derivative of $u - I_m u$ in one dimension taken from the proof of [1, Theorem 3.3.1],

\begin{equation}
\|(u - I_m u)'\|_{\infty, [-1, 1]} \leq \left( \frac{1}{(r - 1)!} + \frac{1}{r!} C(m) \right) \|u^{(r)}\|_{\infty, [-1, 1]} \quad \text{for } u \in C^r[-1, 1],
\end{equation}

with $1 \leq r \leq m + 1$ and $C(m)$ a constant which may be estimated by $C(m) \leq \Lambda_m m^2$, cf. [1]. Affine transformation then yields

\begin{equation}
\|(u - I_m u)'\|_{\infty, I_j} \leq 2^{-(r-1)}|I_j|^{r-1} \left( \frac{1}{(r - 1)!} + \frac{1}{r!} \Lambda_m m^2 \right) \|u^{(r)}\|_{\infty, I_j}
\end{equation}

for $I_j := [a_j, b_j] \subset \mathbb{R}$.

Furthermore, we need an estimate for the norms of the derivatives of algebraic polynomials known as Markov’s Theorem [11, Theorem 1.4]:

\begin{equation}
\|p'\|_{\infty, I_j} \leq m^2 2|I_j|^{-1} \|p\|_{\infty, I_j} \quad \text{for } p \in \mathcal{P}_m.
\end{equation}

This estimate cannot be improved; in particular, it is sharp for the Chebyshev polynomials.

To begin with, we state a result concerning the first derivatives of $\kappa - I_m^{(\sigma, \tau)}\kappa$.

Lemma 3.5. Provided $\text{diam}(B_\sigma \times B_\tau) \leq \eta \text{ dist}(B_\sigma, B_\tau)$ with $\eta > 0$, an asymptotically smooth $\kappa \in C^{m+2}(B)$ on $B := B_\sigma \times B_\tau$ satisfies

\begin{equation}
\|\partial_\alpha (\kappa - I_m^{(\sigma, \tau)}\kappa)\|_{\infty, B} \leq c_1 c_2 2^{-m} c_{asm}^{-(m+1)} \eta^{m+1} \quad \text{for } 1 \leq \alpha \leq 2d,
\end{equation}

where $c_1$ is a numerical constant depending on $d$, $B$, and $\kappa$ and $c_2 = (\Lambda_m m^2 + m + 1)\Lambda_m^{2d-1}$ grows polynomially in $m$.

Proof. As in [2, 3], we write

\begin{equation}
\|\partial_\alpha (\kappa - I_m^{(\sigma, \tau)}\kappa)\|_{\infty, B} \leq \sum_{j=1}^{2d} \|\partial_\alpha \left( \bigotimes_{k=1}^{j-1} I_m^k \otimes (I_d - I_m^j) \otimes \bigotimes_{k=j+1}^{2d} I_d \right) \|_{\infty, B} ;
\end{equation}

\begin{equation}
= \|\partial_\alpha \mathcal{J}_\kappa\|_{\infty, B},
\end{equation}

to obtain estimates for $\|\partial_\alpha \mathcal{J}_\kappa\|_{\infty, B}$, we have to consider three cases.
• First, for \( j < \alpha \), an explicit computation gives \( \partial_n \mathcal{J}_j \kappa = \mathcal{J}_j (\partial_n \kappa) \), whence

\[
\| \partial_n \mathcal{J}_j \kappa \|_{\infty, B} \leq \Lambda_m^{j-1} \left\| \left( \bigotimes_{k=1}^{j-1} \text{Id} \otimes (\text{Id} - \mathcal{I}_m^j) \otimes \bigotimes_{k=j+1}^{2d} \text{Id} \right) \partial_n \kappa \right\|_{\infty, B} \leq \Lambda_m^{j-1} \frac{4^{-m}}{2(m+1)!} |I_j|^{m+1} \| \partial_n^{m+1} \partial_n \kappa \|_{\infty, B};
\]

here we have used (19) on \( \mathcal{I}_m^k \), \( k < j \), and applied the estimate in (27) to \( I_j \). By exploiting the asymptotic smoothness of \( \kappa \) to estimate \( \| \partial_n^{m+1} \partial_n \kappa \|_{\infty, B} \leq C_{\text{asm}} \left( c_{\text{asm}} \text{dist}(B_{\sigma}, B_{\tau}) \right)^{(m+2+s)} \left( m + 2 \right)! \), \( |I_j| \leq \text{diam}(B_{\sigma} \times B_{\tau}) \), and \( \text{diam}(B_{\sigma} \times B_{\tau}) \leq \eta \text{dist}(B_{\sigma}, B_{\tau}) \), we get

\[
\| \partial_n \mathcal{J}_j \kappa \|_{\infty, B} \leq C_{\text{asm}} c_{\text{asm}}^{-(s+1)} \text{dist}(B_{\sigma}, B_{\tau})^{-s} c_{\text{asm}}^{-(m+1)} \eta^m m^2 2^{-m} |I_{\alpha}|^{-1} \Lambda_m^{j-1}.
\]

• Second, for \( j = \alpha \), it follows from (29) with \( r = m + 1 \) that

\[
\| \partial_n \mathcal{J}_j \kappa \|_{\infty, B} \leq \Lambda_m^{\alpha-1} \left\| \left( \bigotimes_{k=1}^{\alpha-1} \text{Id} \otimes (\text{Id} - \mathcal{I}_m^j) \otimes \bigotimes_{k=\alpha+1}^{2d} \text{Id} \right) \kappa \right\|_{\infty, B} \leq \Lambda_m^{\alpha-1} \frac{1}{m!} + \frac{m^2}{(m+1)!} \Lambda_m \| \partial_n^{m+1} \kappa \|_{\infty, B}.
\]

As before, we now obtain

\[
\| \partial_n \mathcal{J}_j \kappa \|_{\infty, B} \leq C_{\text{asm}} c_{\text{asm}}^{-(s+1)} \text{dist}(B_{\sigma}, B_{\tau})^{-s} c_{\text{asm}}^{-(m+1)} \eta^m m^2 2^{-m} |I_{\alpha}|^{-1} \Lambda_m^{\alpha-1}.
\]

• Third, the case \( j > \alpha \) is treated by applying Markov’s Theorem (30) to \( \mathcal{I}_m^j \) and by using the estimate in (27), whence

\[
\| \partial_n \mathcal{J}_j \kappa \|_{\infty, B} \leq \Lambda_m^{j-1} m^2 |I_{\alpha}|^{-1} \left\| \left( \bigotimes_{k=1}^{j-1} \text{Id} \otimes (\text{Id} - \mathcal{I}_m^j) \otimes \bigotimes_{k=j+1}^{2d} \text{Id} \right) \kappa \right\|_{\infty, B} \leq \Lambda_m^{j-1} m^2 \frac{4^{-m}}{(m+1)!} |I_{\alpha}|^{-1} |I_j|^{m+1} \| \partial_n^{m+1} \kappa \|_{\infty, B}
\]

and

\[
\| \partial_n \mathcal{J}_j \kappa \|_{\infty, B} \leq C_{\text{asm}} c_{\text{asm}}^{-(s+1)} \text{dist}(B_{\sigma}, B_{\tau})^{-s} c_{\text{asm}}^{-(m+1)} \eta^m m^2 2^{-m} |I_{\alpha}|^{-1} \Lambda_m^{j-1}.
\]

Collecting the estimates in (33), (34), and (35), we finally have

\[
\| \partial_n (\kappa - \mathcal{I}_m^{(\sigma, \tau)} \kappa) \|_{\infty, B} \leq \Lambda_m^{\alpha-1} m^2 |I_{\alpha}|^{-1} \left( c_{\text{asm}}^{-(m+1)} \eta^m m^2 2^{-m} \text{dist}(B_{\sigma}, B_{\tau})^{-1} \sum_{j<\alpha} \Lambda_m^{j-1} + \right)
\]

\[
\left( \Lambda_m m^2 + m + 1 \right) 2^{-m} |I_{\alpha}|^{-1} \Lambda_m^{\alpha-1} + m^2 2^{-m} |I_{\alpha}|^{-1} \sum_{j>\alpha} \Lambda_m^{j-1} \right) \leq c_{\text{asm}}^{-(m+1)} \eta^m m^2 + m + 1 \right) 2^{-m} \Lambda_m^{2d-1} C_{\text{asm}} c_{\text{asm}}^{-(s+1)} \text{dist}(B_{\sigma}, B_{\tau})^{-s} \times
\]

\[
\left( (\alpha - 1) c_{\text{asm}}^{-1} \text{dist}(B_{\sigma}, B_{\tau})^{-1} + (2d - (\alpha - 1)) \left( \min_{j=1}^{2d} |I_j| \right)^{-1} \right),
\]
which gives the desired result.

For the second derivatives of $\kappa - \mathcal{I}_m^{(\sigma, \tau)}$, we obtain in a similar fashion:

**Lemma 3.6.** Under the assumptions of Lemma 3.5, we have

\begin{equation}
\|\partial_\alpha \partial_\beta (\kappa - \mathcal{I}_m^{(\sigma, \tau)})\|_{\infty, B} \leq c_1 c_2 2^{-(m-1)} c_{\text{asm}}^{-(m+1)} \eta^{m+1} \quad \text{for } 1 \leq \alpha, \beta \leq 2d,
\end{equation}

with a constant $c_1$ depending on $d$, $B$, and $\kappa$ and $c_2 = (\Lambda_m m^2 + m + 1) m^2 \Lambda_m^{2d-1}$.

**Proof.** Without loss of generality, we assume $1 \leq \alpha \leq \beta \leq 2d$ throughout; similar reasoning as in the proof of Lemma 3.5, with $\mathcal{J}_j \kappa$ defined as in (32), then implies the following cases:

- $j < \alpha \leq \beta$: with $\partial_\alpha \partial_\beta \mathcal{J}_j \kappa = \mathcal{J}_j (\partial_\alpha \partial_\beta \kappa)$, one has as in (33)

\[
\|\partial_\alpha \partial_\beta \mathcal{J}_j \kappa\|_{\infty, B} \leq \Lambda_m^{j-1} \frac{4^{-m}}{2(m+1)!} |I_j|^{m+1} \|\partial_j^{m+1} \partial_\alpha \partial_\beta \kappa\|_{\infty, B} \leq \\
\leq \Lambda_m^{j-1} C_{\text{asm}} c_{\text{asm}}^{-(s+2)} \text{dist}(B_\sigma, B_r)^{-(s+2)} c_{\text{asm}}^{-(m+1)} \eta^{m+1} 4^{-m} \frac{(m+2)(m+3)}{2}.
\]

- $j = \alpha < \beta$: from $\partial_\alpha \partial_\beta \mathcal{J}_j \kappa = \partial_\alpha \mathcal{J}_j (\partial_\beta \kappa)$ and (29) for $r = m+1$ it follows that

\[
\|\partial_\alpha \partial_\beta \mathcal{J}_j \kappa\|_{\infty, B} \leq \Lambda_m^{\alpha-1} 2^{-m} |I_\alpha|^m \left(1 + \frac{m^2}{(m-1)!} \Lambda_m\right) \|\partial_j^{m+1} \partial_\beta \kappa\|_{\infty, B} \leq \\
\leq \Lambda_m^{\alpha-1} C_{\text{asm}} c_{\text{asm}}^{-(s+1)} \text{dist}(B_\sigma, B_r)^{-(s+1)} |I_\alpha|^{-1} c_{\text{asm}}^{-(m+1)} \eta^{m+1} 2^{-m} \times \\
\times (m+2)(1 + m + m^2 \Lambda_m),
\]

cf. (34);

- $\alpha < j < \beta$: as $\partial_\alpha \partial_\beta \mathcal{J}_j \kappa = \partial_\alpha \mathcal{J}_j (\partial_\beta \kappa)$, we obtain with (30)

\[
\|\partial_\alpha \partial_\beta \mathcal{J}_j \kappa\|_{\infty, B} \leq \Lambda_m^{\alpha-1} m^2 |I_\alpha|^{-1} \frac{4^{-m}}{2(m+1)!} |I_j|^{m+1} \|\partial_j^{m+1} \partial_\beta \kappa\|_{\infty, B} \leq \\
\leq \Lambda_m^{\alpha-1} C_{\text{asm}} c_{\text{asm}}^{-(s+1)} \text{dist}(B_\sigma, B_r)^{-(s+1)} |I_\alpha|^{-1} c_{\text{asm}}^{-(m+1)} \eta^{m+1} 4^{-m} m^2 (m+2),
\]

see (35);

- $j = \alpha = \beta$: to obtain an estimate for $\partial_\beta^2 \mathcal{J}_j \kappa$, we proceed as in [1, Theorem 3.3.1]: given $u \in C^{m+1}[-1, 1]$ and $1 \leq r \leq m+1$, let $R_r$ be the remainder term of the Taylor series expansion of degree $r-1$ of $u$ about $t = 0$. Using $u - I_m u = R_r - I_m R_r$, $\|R_r\|_{\infty, [-1, 1]} \leq 1/r! \|u^{(r)}\|_{\infty, [-1, 1]}$, and $\|R_r^0\|_{\infty, [-1, 1]} \leq 1/(r-2)! \|u^{(r)}\|_{\infty, [-1, 1]}$, we have by the triangle inequality and with (30)

\[
\|(u - I_m u)^r\|_{\infty, [-1, 1]} \leq \|R_r^0\|_{\infty, [-1, 1]} + \|(I_m R_r)^r\|_{\infty, [-1, 1]} \leq \\
\leq \frac{1}{(r-2)!} \|u^{(r)}\|_{\infty, [-1, 1]} + m^2 (m-1)^2 \Lambda_m \frac{1}{r!} \|u^{(r)}\|_{\infty, [-1, 1]}.
\]

By affine transformation it follows for $u \in C^{m+1}(I_\alpha)$ and $r = m+1$ that

\[
\|(u - I_m u)^r\|_{\infty, I_\alpha} \leq 2^{-(m-1)} |I_\alpha|^{m-1} \left(\frac{1}{(m-1)!} + \frac{1}{(m+1)!} (m-1)^2 \Lambda_m\right) \|u^{(m+1)}\|_{\infty, I_\alpha}.
\]
whence
\[
\|\partial_\alpha^2 \mathcal{J}_\alpha \kappa\|_{\infty, \mathcal{B}} \leq \Lambda_m^{\alpha-1} \left( \sum_{k=1}^{\infty} \frac{1}{k!} (\mathbb{I} \otimes \partial_\alpha^2 (\mathbb{I} - I_m^{(s)}) \otimes \mathbb{I}) \right) \kappa \leq \Lambda_m^{\alpha-1} 2^{-(m-1)}|I_\alpha|^{m-1} \left( \frac{1}{(m-1)!} + \frac{1}{(m+1)!} \right) \|\partial_\alpha^{m+1} \kappa\|_{\infty, \mathcal{B}} \leq \Lambda_m^{\alpha-1} C_{\text{asm}} c_{\text{asm}} \text{dist}(I_\sigma, I_\tau)^{-s} |I_\alpha|^{-2} c_{\text{asm}}^{-1} \eta^{m+1} \|\partial_\alpha^{m+1} \kappa\|_{\infty, \mathcal{B}} \leq \Lambda_m^{\alpha-1} C_{\text{asm}} c_{\text{asm}} \text{dist}(I_\sigma, I_\tau)^{-s} |I_\alpha|^{-1} |I_\beta|^{-1} c_{\text{asm}}^{-1} \eta^{m+1} 2^{-(m-1)} \times (m(m+1) + (m-1)2m^2 \Lambda_m);
\]

- \(\alpha < j = \beta\): with \(\partial_\alpha \partial_\beta \mathcal{J}_\beta \kappa = \partial_\alpha (\partial_\beta \mathcal{J}_\beta \kappa)\), (29) and (30) give

\[
\|\partial_\alpha \partial_\beta \mathcal{J}_\beta \kappa\|_{\infty, \mathcal{B}} \leq \Lambda_m^{\beta-1} m^2 |I_\alpha|^{-1} m^2 |I_\beta|^{-1} \frac{2-k}{(m+1)!} |I_j|^{m+1} \|\partial_\alpha^{m+1} \partial_\beta \kappa\|_{\infty, \mathcal{B}} \leq \Lambda_m^{\beta-1} C_{\text{asm}} c_{\text{asm}} \text{dist}(I_\sigma, I_\tau)^{-s} |I_\alpha|^{-1} |I_\beta|^{-1} c_{\text{asm}}^{-1} \eta^{m+1} 4^{-(m-1)} \times m^2 (1 + m + m^2 \Lambda_m);
\]

- \(\alpha \leq \beta < j\): applying Markov’s Theorem (30) twice yields

\[
\|\partial_\alpha \partial_\beta \mathcal{J}_\beta \kappa\|_{\infty, \mathcal{B}} \leq \Lambda_m^{\beta-1} m^2 |I_\alpha|^{-1} m^2 |I_\beta|^{-1} \frac{4-m}{2(m+1)!} |I_j|^{m+1} \|\partial_\alpha^{m+1} \partial_\beta \kappa\|_{\infty, \mathcal{B}} \leq \Lambda_m^{\beta-1} C_{\text{asm}} c_{\text{asm}} \text{dist}(I_\sigma, I_\tau)^{-s} |I_\alpha|^{-1} |I_\beta|^{-1} c_{\text{asm}}^{-1} \eta^{m+1} 4^{-(m-1)} \frac{m^4}{2}.
\]

Let us first consider \(\alpha < \beta\): by collecting the above estimates, we obtain as in Lemma 3.5

\[
\|\partial_\alpha \partial_\beta (\kappa - \mathcal{J}_m^{(\sigma, \tau)} \kappa)\|_{\infty, \mathcal{B}} \leq \sum_{k=1}^{\infty} \frac{1}{k!} (\mathbb{I} \otimes \partial_\alpha^2 (\mathbb{I} - I_m^{(s)}) \otimes \mathbb{I}) \kappa \leq \sum_{k=1}^{\infty} \frac{1}{k!} \left( \frac{1}{(m-1)!} + \frac{1}{(m+1)!} \right) \|\partial_\alpha^{m+1} \kappa\|_{\infty, \mathcal{B}} \leq \Lambda_m^{\alpha-1} C_{\text{asm}} c_{\text{asm}} \text{dist}(I_\sigma, I_\tau)^{-s} |I_\alpha|^{-1} |I_\beta|^{-1} c_{\text{asm}}^{-1} \eta^{m+1} 4^{-(m-1)} \frac{m^4}{2},
\]

from which the result follows. An analogous computation for \(\alpha = \beta\) shows that the estimate then still holds, with the same constants \(c_1\) and \(c_2\). This concludes the proof.

**Remark 7.** Lemmas 3.5 and 3.6 ensure exponential convergence with respect to \(m\) provided at least \(2c_{\text{asm}} > \eta\). For \(\kappa\) the Newtonian kernel \(G\), this implies exponential convergence for \(\eta \in (0, 2)\), cf. Remark 4.

### 4. \(H^2\)-Matrix Techniques

In this section we motivate the concept of hierarchical matrices and give the corresponding definitions.
4.1. Motivation. We consider a bilinear form $a(\cdot, \cdot)$ on $L^2(\Omega)$ given by a double integration with an asymptotically smooth kernel $\kappa$,

$$
a(u, v) := \int_\Omega \int_\Omega u(x) \kappa(x, y) v(y) \, dy \, dx \quad \text{for } u, v \in L^2(\Omega),
$$

with $\Omega \subset \mathbb{R}^d$ a bounded domain. In the following we require a partition $T$ of $\Omega$ and a block partitioning $\mathcal{P}$ of $T \times T$. For $\eta > 0$ fixed, a block $(\sigma, \tau) \in \mathcal{P}$ is then called admissible provided

$$
diam(B_\sigma \times B_\tau) \leq \eta \text{dist}(B_\sigma, B_\tau),
$$

where $B_\sigma$ and $B_\tau$ denote axially parallel boxes in $\mathbb{R}^d$ of minimal size containing $\cup \sigma$ and $\cup \tau$, respectively; otherwise $(\sigma, \tau)$ is called inadmissible. Here $\cup \sigma := \{ x \in T \mid T \in \sigma \subseteq T \}$ (resp. $\cup \tau := \{ y \in T \mid T \in \tau \subseteq T \}$) denotes the union of all elements $T \in T$ contained in $\sigma$ (resp. in $\tau$). $\mathcal{P}$ is thus split into two subsets $\mathcal{P}_{\text{far}}$ and $\mathcal{P}_{\text{near}}$: the subset of all admissible blocks is called far field and denoted by $\mathcal{P}_{\text{far}}$; the inadmissible blocks are collected in the near field $\mathcal{P}_{\text{near}} := \mathcal{P} \setminus \mathcal{P}_{\text{far}}$.

The approximate bilinear form $\tilde{a}(\cdot, \cdot)$ is obtained by replacing the kernel function on admissible blocks by an approximate but degenerate kernel obtained by interpolation. More precisely,

$$
\tilde{a}(u, v) := \sum_{(\sigma, \tau) \in \mathcal{P}_{\text{near}}} \int_{\cup \sigma} \int_{\cup \tau} u(x) \kappa(x, y) v(y) \, dy \, dx + \sum_{(\sigma, \tau) \in \mathcal{P}_{\text{far}}} \int_{\cup \sigma} \int_{\cup \tau} u(x) (T_{m_1, m_2}^{(\sigma, \tau)}(y)) v(y) \, dy \, dx,
$$

where $T_{m_1, m_2}^{(\sigma, \tau)}$ denotes the tensor interpolation operator with respect to the bounding box $B := B_\sigma \times B_\tau$.

For each $\tau \subseteq T$ with corresponding bounding box $B_\tau$, define a family $(x_j)_{j=0}^{M_\tau}$ of interpolation points plus the associated tensor Lagrange polynomials $(L_j^\tau)_{j=0}^{M_\tau}$, where $M_\tau \in \mathbb{N}$. Given

$$
T_{m_1, m_2}^{(\sigma, \tau)} \kappa(x, y) = \sum_{m_1=0}^{M_\sigma} \sum_{m_2=0}^{M_\tau} \kappa(x_{m_1}, x_{m_2}^\tau) L_{m_1}^\sigma(x) L_{m_2}^\tau(y) \quad \text{for } (x, y) \in \cup \sigma \times \cup \tau,
$$

the second term from (39) can then be written as

$$
\sum_{(\sigma, \tau) \in \mathcal{P}_{\text{far}}} \sum_{m_1=0}^{M_\sigma} \sum_{m_2=0}^{M_\tau} \kappa(x_{m_1}, x_{m_2}^\tau) \int_{\cup \sigma} u(x) L_{m_1}^\sigma(x) \, dx \int_{\cup \tau} v(y) L_{m_2}^\tau(y) \, dy.
$$

The advantage of this new representation becomes obvious if we discretize $a(\cdot, \cdot)$: consider the basis $\{\varphi_1, \ldots, \varphi_N\}$ of the space $P_0(T)$ of piecewise constant functions on $T$ given by $\varphi_j := \chi_{T_j}$, where $\chi_{T_j}$ is the characteristic function on $T_j \in T$ and $N = |T|$, and define $A \in \mathbb{R}^{N \times N}$ by $A_{jk} := a(\varphi_j, \varphi_k)$ and the approximate matrix $\tilde{A} \in \mathbb{R}^{N \times N}$ by $\tilde{A}_{jk} := \tilde{a}(\varphi_j, \varphi_k)$ for all $1 \leq j, k \leq N$. On inadmissible blocks $(\sigma, \tau) \in \mathcal{P}_{\text{near}}$, we then simply have $\tilde{A}|_{\sigma \times \tau} = A|_{\sigma \times \tau}$, whereas for $(\sigma, \tau) \in \mathcal{P}_{\text{far}}$, $\tilde{A}$ is given by

$$
\tilde{A}|_{\sigma \times \tau} = V^\sigma S^\tau V^{\tau T} \approx A|_{\sigma \times \tau},
$$
with $V_{j,m_1}^\sigma := V_{m_1}^\sigma (\varphi_j)$ and $V_{k,m_2}^\tau := V_{m_2}^\tau (\varphi_k)$ as defined in (40).

4.2. Block Partitioning. A simple method to find a hierarchical partition $\mathbb{P}$ of $T \times T$ is to first construct a cluster tree from $T$ by binary space partitioning: one starts with the root cluster containing all $T \in T$, splits it into two son clusters and repeats the procedure recursively until each cluster contains less than a given number of elements $C_{il}$. Geometrically speaking, for every $T \in T$, one chooses a coordinate axis and splits the set along this axis.

One can then use the admissibility condition (38) in combination with the cluster tree structure to construct $\mathbb{P}$: starting with $T \times T$, one splits each pair of clusters as long as it is not admissible. This gives a $\mathbb{P}$ satisfying $T \times T = \bigcup_{(\sigma,\tau) \in \mathbb{P}} \sigma \times \tau$. Clearly, a pair $(\sigma,\tau)$ can only appear in $\mathbb{P}$ if it is admissible or if either $\sigma$ or $\tau$ is a leaf. The above procedure can easily be formalized, see e.g. [4, 3, 16] for formal definitions and algorithms.

4.3. $\mathcal{H}$-Matrices vs. $\mathcal{H}^2$-Matrices. Based on the concepts of cluster tree and block partitioning, the matrix approximation approach outlined in Section 4.1 can be generalized by introducing a class of data-sparse matrices, the so-called $\mathcal{H}$-matrices. Given a block partitioning $\mathbb{P}$ of $T \times T$ and some $k \in \mathbb{N}$, a matrix $A \in \mathbb{R}^{N \times N}$ is called $\mathcal{H}$-matrix of rank $k$ provided $\text{rank}(A|_{\sigma \times \tau}) \leq k$ for each $(\sigma,\tau) \in \mathbb{P}$. Moreover, if a factorization of the form (41) holds for a family $V = (V^r)_r \subset T$ and some multiplication matrices $S^{\sigma\tau}$, $A$ is called uniform $\mathcal{H}$-matrix with respect to the cluster basis $V$, cf. [4, 16].

Additional structure can be gained by considering uniform $\mathcal{H}$-matrices for which the corresponding cluster bases are nested: if the space $\mathcal{Q}_m$ defined in (21) is used for interpolation on all clusters, polynomials corresponding to father clusters can be expressed exactly in terms of polynomials corresponding to son clusters. For $\tau \subseteq T$ and $\tau' \in \text{sons}(\tau)$, we have

$$L_j^\tau (x) = \sum_{m=0}^{M} L_j^\tau (x_m^\tau) L_m^\tau (x) \quad (42)$$

Defining the transfer matrix $B^\tau \tau' \in \mathbb{R}^{M \times M}$ by $B_{m,j}^{\tau \tau'} := L_j^\tau (x_m^\tau)$, we obtain

$$V_{ij}^{\tau} = \int_{\cup \tau} \chi_{\tau'} L_j^\tau (x) dx = \sum_{\tau' \subseteq \text{sons}(\tau)} \sum_{m=0}^{M} L_j^\tau (x_m^\tau) \int_{\cup \tau'} \chi_{\tau'} L_m^\tau (x) dx = \sum_{\tau' \subseteq \text{sons}(\tau)} \sum_{m=0}^{M} B_{m,j}^\tau \tau' V_m^{\tau'} \quad (43)$$

for all $i \in \tau' \subseteq \tau$, i.e., $V^\tau |_{\tau'} = V^\tau B^\tau \tau'$. $A$ is called $\mathcal{H}^2$-matrix with respect to $V$ if it is a uniform $\mathcal{H}$-matrix with respect to $V$ and if $V$ is nested.

Remark 8. For $\mathcal{H}^2$-matrices, one only has to store the multiplication matrices $S^{\sigma\tau}$ for all admissible blocks $(\sigma,\tau) \in \mathbb{P}_{\text{far}}$, the cluster matrices $V^\tau$ for all leaves $\tau \subseteq T$, the transfer matrices $B^\tau \tau'$ for all father-son pairs, and $A|_{\sigma \times \tau}$ on all non-admissible blocks $(\sigma,\tau) \in \mathbb{P}_{\text{near}}$.

Remark 9. To evaluate (39), fast and efficient algorithms for matrix-vector multiplication are required. We assume the underlying partitioning $\mathbb{P}$ to be sparse in the sense of [13], with some sparsity constant $C_{sp}$. Given an $\mathcal{H}^2$-matrix $A \in \mathbb{R}^{N \times N}$ and $x, y \in \mathbb{R}^{N}$, it can then be shown that the computation of $y = Ax$ needs only $O(Nm^d)$ operations to complete. Similarly, both the number of operations required to build an $\mathcal{H}^2$-matrix approximation and the amount of storage needed to store it are of order $O(Nm^d)$, cf. [4, 12, 16]. Note that $C_{sp}$ enters all the above complexity estimates; for $C_{sp} \to \infty$, the complexity of the problem will become unbounded, as well. Estimates on $C_{sp}$ can be found in [14].
4.4. Global Error Analysis for the Approximate Bilinear Form. By definition of the admissibility condition (38), one can apply the approximation results of Section 3.4 on each admissible block $(\sigma, \tau) \in \mathbb{P}_{\text{far}}$. Indeed, given some constant $c_3$, Lemma 3.4 shows that we can choose an approximation order $m \in \mathbb{N}$ so that

$$
\|\kappa - \mathcal{T}_m^{(\sigma, \tau)} \kappa\|_{L^\infty(B_2 \times B_2)} \leq c_3
$$

for all $(\sigma, \tau) \in \mathbb{P}_{\text{far}}$.

**Theorem 4.1.** Under the above assumptions, we have

$$
|a(u, v) - \tilde{a}(u, v)| \leq c_3 \|u\|_{L^1(\Omega)} \|v\|_{L^1(\Omega)} \leq c_3 \|\Omega\| \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}
$$

for all $u, v \in L^2(\Omega)$.

**Proof.** For almost all $(x, y) \in \Omega \times \Omega$, we define an integral kernel $\tilde{\kappa}(x, y)$ as follows. Let $\mathcal{S} := \{\partial T \mid T \in \mathcal{T}\}$ denote the skeleton of the partition. Note that $\mathcal{S} \subset \mathbb{R}^d$ is a set of measure zero. For $x, y \in \Omega \setminus \mathcal{S}$, there are unique elements $T_x, T_y \in \mathcal{T}$ satisfying $x \in T_x$ and $y \in T_y$, respectively. Since $\mathbb{P}$ is a partition of $\mathcal{T} \times \mathcal{T}$, there is a unique block $(\sigma, \tau) \in \mathbb{P}$ with $(T_x, T_y) \in \sigma \times \tau$. Consequently, we may define

$$
\tilde{\kappa}(x, y) := \begin{cases} 
\kappa(x, y) & \text{if } (\sigma, \tau) \text{ is not admissible}, \\
\mathcal{T}_m^{(\sigma, \tau)} \kappa(x, y) & \text{else}.
\end{cases}
$$

Since $a(\cdot, \cdot)$ and $\tilde{a}(\cdot, \cdot)$ differ only on the far-field blocks, we have

$$
|a(u, v) - \tilde{a}(u, v)| = \left| \int_{\Omega} \int_{\Omega} u(x)(\kappa(x, y) - \tilde{\kappa}(x, y)) v(y) \, dy \, dx \right|,
$$

and a Hölder inequality yields

$$
|a(u, v) - \tilde{a}(u, v)| \leq \|\kappa - \tilde{\kappa}\|_{L^\infty(\Omega \times \Omega)} \|u\|_{L^1(\Omega)} \|v\|_{L^1(\Omega)},
$$

since $u(x)$ and $v(y)$ decouple on $\Omega \times \Omega$. Using $\|u\|_{L^1(\Omega)} \leq \|\Omega\|^{1/2} \|u\|_{L^2(\Omega)}$, we obtain the desired estimate. \(\blacksquare\)

4.5. Global Error Analysis for the Corresponding Matrix. Let $U_h \leq L^2(\Omega)$ be a finite-dimensional space, and define the stiffness matrix $A \in \mathbb{R}^{N \times N}$ and its $\mathcal{H}^2$-approximation $\tilde{A} \in \mathbb{R}^{N \times N}$ by $A_{jk} := a(u_j, u_k)$ and $\tilde{A}_{jk} := \tilde{a}(u_j, u_k)$, respectively, for a fixed basis $\{u_1, \ldots, u_N\}$ of $U_h$. Then, there holds

**Corollary 4.2.** The approximation error for the stiffness matrix $A$ is bounded by

$$
\|A - \tilde{A}\|_F \leq Nc_3 \left( \max_{j=1}^N \|u_j\|_{L^1(\Omega)} \right)^2.
$$

In particular, the error decreases exponentially with the approximation order $m$. Provided $A$ is a regular matrix, $\tilde{A}$ also is regular for large approximation orders $m$.

**Proof.** The first assertion follows by applying Theorem 4.1 to each entry of $A$. Regularity is immediate, as the regular matrices $GL(N)$ form an open subset of $\mathbb{R}^{N \times N}$.

\(\blacksquare\)
5. \textbf{Galerkin Discretization of the Potential Equation}

In this section we provide two different $H^2$-matrix approaches to obtain a reasonable data sparse approximation of the stiffness matrix

\begin{equation}
\mathbf{A} \in \mathbb{R}^{dN \times dN} \quad \text{with} \quad A_{jk} := a(\varphi_j, \varphi_k)
\end{equation}

for a fixed basis \{\varphi_1, \ldots, \varphi_{dN}\} of $\mathcal{P}_0(T)^d$, where the bilinear form $a(\cdot, \cdot)$ is defined as in (15). We recall a result from [25].

\begin{theorem}
For bounded Lipschitz domains $\omega, \tilde{\omega} \subset \mathbb{R}^d$ and vectors $\mathbf{m}, \tilde{\mathbf{m}} \in \mathbb{R}^d$, we have
\begin{equation}
\begin{aligned}
a(\chi_\omega \mathbf{m}, \chi_\tilde{\omega} \tilde{\mathbf{m}}) &= - \int_{\partial \omega} \int_{\partial \tilde{\omega}} G(x-y)(\mathbf{v}(x) \cdot \mathbf{m})(\mathbf{\tilde{v}}(y) \cdot \tilde{\mathbf{m}}) \, ds_y \, ds_x,
\end{aligned}
\end{equation}
where $\mathbf{v}$ and $\mathbf{\tilde{v}}$ denote the outer normal vectors on $\partial \omega$ and $\partial \tilde{\omega}$, respectively. Furthermore, we have the symmetry properties
\begin{equation}
a(\chi_\omega \mathbf{m}, \chi_\tilde{\omega} \tilde{\mathbf{m}}) = a(\chi_\tilde{\omega} \tilde{\mathbf{m}}, \chi_\omega \mathbf{m}),
\end{equation}
and in the case $\text{dist}(\omega, \tilde{\omega}) > 0$ there holds
\begin{equation}
a(\chi_\omega \mathbf{m}, \chi_\tilde{\omega} \tilde{\mathbf{m}}) = \int_{\tilde{\omega}} \int_\omega \mathbf{m} \cdot H_G(x-y) \tilde{\mathbf{m}} \, dx \, dy
\end{equation}
with the Hessian $H_G$ of the Newtonian kernel $G$.
\end{theorem}

Now, a reasonable choice for a basis of $\mathcal{P}_0(T)^d$ is
\begin{equation}
\varphi_j := \chi_{T_j} \mathbf{e}_1, \quad \varphi_{j+N} := \chi_{T_j} \mathbf{e}_2 \quad \text{etc. for } 1 \leq j \leq N,
\end{equation}
as is shown in the following. This basis gives rise to the definition of the matrices
\begin{equation}
A^{\alpha\beta} \in \mathbb{R}^{N \times N}_{\text{sym}} \quad \text{for fixed } 1 \leq \alpha, \beta \leq d, \quad A^{\alpha\beta}_{jk} := a(\chi_{T_j} \mathbf{e}_\alpha, \chi_{T_k} \mathbf{e}_\beta),
\end{equation}
where the symmetry of $A^{\alpha\beta}$ (i.e., an additional symmetry of $\mathbf{A}$) follows from (48). Note that again by equation (48) – we have $A^{\alpha\beta} = A^{\beta\alpha}$. Therefore, $\mathbf{A}$ is a symmetric $d \times d$ block matrix with symmetric blocks $A^{\alpha\beta}$ of dimension $N \times N$,
\begin{equation}
\begin{aligned}
\mathbf{A} &= \begin{pmatrix}
A^{11} & A^{12} \\
A^{21} & A^{22}
\end{pmatrix} \quad \text{for } d = 2 \quad \text{and} \quad \mathbf{A} = \begin{pmatrix}
A^{11} & A^{12} & A^{13} \\
A^{21} & A^{22} & A^{23} \\
A^{31} & A^{32} & A^{33}
\end{pmatrix} \quad \text{for } d = 3,
\end{aligned}
\end{equation}
respectively. The idea is to approximate each block $A^{\alpha\beta}$ by an appropriate $H^2$-matrix.

\subsection{5.1. Approximation of $A$ on Far Field Blocks via (49)}

A direct $H^2$-matrix approach stems from the far field representation (49) for which the Hessian (i.e., the second derivatives) of the Newtonian kernel enters,
\begin{equation}
\frac{\partial^2 G}{\partial x_\alpha \partial x_\beta}(x) = \frac{1}{\gamma_d} \frac{\delta_{\alpha\beta}|x|^2 - dx_\alpha x_\beta}{|x|^{d+2}} \quad \text{for } x \in \mathbb{R}^d \setminus \{0\},
\end{equation}
where $1 \leq \alpha, \beta \leq d$ and $\delta_{\alpha\beta}$ denotes the Kronecker delta. Obviously, these kernels are asymptotically smooth with singularity order $s = -d$, cf. Remark 4. Let $\mathcal{P}$ be a block partitioning with respect to the given triangulation $T$. To abbreviate notation, let $\mathbf{v}^j$ denote the outer normal vector on the boundary $\partial T_j$ of an element $T_j$, and let $\mathbf{m}_h^j := \mathbf{m}_h|_{T_j} \in \mathbb{R}^d$. 

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be a discrete magnetization $m_h \in P_0(T)^d$. In analogy to the previous section and according to (47) and (49), the bilinear form $a(\cdot, \cdot)$ on $P_0(T)^d$ reads

$$a(m_h, \tilde{m}_h) = -\sum_{j,k=1}^{N} \int_{\partial T_j} \int_{\partial T_k} G(x-y)(\nu^j(x) \cdot m_h^j) (\nu^k(y) \cdot \tilde{m}_h^k) \, ds_y \, ds_x =$$

$$= -\sum_{(\sigma, \tau) \in \mathbb{P}_{\text{near}}} \sum_{T_j \sigma} \sum_{T_k \in \tau} \int_{\partial T_j} \int_{\partial T_k} G(x-y)(\nu^j(x) \cdot m_h^j) (\nu^k(y) \cdot \tilde{m}_h^k) \, ds_y \, ds_x +$$

$$+ \sum_{(\sigma, \tau) \in \mathbb{P}_{\text{far}}} \int_{\bigcup \sigma} \int_{\bigcup \tau} m_h(x) \cdot H_G(x-y) \tilde{m}_h(y) \, dy \, dx.$$

As in the previous section, we obtain the approximate bilinear form by replacing the exact kernel $H_G$ on far field blocks by tensor interpolation $T^{(\sigma, \tau)}_m H_G$ which is now understood coefficient-wise [since we are dealing with a matrix kernel $H_G(x-y) \in \mathbb{R}^{d \times d}$],

$$\tilde{a}(m_h, \tilde{m}_h) := -\sum_{(\sigma, \tau) \in \mathbb{P}_{\text{near}}} \sum_{T_j \sigma} \sum_{T_k \in \tau} \int_{\partial T_j} \int_{\partial T_k} G(x-y)(\nu^j(x) \cdot m_h^j) (\nu^k(y) \cdot \tilde{m}_h^k) \, ds_y \, ds_x +$$

$$+ \sum_{(\sigma, \tau) \in \mathbb{P}_{\text{far}}} \int_{\bigcup \sigma} \int_{\bigcup \tau} m_h(x) \cdot (T^{(\sigma, \tau)}_m H_G(x-y)) \tilde{m}_h(y) \, dy \, dx.$$

The error analysis is completely straightforward following the arguments of Section 4.4. We denote by $\kappa_{\alpha\beta}, \ 1 \leq \alpha, \beta \leq d$, the second derivatives of the Newtonian kernel, cf. (53). As before, we assume that the approximation degree $m \in \mathbb{N}$ is large enough so that

$$\|\kappa_{\alpha\beta} - T^{(\sigma, \tau)}_m \kappa_{\alpha\beta} \|_{L^\infty(B_\sigma \times B_\tau)} \leq c_3,$$

with a constant $c_3$ which is independent of $(\sigma, \tau) \in \mathbb{P}_{\text{far}}$.

**Theorem 5.2.** Under the above assumptions, we have

$$\|a(m_h, \tilde{m}_h) - \tilde{a}(m_h, \tilde{m}_h)\| \leq c_3 d|\Omega| \|m_h\|_{L^2(\Omega)} \|\tilde{m}_h\|_{L^2(\Omega)}$$

for all $m_h, \tilde{m}_h \in P_0(T)^d$.

**Proof.** With $\kappa(x,y) := H_G(x-y)$ and $\tilde{\kappa}$ as in the proof of Theorem 4.1, we obtain

$$\|a(m_h, \tilde{m}_h) - \tilde{a}(m_h, \tilde{m}_h)\| \leq |\Omega|\|\kappa - \tilde{\kappa}\|_{L^\infty(\Omega \times \Omega, \mathbb{R}^{d \times d})} \|m_h\|_{L^2(\Omega, \mathbb{R}^d)} \|\tilde{m}_h\|_{L^2(\Omega, \mathbb{R}^d)},$$

where we consider the usual (Euclidean) operator norm $\| \cdot \|$ on $\mathbb{R}^{d \times d}$. Recalling that the Frobenius norm satisfies $\| A \| \leq \| A \|_F := (\sum_{j,k=1}^{d} A_{jk}^2)^{1/2}$, it follows that

$$\|\kappa(x,y) - \tilde{\kappa}(x,y)\| \leq \|\kappa(x,y) - \tilde{\kappa}(x,y)\|_F \leq dc_3.$$

This concludes the proof. 

Finally, we explicitly state the approximation $\tilde{A}^{\alpha \beta} \in \mathbb{R}^{N \times N}$ to $A^{\alpha \beta}$ to clarify what has to be implemented. Recall that $A^{\alpha \beta} \in \mathbb{R}^{N \times N}$ is defined by $A_{jk}^{\alpha \beta} = \tilde{a}(\chi_{T_j} e_\alpha, \chi_{T_k} e_\beta)$. The computation of $\tilde{A}^{\alpha \beta}$ is performed separately on the admissible and the inadmissible blocks of $P$: 

```
First, let \((\sigma, \tau) \in \mathbb{P}_{\text{far}}\) be admissible; given the degenerate kernel
\[
\mathcal{I}^{(\sigma, \tau)}_m \kappa_{\alpha \beta}(x, y) = \sum_{m_1=0}^{M} \sum_{m_2=0}^{M} \kappa_{\alpha \beta}(x^\sigma_{m_1}, x^\tau_{m_2}) \mathcal{L}^\sigma_{m_1}(x) \mathcal{L}^\tau_{m_2}(y) \quad \text{for} \ (x, y) \in \cup \sigma \times \cup \tau,
\]
where \(M = m^d\) and \(\mathcal{L}^\sigma_{m_1}\) and \(\mathcal{L}^\tau_{m_2}\) are the appropriate tensor Lagrange polynomials, this implies
\[
\tilde{A}^{\alpha \beta}_{jk} = \int_{T_j} \int_{T_k} \kappa_{\alpha \beta}(x, y) \, dy \, dx = \sum_{m_1=0}^{M} \sum_{m_2=0}^{M} \kappa_{\alpha \beta}(x^\sigma_{m_1}, x^\tau_{m_2}) \int_{T_j} \mathcal{L}^\sigma_{m_1}(x) \, dx \int_{T_k} \mathcal{L}^\tau_{m_2}(y) \, dy
\]
for \(T_j \in \sigma\) and \(T_k \in \tau\). With the matrices \(V^\sigma \in \mathbb{R}^{[\sigma] \times M}, V^\tau \in \mathbb{R}^{[\tau] \times M}\), and \(S^{\alpha \beta \sigma \tau} \in \mathbb{R}^{M \times M}\), the submatrix \(A^{\alpha \beta}_{\sigma \times \tau}\) of \(A^{\alpha \beta}\) can be computed approximately by a matrix product
\[
A^{\alpha \beta}_{\sigma \times \tau} \approx V^\sigma S^{\alpha \beta \sigma \tau} V^\tau =: \tilde{A}^{\alpha \beta}_{\sigma \times \tau}.
\]
Second, for an inadmissible block \((\sigma, \tau) \in \mathbb{P}_{\text{near}}\), we have \(\tilde{A}^{\alpha \beta}_{\sigma \times \tau} = a(\chi_{T_j} \mathbf{e}_\alpha, \chi_{T_k} \mathbf{e}_\beta) = A^{\alpha \beta}_{\sigma \times \tau}\); these entries are computed by (47). Double boundary integrals as in (47) occur in the context of boundary integral methods with piecewise constant ansatz and test functions. For simple geometries of the elements, analytic formulae are known, cf. [15, 22, 23].

Remark 10. Note that only the multiplication matrices \(S^{\alpha \beta \sigma \tau}\) and \(A^{\alpha \beta}\) in (57) do depend on the indices \(\alpha, \beta\). Therefore, on admissible blocks \((\sigma, \tau) \in \mathbb{P}_{\text{far}}\), the matrices \(\tilde{A}^{\alpha \beta}_{\sigma \times \tau}\) should be treated simultaneously for all \(1 \leq \alpha \leq \beta \leq d\).

Remark 11. For inadmissible blocks \((\sigma, \tau) \in \mathbb{P}_{\text{near}}\), the matrices \(\tilde{A}^{\alpha \beta}_{\sigma \times \tau} = A^{\alpha \beta}_{\sigma \times \tau}\) should also be assembled simultaneously, as the entries only differ on the components of the normal vectors, cf. (47). Since the block partitioning is symmetric and we are using constant approximation order, \(\tilde{A}^{\alpha \beta}\) also is symmetric; therefore \(\tilde{A}^{\alpha \beta}_{jk}\) should only be computed and stored for \(1 \leq j \leq k \leq N\).

5.2. Approximation of A on Far Field Blocks via (47). A different \(H^2\)-matrix approach can be realized by use of (47) and by replacing \(G\) by \(\mathcal{I}^{(\sigma, \tau)}_m G\) on admissible blocks. We start out from (54) and consider the following discretization of the bilinear form \(a(\cdot, \cdot)\), where \(\kappa(x, y) := G(x - y)\) now:
\[
\tilde{a}(\mathbf{m}_h, \mathbf{m}_h) = - \sum_{(\sigma, \tau) \in \mathbb{P}_{\text{near}}} \sum_{T_j \in \sigma} \sum_{T_k \in \tau} \int_{T_j} \int_{T_k} (\mathbf{v}^j(x) \cdot \mathbf{m}^j_h(y)) \kappa(x, y) (\mathbf{v}^k(y) \cdot \mathbf{m}^k_h) \, dy \, ds_x - \sum_{(\sigma, \tau) \in \mathbb{P}_{\text{far}}} \sum_{T_j \in \sigma} \sum_{T_k \in \tau} \int_{T_j} \int_{T_k} (\mathbf{v}^j(x) \cdot \mathbf{m}^j_h) \mathcal{I}^{(\sigma, \tau)}_m \kappa(x, y) (\mathbf{v}^k(y) \cdot \mathbf{m}^k_h) \, dy \, ds_x.
\]
We state the implementational details first. The only difference to Section 5.1 is in the way how \(\tilde{A}\) is computed on admissible blocks \((\sigma, \tau) \in \mathbb{P}_{\text{far}}\). Given the degenerate kernel
\[
\mathcal{I}^{(\sigma, \tau)}_m \kappa(x, y) = \sum_{m_1=0}^{M} \sum_{m_2=0}^{M} \kappa(x^\sigma_{m_1}, x^\tau_{m_2}) \mathcal{L}^\sigma_{m_1}(x) \mathcal{L}^\tau_{m_2}(y),
\]
we obtain for $T_j \in \sigma, T_k \in \tau$ using integration by parts
\[
\tilde{A}_{jk}^{\alpha\beta} = \tilde{a}(x_\gamma, e_\alpha, x_\gamma e_\beta) = -\int_{\partial T_j} \int_{\partial T_k} \nu_\alpha^i(x) T_m^{(\alpha, \tau)} \kappa(x, y) \nu_\beta^k(y) \, ds_y \, ds_x = \]
\[
= -\sum_{m_1=0}^M \sum_{m_2=0}^M \kappa(x_{m_1}, x_{m_2}) \int_{\partial T_j} L_{m_1}^{\alpha}(x) \, ds_x \int_{\partial T_k} L_{m_2}^{\tau}(y) \nu_\beta^k(y) \, ds_y = \]
\[
= -\sum_{m_1=0}^M \sum_{m_2=0}^M \kappa(x_{m_1}, x_{m_2}) \int_{T_j} \frac{\partial L_{m_1}^{\alpha}}{\partial x_\alpha} \, dx \int_{T_k} \frac{\partial L_{m_2}^{\tau}}{\partial y_\beta} \, dy .
\]

**Remark 12.** The integrands in $V_{j m_1}^{\sigma}$ and $V_{k m_2}^{\tau}$ are computable, as formulae for the first derivatives of the Lagrange polynomials in one dimension can easily be derived by induction: given $L_j \in \mathcal{P}_m$, we have

\[
L_j'(t) = \sum_{k=1}^m \frac{1}{t_j - t_k} \prod_{l=1 \atop l \neq j}^m \frac{t - t_l}{t_j - t_l} .
\]

**Remark 13.** Note that the multiplication matrices $S^{\sigma\tau}$ do not depend on $\alpha, \beta$ here. Thus, we now have to assemble and store $S^{\sigma\tau}$, the matrices $V^{\sigma\alpha}$ and $V^{\tau\beta}$ for all leaves $\sigma, \tau$ of the cluster tree only, and the father-son transformation matrices $B^{\sigma\tau}$. This is a significant difference to Section 5.1, where the computation of $S^{\alpha\beta\sigma\tau}$, of $V^{\tau}$ for all leaves $\tau$, and of $B^{\sigma\tau}$ is required.

The results of Section 5.1 now carry over immediately; however, we have to assume that $m$ is large enough so that

\[
\| \partial_\alpha \partial_\beta (\kappa - T_m^{(u, \tau)} \kappa) \|_{L^\infty(B_x \times B_y)} \leq c_3
\]

for $\kappa(x, y) = G(x - y)$, with a constant $c_3$ which is independent of $(\sigma, \tau) \in \mathcal{P}_{\text{far}}$ and $1 \leq \alpha, \beta \leq d$. The existence of such a constant is a consequence of Lemma 3.6. In complete analogy to Theorem 5.2 we conclude:

**Theorem 5.3.** Under the above assumptions, we have

\[
|a(m_h, \tilde{m}_h) - a(m_h, \tilde{m}_h) | \leq c_3 d |\Omega| \|m_h\|_{L^2(\Omega)} \|\tilde{m}_h\|_{L^2(\Omega)}
\]

for all $m_h, \tilde{m}_h \in \mathcal{P}_0(T)^d$. \hfill \blacksquare

5.3. Solvability of the Approximate Discrete Model ($\widetilde{RP}_{\epsilon, h}$). Suppose a triangulation $T$ of $\Omega$ by rectangular, axis-parallel boxes to be given. The Gauss Divergence Theorem then shows that the restriction of $\mathcal{P}, \mathcal{P} \big|_{\mathcal{P}_0(T)^d}$, is injective, cf. [7]. In particular, there exists a constant $c_4$ such that

\[
a(m_h, m_h) = \|\mathcal{P} m_h\|_{L^2(\mathbb{R}^d)}^2 \geq c_4 \|m_h\|_{L^2(\Omega)}^2or all $m_h \in \mathcal{P}_0(T)^d$.

**Theorem 5.4.** Given the above assumptions, take $m$ large enough so that $C_{\text{diff}} := c_4 - c_3 d |\Omega| \geq 0$, with $c_3$ from Theorems 5.2 and 5.3, respectively. Then, the approximate discrete model ($\widetilde{RP}_{\epsilon, h}$) has solutions. In case $C_{\text{diff}} > 0$, the solution to ($\widetilde{RP}_{\epsilon, h}$) is unique.
Proof. As
\[ \tilde{a}(m_h, m_h) \geq a(m_h, m_h) - |a(m_h, m_h) - \tilde{a}(m_h, m_h)| \geq C_{df} \|m_h\|_{L^2(\Omega)}^2, \]
one concludes with \( C_{df} \geq 0 \) that the approximate bilinear form \( \tilde{a}(\cdot, \cdot) \) is positive semidefinite, and even positive definite if \( C_{df} > 0 \). Thus, the symmetry of \( \tilde{a}(\cdot, \cdot) \) shows that \( m_h \mapsto \tilde{a}(m_h, m_h) \) is a convex functional. In sum, the approximate energy functional
\[ (59) \quad \tilde{E}(m_h) := \int_\Omega \phi^{**}(m_h) \, dx - \int_\Omega f \cdot m_h \, dx + \frac{1}{2} \tilde{a}(m_h, m_h) + \frac{1}{2} \int_\Omega \varepsilon^{-1}(|m_h| - 1)^2 \, dx \]
is continuous and convex. Coercivity of \( \tilde{E} \) follows from the last term in (59), i.e., from the penalization energy contribution.

The Direct Method of the Calculus of Variations now proves the existence of (global) minimizers for \( \tilde{E}(\cdot) \). As minimizers of \( \tilde{E} \) in \( P_0(\mathcal{T})^d \) are zeros of the corresponding Gâteaux derivatives, which read (60) now. One immediately finds
\[ (60) \quad \tilde{a}(m_h - \tilde{m}_h, m_h - \tilde{m}_h) + \langle D\phi^{**}(m_h) - D\phi^{**}(\tilde{m}_h) ; m_h - \tilde{m}_h \rangle_{L^2(\Omega)} + \langle \lambda_h m_h - \tilde{\lambda}_h \tilde{m}_h ; m_h - \tilde{m}_h \rangle_{L^2(\Omega)} = 0. \]
As \( \phi^{**} \) is convex, the second term in (60) cannot be negative, whereas the third term is non-negative by a direct calculation [9]. Hence, \( \tilde{a}(m_h - \tilde{m}_h, m_h - \tilde{m}_h) = 0 \), which implies \( m_h = \tilde{m}_h \) due to the definiteness of \( \tilde{a}(\cdot, \cdot) \).

Remark 14. The a priori and a posteriori error analysis for the approximate model (\( \tilde{R}\mathcal{P}_{\varepsilon,h} \)) is the topic of ongoing research and will appear in a subsequent work.

6. Numerical Experiments

In the following section we collect the results of our numerical experiments. We compare the performance of the two \( H^2 \)-matrix approximations introduced in Sections 5.1 and 5.2 to that of the standard approach using the full stiffness matrices.

6.1. Implementational Details. All numerical experiments were conducted using the HLib software package provided by S. Börn and L. Grasedyck of the Max-Planck-Institute for Mathematics in the Sciences (Leipzig). We utilized a Compaq/HP AlphaServer ES45 under Unix, with four Alpha EV68 CPUs running at 1 GHz each and 32 GBytes of RAM.

For our experiments, we varied the interpolation order \( m \) in the \( H^2 \)-matrix approximation between 2 and 6. As for the parameters, we fixed the maximum leaf size \( C_{il} = 20 \) and the admissibility parameter \( \eta = 1 \) throughout. However, to avoid having to vary \( C_{il} \) with \( m \), we decided to store admissible blocks \( (\sigma, \tau) \in \mathcal{P}_{\text{full}} \) as full matrices whenever \( |\sigma||\tau| \leq M_{\sigma}M_{\tau} \), i.e., whenever storing the full matrix was less expensive than storing the corresponding multiplication matrix \( S^{\sigma\tau} \).

Moreover, to be able to use the supplied HLib routines with as few modifications as possible, we did exploit the symmetry of \( A^{\alpha\beta} \) when setting up the matrices, but neglected it in the process of storing them.
6.2. Full Matrices vs. $H^2$-Matrices. In a first example, we restrict ourselves to a comparison of the properties of $A$ and its $H^2$-approximation $\tilde{A}$. For simplicity, we assume the domain $\Omega$ to be the unit square, $\Omega := [0, 1] \times [0, 1] \subset \mathbb{R}^2$, and consider a uniform triangulation of $\Omega$ consisting of rectangular elements. The number $N$ of degrees of freedom is varied between 256 and 1048576. Note that for $N \geq 16384$, we have no longer set up the full matrix $A$, but have approximated it by an $H^2$-matrix with $m = 10$; all subsequent references are made to this approximation.

We compare the results of our two $H^2$-matrix approaches for fixed approximation orders $m$. First, we give the relative approximation errors $\|A - \tilde{A}\|_2/\|A\|_2$; the values are collected in the following Tables 1 and 2.

**Table 1.** Relative approximation errors (first $H^2$-matrix approach, cf. Section 5.1).

<table>
<thead>
<tr>
<th>$N/m$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>256</td>
<td>2.38605 -3</td>
<td>2.42547-4</td>
<td>2.22014 -16</td>
<td>2.36817 -16</td>
<td>2.10329-16</td>
</tr>
<tr>
<td>1024</td>
<td>4.66221 -3</td>
<td>4.74404-4</td>
<td>4.42905 -5</td>
<td>4.45515 -6</td>
<td>3.85638-7</td>
</tr>
<tr>
<td>4096</td>
<td>6.08654 -3</td>
<td>6.08069-4</td>
<td>6.14339 -5</td>
<td>5.88869 -6</td>
<td>5.21420-7</td>
</tr>
<tr>
<td>16384</td>
<td>7.28966 -3</td>
<td>7.41998-4</td>
<td>7.76218 -5</td>
<td>7.11627 -6</td>
<td>7.26753-7</td>
</tr>
<tr>
<td>65536</td>
<td>8.42299 -3</td>
<td>8.57423-4</td>
<td>8.83683 -5</td>
<td>8.26648 -6</td>
<td>8.21227-7</td>
</tr>
</tbody>
</table>

**Table 2.** Relative approximation errors (second $H^2$-matrix approach, cf. Section 5.2).

<table>
<thead>
<tr>
<th>$N/m$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>256</td>
<td>1.47377 -2</td>
<td>1.80771 -3</td>
<td>2.46549 -16</td>
<td>2.25802 -16</td>
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<td>3.47728 -3</td>
<td>2.38627 -4</td>
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<td>5.48085 -2</td>
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<td>5.19552 -4</td>
<td>5.21367 -5</td>
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<td>6.14949 -4</td>
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<td>6.88747-6</td>
</tr>
</tbody>
</table>

In summary, the errors in the second approach (cf. Section 5.2) seem to be larger by one order of magnitude. The convergence rates, however, are optimal in both cases: every increase of $m$ by one reduces the error by an order of magnitude. Note that for $N = 256$ and $m \geq 4$, our choice of $C_H$ implies that there are no admissible blocks; the error in this case is due to rounding. The error estimates themselves are computed by a power iteration, with a maximum of 100 iterative steps.

Second, we consider the times needed for building the two $H^2$-matrix approximations and compare them to the setup times of the full matrix $A$, see Tables 3 and 4.

The time required for setting up the full matrix by far surpasses the setup times of the approximations; the gap increases with the number $N$ of degrees of freedom. For $m$ fixed, the difference between the two approaches is negligible here.

Third, of particular interest is the amount of memory required for storing the matrix approximations, as compared to the storage requirements of the original matrices; these are listed in Tables 5 and 6.
Table 3. Setup time in seconds (first approach).

<table>
<thead>
<tr>
<th>N/m</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>full</th>
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<tbody>
<tr>
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<td>0.7</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
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<td>1024</td>
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<td>3.8</td>
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<td>9.0</td>
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<td>13.5</td>
</tr>
<tr>
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<td>18.0</td>
<td>50.9</td>
<td>51.1</td>
<td>60.1</td>
<td>215.6</td>
</tr>
<tr>
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<td>78.5</td>
<td>237.4</td>
<td>237.9</td>
<td>286.1</td>
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</tr>
<tr>
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<td>330.9</td>
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<td>1042.2</td>
<td>1263.9</td>
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<tr>
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<td>1361.8</td>
<td>4303.4</td>
<td>4308.8</td>
<td>5248.6</td>
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</tr>
<tr>
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<td>5925.1</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
<td>n/a</td>
</tr>
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</table>

Table 4. Setup time in seconds (second approach).

<table>
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<tr>
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<th>6</th>
<th>full</th>
</tr>
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<tr>
<td>256</td>
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<td>0.6</td>
<td>0.8</td>
<td>0.9</td>
<td>0.9</td>
<td>0.8</td>
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<tr>
<td>1024</td>
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<td>3.9</td>
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<td>9.1</td>
<td>10.4</td>
<td>13.5</td>
</tr>
<tr>
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<td>18.0</td>
<td>51.1</td>
<td>51.4</td>
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</tr>
<tr>
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<td>238.5</td>
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<td>n/a</td>
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Table 5. Memory requirement in KBytes/N (first approach).

<table>
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<th>N/m</th>
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<td>n/a</td>
<td>n/a</td>
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<td>24576.0</td>
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</table>

Table 6. Memory requirement in KBytes/N (second approach).

<table>
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<th>N/m</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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</tr>
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<tbody>
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<td>7.2</td>
<td>8.1</td>
<td>9.2</td>
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<tr>
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<td>19.9</td>
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<tr>
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<td>n/a</td>
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</tr>
</tbody>
</table>

Overall, the figures compare very favourably with the storage required by the full matrices; here the second approach is clearly superior to the first one for m fixed and N large. This
is probably due to the fact that the multiplication matrices on admissible blocks only have
to be stored once instead of for each block. Note that on meshes with $N$ small, the memory
requirements slightly favour the full matrix approach. This is due to the organizational effort
involved in constructing the cluster tree of the $H^2$-matrix approximation and in allocating
memory blockwise for subblocks instead of for the whole matrix.

 Altogether, these numerical experiments underline the applicability of $H$-matrix techniques
to the discretized potential operator $P$ from $(RP_{x,h})$. For given approximation order $m$,
the first approach leads to lesser approximation errors, but is otherwise also more costly
numerically, as is reflected by the much higher storage requirements.

 In a certain sense, however, the two approaches seem almost equivalent: if we require some
fixed accuracy, the approximation order $m$ always has to be higher by one in the second
approach, as the errors lag behind by one order of magnitude. A comparison of the respective
setup times and memory requirements then shows the numerical cost to be almost even.

6.3. An Example With Known Exact Solution. In our second example, we consider
a model problem for the relaxed Landau-Lifshitz problem $(RP)$ taken from [6]. As above,
let the domain $\Omega$ be the unit square; assume $\Omega$ to be filled with some uniaxial magnetized
material, with the easy axis given by $e = (-1,1)/\sqrt{2}$ and the corresponding normal by
$z = (1,1)/\sqrt{2}$, see Remark 1. Define $(m, \lambda) \in W^{1,\infty}(\Omega; \mathbb{R}^2) \times L^\infty(\Omega)$ as

\[
m(x) := \begin{cases} 
  x & \text{for } |x| \leq 1, \\
  x/|x| & \text{for } |x| > 1
\end{cases}
\quad \text{and} \quad
\lambda(x) := \begin{cases} 
  0 & \text{for } |x| \leq 1, \\
  1 & \text{for } |x| > 1.
\end{cases}
\]

Then, $(m, \lambda)$ solves $(RP)$ with given right-hand side

\[
f := Pm + (m \cdot z)z + \lambda m \quad \text{in } L^2(\Omega; \mathbb{R}^2),
\]

cf. (11),(12). In the following, we replace $Pm$ in (62) by $Pm_T$, where $m_T$ denotes the
piecewise integral mean of $m$. Note that there are no fully analytic examples for $(RP)$ with
known solutions, which is why we have to restrict ourselves to the present model.

As $m$ is Lipschitz continuous and therefore in $W^{1,\infty}(\Omega; \mathbb{R}^2)$, the a priori analysis from [6]
yields $\|m - m_h\|_{L^2(\Omega)} = O(\varepsilon + h)$, with $\varepsilon$ the penalization parameter from (14). For
our experiments, we choose $\varepsilon = h$ and compute the discrete solution $m_h = \sum_{j=1}^{2N} x_j \varphi_j$
with respect to the basis $\{\varphi_1, \ldots, \varphi_{2N}\}$ from (50) by a classical Newton-Raphson scheme: the
unknown coefficient vector $x \in \mathbb{R}^{2N}$ is determined as the unique zero of

\[
F(x) := \left( (Pm_h + D\phi^x(m_h) + \lambda_h m_h - f ; \varphi_j) \right)_{j=1}^{2N} = 0.
\]

[An detailed discussion on the relevance of $\varepsilon$ can be found in [6].] Note that the convergence of
the Newton-Raphson method is not guaranteed mathematically, since $F$ is only differentiable
almost everywhere. The Jacobian of $F$ can be written as a finite sum $DF(x) = A + \sum_{j=1}^{N} D_j(x)$
with symmetric positive semidefinite matrices $D_j(x) \in \mathbb{R}^{2N \times 2N}$, cf. [7]. For
a triangulation by rectangular elements such as ours, the matrix $A$ can be shown to be
positive definite by applying the Gauss Divergence Theorem, see [6]. We therefore employ
a preconditioned conjugate gradient method, with the LU decomposition of a coarsened
$H$-matrix version of $A$ as preconditioner.
In Table 7, we summarize the number of Newton steps required for finding the coefficient vector $x$ of $m_h$ in (63). One sees that the number of steps is nearly constant, i.e., independent of $m$ and growing only slightly with $N$.

Figure 2 gives the convergence history of the full error $\|m - m_h\|_{L^2(\Omega)}$ for different choices of the order $m$ in the $H^2$-matrix approximation. The convergence rate is almost the optimal $1/2$, up to minimal deviations which are presumably due to the fact that we approximate $Pm$ by $Pm_T$ in (62). As was to be expected, the interpolation order $m$ has to increase with the number $N$ of degrees of freedom in order to maintain optimal convergence; the effect is clearly more pronounced for the second approach.

Note that we consider the full $L^2$ norm instead of $\|D\phi^{\ast\ast}(m) - D\phi^{\ast\ast}(m_h)\|_{L^2(\Omega)}$, although only the latter is covered by the available a priori error analysis. Recently it has been shown that one always obtains weak $L^2$ convergence of $m_h \rightharpoonup m$; more precisely, there holds $\|m - m_h\|_{H^{-1}(\Omega)} = O(\varepsilon + h)$, cf. [8]. Nevertheless, the numerical experiments available from [6] indicate that one may hope for sharper results.

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- **Carsten Carstensen, Nikola Popović, Dirk Praetorius:** *Applications of H-Matrix Techniques for the Effective Simulation of Stationary Micromagnetics.*
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