BACHELORARBEIT

A Virtual Element Method of arbitrary regularity

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durch

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Bachelorarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am June 3, 2019

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1 Introduction

In this Bachelor thesis we investigate Virtual Element Methods (VEMs) with arbitrary regularity for stationary diffusion equations on bounded two-dimensional domains with homogenous Dirichlet boundary conditions. VEM is a relatively modern approach to the numerical approximation of partial differential equations. It is akin to the Finite Element Method (FEM), but uses approximation functions, that although not known explicitly (they are *virtual*), still make the computation of the stiffness matrix possible.

The goal of this thesis is to give an overview of VEM for two-dimensional, stationary diffusion problems. The main contribution lies in the extended and comparably rigorous proofs of all claims, which, compared to the literature, may make this thesis an easier introduction to VEM. Furthermore, we also prove the reliability and efficiency of the jumping term η_P^{j} (see Definition 5.4) introduced in [BdV M 15] for the treatment of a discretization with locally reduced regularity.

This thesis is mainly based on two papers: [BdV M 14], in which VEMs with arbitrary regularity were first introduced, and [BdV M 15], which developed A POSTERIORI estimates for the method. The structure of this thesis is inspired by the lecture notes [Praet], which give an introduction to classical FEM.

This thesis is organized as follows: In the remainder of this chapter, required results from the literature are cited and essential notations are introduced.
Chapter 2 first recalls the basic solution theory of stationary diffusion equations, then develops a basic approximation scheme on which VEM is based, and finally briefly discusses the theory of polyharmonic equations.
Chapter 3 starts with the definition of the mesh. Then, the VEM space is introduced and the degrees of freedom are defined. After that the discrete bilinear form and the loading term are defined. Finally, the construction of the stiffness matrix and discrete spaces of locally reduced regularity are discussed.
In Chapter 4, we develop A PRIORI estimates for the VEM. Chapter 5 describes A POSTERIORI estimates and is based on [BdV M 15].

1.1 Auxiliary results and basic notations

Let $n \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^n$.

1.1.1 Matrices and vectors

Definition 1.1. *We denote the n-dimensional identity matrix by* I_n. *Furthermore we will often denote the identity as function by* I.

Definition 1.2 (ellipticity)**.** *Let* $\kappa > 0$. *A matrix* $A \in \mathbb{R}^{n\times n}$ *is called* κ*-elliptic if*

$$x^T Ax \geq \kappa \|x\|^2 \quad \text{for all } x \in \mathbb{R}^n$$

A matrix-valued function $A : \Omega \to \mathbb{R}^{n\times n}$ *is called uniformly* κ*-elliptic, if*

$$x^T A(\xi)x \geq \kappa \|x\|^2 \quad \text{for all } \xi \in \Omega \text{ and } x \in \mathbb{R}^n.$$

For any uniformly κ*-elliptic, symmetric matrix* A*, there exists a symmetric, positive definite matrix* $\sqrt{A}$*, such that*

$$A = \sqrt{A}\sqrt{A}.$$

1.1.2 Geometry

Definition 1.3 (straight line)**.** *Let* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. *Then the straight line* $\overline{\boldsymbol{xy}}$ *between* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *is defined by*

$$\overline{\boldsymbol{xy}} := \{\boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x}) \mid t \in [0, 1]\}.$$

Definition 1.4 (diameter and measure)**.** *Let* $P \subseteq \mathbb{R}^n$ *be a bounded set. The diameter of* P *is defined by*

$$h_P := \operatorname{diam}(P) := \sup_{\boldsymbol{x},\boldsymbol{y}\in P} |\boldsymbol{x} - \boldsymbol{y}|.$$

Furthermore, we denote the n-dimensional Lebesgue measure of a Lebesgue-measurable set P *by* $|P|$.

Definition 1.5. *A bounded domain* Ω *is called a Lipschitz domain (or a domain with Lipschitz boundary), if it has an orientable boundary* $\partial\Omega$*, meaning that* Ω *is on one side of the boundary and if the boundary is locally parametrizable by a Lipschitz continuous function. The outer unit normal vector of the boundary is denoted by* $\boldsymbol{n}_\Omega : \partial\Omega \to \mathbb{R}^n$. *Often we will omit the* Ω *subscript and just write* $\boldsymbol{n}$.

Definition 1.6 (polygon, $n = 2$)**.** *A compact subset $P \subseteq \mathbb{R}^2$ is called polygon, if there are $d \geq 3$ pairwise different points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{d-1} \in \mathbb{R}^2$ such that, if we define $\boldsymbol{x}_d := \boldsymbol{x}_0$, for all $i, j = 0, \ldots, d-1$ with $i \neq j$, it holds that*

$$\overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}} \bigcap \overline{\boldsymbol{x}_j \boldsymbol{x}_{j+1}} = \begin{cases} \boldsymbol{x}_i & \text{for } j = i-1, \\ \boldsymbol{x}_{i+1} & \text{for } j = i+1, \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$\partial P = \bigcup_{i=0,\ldots,d-1} \overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}}.$$

The points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{d-1}$ are called vertices of P and the straight connections $\overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}}$, $i = 0, \ldots, d-1$ are the edges of P.

Note that a polygon is a Lipschitz domain.

Furthermore, we define

$$\begin{aligned} \operatorname{vertices}(P) &:= \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{d-1}\}, \\ \operatorname{edges}(P) &:= \{\overline{\boldsymbol{x}_0 \boldsymbol{x}_1}, \overline{\boldsymbol{x}_1 \boldsymbol{x}_2}, \ldots, \overline{\boldsymbol{x}_{d-1} \boldsymbol{x}_d}\}. \end{aligned}$$

Let $\boldsymbol{e} = \overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}} \in \operatorname{edges}(P)$, then we define the normal vector to $\boldsymbol{e}$ by

$$\boldsymbol{n}_{\boldsymbol{e}} := \boldsymbol{n}|_{\boldsymbol{e}}$$

and the tangential vector by

$$\tau_{\boldsymbol{e}} := \frac{1}{|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i|} (\boldsymbol{x}_{i+1} - \boldsymbol{x}_i).$$

The tangential vector $\tau_P : \partial P \to \mathbb{R}^2$ of P is defined by $\tau_P|_{\boldsymbol{e}} = \tau_{\boldsymbol{e}}$ for every $\boldsymbol{e} \in \operatorname{edges}(P)$. Often we will omit the subscript and simply write τ.

The barycenter of P is defined by

$$\boldsymbol{x}_P = \begin{pmatrix} x_P^1 \\ x_P^2 \end{pmatrix},$$

where

$$\begin{aligned} x_P^1 &:= \frac{1}{6\sigma(P)} \sum_{j=0}^{d-1} (x_j^1 + x_{j+1}^1)(x_j^1 x_{j+1}^2 - x_{j+1}^1 x_j^2), \\ x_P^2 &:= \frac{1}{6\sigma(P)} \sum_{j=0}^{d-1} (x_j^2 + x_{j+1}^2)(x_j^1 x_{j+1}^2 - x_{j+1}^1 x_j^2), \end{aligned}$$

with

$$\boldsymbol{x}_j = \begin{pmatrix} x_j^1 \\ x_j^2 \end{pmatrix} \quad \textit{and}$$

$$\sigma(P) := \frac{1}{2}\sum_{j=0}^{d-1}(x_j^1 x_{j+1}^2 - x_{j+1}^1 x_j^2)$$

denoting the signed area of P*.*

Definition 1.7 (triangle)**.** *A triangle* $T \subseteq \mathbb{R}^2$ *is a polygon with three vertices* $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2$*. It holds that* $T = \operatorname{conv}\{\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2\}$*.*

Definition 1.8 (ball)**.** *Let* $\delta > 0$ *and* $x \in \mathbb{R}^n$*. We denote the closed ball of radius* δ *around* x *by*

$$B_\delta(x) := \left\{ y \in \mathbb{R}^n \;\middle|\; |x - y| \leq \delta \right\}.$$

1.1.3 Bilinear forms and scalar products

Definition 1.9 (coercivity and continuity of bilinear forms)**.** *Let* H *be a Hilbert space with scalar product* $\langle \cdot, \cdot \rangle$ *and induced norm* $\| \cdot \|$*. A bilinear form* $a : H \times H \to \mathbb{R}$ *is continuous, if there exists a constant* $C_{\text{cont}} > 0$ *such that*

$$|a(u, v)| \leq C_{\text{cont}} \|u\| \|v\| \quad \textit{for all } u, v \in H.$$

The bilinear form is coercive, if there exists a constant $C_{\text{coer}} > 0$ *such that*

$$a(u, u) \geq C_{\text{coer}} \|u\|^2 \quad \textit{for all } u \in H.$$

Theorem 1.10 (Lax–Milgram theorem, [Juengel, Satz 5.18])**.** *Let* H *be a Hilbert space and let* $a : H \times H \to \mathbb{R}$ *be a bilinear form. If* a *is continuous and coercive, there exists a unique* $u \in H$ *for every continuous linear form* $F : H \to \mathbb{R}$ *such that*

$$a(u, v) = F(v) \quad \textit{for all } v \in H.$$

Additionally, it holds that

$$\|u\| \leq \frac{1}{C_{\text{coer}}} \|F\|.$$

□

Definition 1.11 (orthogonal projection)**.** *Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and let $V \subseteq H$ be a closed subspace. A linear map $\pi_V : H \longrightarrow V$ is called the orthogonal projection onto V if*

$$\pi_V(H) = V, \ \pi_V \circ \pi_V = \pi_V \ \text{and}$$
$$\langle \pi_V u, (\mathrm{I} - \pi_V)v \rangle = 0 \quad \text{for all } v \in H.$$

Lemma 1.12. *Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. If $\{b_1, \dots, b_N\} \subseteq H$ is an orthonormal basis of a subspace $V \subseteq H$, i.e. $V = \operatorname{span}\{b_1, \dots, b_N\}$ and*

$$\langle b_i, b_j \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise,} \end{cases}$$

then the orthogonal projection $\pi_V : H \to V$ is given by

$$\pi_V(v) = \sum_{j=1}^{N} \langle v, b_j \rangle b_j \quad \text{for all } v \in H.$$

□

Theorem 1.13 (Gram–Schmidt, [Havlicek, Satz 11.5.7])**.** *Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let the $\{b_1, \dots, b_N\} \subseteq H$ be a basis of the linear subspace $V \subseteq H$.*
Then, there exists an orthonormal basis $\{c_1, \dots, c_N\}$ of V. A possible orthonormal basis of V is given by

$$c_j := \frac{1}{\|\tilde{c}_j\|} \tilde{c}_j, \ j = 1, \dots, N \ \text{with} \ \tilde{c}_j := b_j - \sum_{i=1}^{j-1} \frac{\langle \tilde{c}_i, b_j \rangle}{\|\tilde{c}_i\|^2} \tilde{c}_i.$$

□

1.1.4 Polynomials

Definition 1.14 (multi-indices)**.** *Let $s = (s_1, \dots, s_n) \in \mathbb{N}_0^n$ be a multi-index. We define its length by*

$$|s| := s_1 + \ldots + s_n.$$

Definition 1.15. *Let $m \in \mathbb{N}$. The space of polynomials with maximal degree m is denoted by*

$$\mathbb{P}_m(\Omega) := \left\{ (x_1, \dots, x_d) \mapsto \sum_{|s| \le m} \alpha_s \ x_1^{s_1} \cdots x_d^{s_d} \ \middle| \ \alpha_s \in \mathbb{R}, \ s \in \mathbb{N}_0^n, \ |s| \le m \right\}.$$

Furthermore we define $\mathbb{P}_{-1}(\Omega) := \{0\}$.

Definition 1.16. *For a line* $\boldsymbol{e} := \overline{\boldsymbol{x}_1\boldsymbol{x}_2} \subseteq \mathbb{R}^n$ *the polynomial space* $\mathbb{P}_m(\boldsymbol{e})$ *is given by the restriction of all polynomials* $\mathbb{P}_m(\mathbb{R}^n)$ *to* $\boldsymbol{e}$.

Theorem 1.17 (Hermite interpolation, [DeuHoh, Satz 7.6])**.** *Let* $m \in \mathbb{N}$, $\boldsymbol{e} := \overline{\boldsymbol{x}_1\boldsymbol{x}_2} \subseteq \mathbb{R}^n$ *and* $\boldsymbol{y}_1, \dots, \boldsymbol{y}_k \in \boldsymbol{e}$ *pairwise different. Let* $n_1, \dots, n_k \in \mathbb{N}$ *be such that*

$$\sum_{j=1}^{k} n_j = m + 1.$$

For any family of real numbers $\gamma_j^i \in \mathbb{R}$, $i = 1, \dots, n_j$, $j = 1, \dots, k$, *there exists exactly one polynomial* $p \in \mathbb{P}_m(\boldsymbol{e})$ *such that*

$$\frac{\partial^i p(\boldsymbol{y}_j)}{\partial \tau^i} = \gamma_j^i \quad \text{for all } i = 1, \dots, n_j \text{ and all } j = 1, \dots, k. \tag{1.1}$$

The tuples $(\boldsymbol{y}_j, \gamma_j^i)$, $i = 1, \dots, n_j, j = 1, \dots, k$ *are called datapoints of the interpolation. If a polynomial satisfies* (1.1), *it is said to interpolate the datapoints.*

□

Remark 1.18. *Note that in [DeuHoh, Satz 7.6] Theorem 1.17 is only proved for polynoms in one dimension. However it is easy to see that* $\Psi : \mathbb{P}_m(\boldsymbol{e}) \to \mathbb{P}_m([0,1])$ *with*

$$\Psi(p) := (t \mapsto p(\boldsymbol{x}_1 + t(\boldsymbol{x}_2 - \boldsymbol{x}_1)))$$

defines an isomorphism between $\mathbb{P}_m(\boldsymbol{e})$ *and* $\mathbb{P}_m([0,1])$ *and thus Theorem 1.17 is correct.*

1.1.5 Derivatives and integrals

Definition 1.19. *Let* $\Omega \subseteq \mathbb{R}^n$ *be open. By* $C^k(\Omega)$, *with* $k \in \mathbb{N}_0 \bigcup \{\infty\}$, *we denote the space of functions* $\Omega \to \mathbb{R}$ *that are* k *times continuously differentiable.*

Definition 1.20. *Let* $\Omega \subseteq \mathbb{R}^n$ *be open. By* $C_{00}^\infty(\Omega)$ *we denote the space of functions that are infinitely often differentiable and have compact support.*

Definition 1.21. *Let* $d \in \mathbb{N}$. *For any open set* $\Omega \subseteq \mathbb{R}^n$, *any function* $u : \Omega \longrightarrow \mathbb{R}^d$ *and any* $j \in \mathbb{N}_0$, *we set* $D^j u$ *to be the collection of all weak derivatives of order* j, *i.e.*

$$D^j u := \left(\frac{\partial^j u}{\partial x_1^{\alpha_1} \dots \partial x_1^{\alpha_n}} \right)_{\substack{|\alpha|=j \\ \alpha \in \mathbb{N}_0^n}}$$

Throughout this thesis we will treat $D^j u$ as a vector-valued function. Sometimes however we will also use it in the sense of Fréchet derivatives, meaning that the term $D^j u(x)(v_1, \ldots, v_j)$ is the derivative of u in the directions $v_1, \ldots, v_j \in \mathbb{R}^n$ evaluated at the point $x \in \Omega$.

As usual the (weak) gradient of u is denoted by $\nabla u := Du := D^1 u$.

The divergence of a function $u \in C^1(\Omega)$ is denoted by $\operatorname{div}(u) := \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}$. The Laplacian of a function $u \in C^2(\Omega)$ is defined by $\Delta u := \operatorname{div}(\nabla u) = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$.

Note that, for sufficiently regular functions, weak derivatives coincide with strong derivatives.

As usual the normal derivative is denoted by $\frac{\partial u}{\partial \boldsymbol{n}} = \nabla u \cdot \boldsymbol{n}$. The j-th normal derivative is defined as

$$\frac{\partial^j u}{\partial \boldsymbol{n}^j} := \begin{cases} \underbrace{\frac{\partial}{\partial \boldsymbol{n}} \cdots \frac{\partial}{\partial \boldsymbol{n}}}_{j \text{ times}} u & \text{for } j \geq 1, \\ u|_{\partial\Omega} & \text{for } j = 0. \end{cases}$$

Definition 1.22. *Let $\Omega \subset \mathbb{R}^n$ be a domain (or a set with non empty interior). We denote by*

$$\int_\Omega u \,\mathrm{d}x \tag{1.2}$$

the Lebesgue integral of an integrable function $u : \Omega \to \mathbb{R}$.

If $\Gamma \subseteq \mathbb{R}^n$ is the finite union of $(n-1)$-dimensional manifolds, we denote the surface integral of u on Γ by

$$\int_\Gamma u \,\mathrm{d}s.$$

Theorem 1.23 (Fubini)**.** *Let $u : \mathbb{R}^n \to \mathbb{R}$ be integrable and σ be any permutation of $\{1, \ldots, n\}$, then it holds that*

$$\int_{\mathbb{R}^n} u \,\mathrm{d}x = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} u \,\mathrm{d}x_{\sigma(1)} \ldots \mathrm{d}x_{\sigma(n)}.$$

1.1.6 Sobolev spaces

Definition 1.24. *For $n, d \in \mathbb{N}$ and a Borel measurable set $\Omega \subseteq \mathbb{R}^n$ the space of functions that are bounded almost everywhere is denoted by*

$$L^\infty(\Omega, \mathbb{R}^d) := \left\{ u : \Omega \to \mathbb{R}^d \;\middle|\; u \text{ is measurable and } \inf_{\substack{N \subset \Omega \\ |N|=0}} \sup_{x \in \Omega \setminus N} |u(x)| < +\infty \right\}$$

We will usually omit $\mathbb{R}^d$ and simply write $L^\infty(\Omega) := L^\infty(\Omega, \mathbb{R}^d)$.

Definition 1.25 (L^2-spaces)**.** *Let $d \in \mathbb{N}$, $p \in [1, +\infty)$ and $\langle \mathcal{S}, \mathfrak{S}, \mu \rangle$ be a measure space. The L^p-norm of a measurable function $u : \mathcal{S} \to \mathbb{R}^d$ is defined by*

$$\|u\|_{L^p(\mathcal{S}, \mathfrak{S}, \mu, \mathbb{R}^d)} := \left(\int_{\mathcal{S}} |u(x)|^p \, \mathrm{d}\mu(x) \right)^{1/p}.$$

The L^p-space is defined by

$$L^p(\mathcal{S}, \mathfrak{S}, \mu, \mathbb{R}^d) := \left\{ u : \mathcal{S} \longrightarrow \mathbb{R}^d \;\middle|\; u \text{ is measurable, } \|u\|_{L^p(\mathcal{S}, \mathfrak{S}, \mu, \mathbb{R}^d)} < \infty \right\}.$$

Furthermore, the L^2-scalar product of two functions $u, v \in L^2(\mathcal{S}, \mathfrak{S}, \mu, \mathbb{R}^d)$ shall be denoted by

$$\langle u, v \rangle_{L^2(\mathcal{S}, \mathfrak{S}, \mu, \mathbb{R}^d)} := \int_{\mathcal{S}} u(x) \cdot v(x) \; \mathrm{d}\mu(x).$$

For the sake of simplicity, we will never explicitly write the terms $\mathfrak{S}, \mu$, and $\mathbb{R}^d$. The specific L^p space or norm under consideration will always be clear from the context, i.e. we write

$$L^p(\mathcal{S}) := L^p(\mathcal{S}, \mathfrak{S}, \mu, \mathbb{R}^d). \tag{1.3}$$

Specifically if $\Omega \subseteq \mathbb{R}^n$ is an open set (or a set with a nonempty interior), we denote its Borel Algebra by $\mathfrak{B}(\Omega)$ and let

$$L^p(\Omega) := L^p(\Omega, \mathfrak{B}(\Omega), |\cdot|, \mathbb{R}^d).$$

If $\mathcal{S} \subseteq \mathbb{R}^n$ is the finite union of $(n-1)$-dimensional manifolds (e.g. if $\mathcal{S} = \partial\Omega$ for an open set Ω with regular boundary or if $\mathcal{S} = \overline{\boldsymbol{xy}}$ is a straight line in $\mathbb{R}^2$), we set μ to be the surface measure on the manifold and define $L^p(\mathcal{S})$ as in (1.3).

Theorem 1.26 (Fundamental theorem of Calculus of Variations, [Praet, Theorem 2.1])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and $v, w \in L^2(\Omega)$, then $v = w$ if and only if*

$$\langle v, \phi \rangle_{L^2(\Omega)} = \langle w, \phi \rangle_{L^2(\Omega)} \text{ for all } \phi \in C^\infty_{00}(\Omega).$$

□

Definition 1.27 (Sobolev spaces on open domains). *For any $k \in \mathbb{N}$, we define the Sobolev space of order $k \in \mathbb{N}$ as*

$$H^k(\Omega) := \left\{ u \in L^2(\Omega) \;\middle|\; D^j u \in L^2(\Omega) \text{ for } j = 1, \dots, k \right\}.$$

The Sobolev norm of a function $u \in H^k(\Omega)$ is defined by

$$\|u\|_{H^k(\Omega)} = \sqrt{\sum_{j=0}^{k} \|D^j u\|^2_{L^2(\Omega)}}.$$

Furthermore, we define the subspace $H_0^k(\Omega) \subseteq H^k(\Omega)$ by

$$H_0^k(\Omega) := \overline{C_{00}^\infty(\Omega)}^{\|.\|_{H^k(\Omega)}}.$$

For general $s \in (0, +\infty)$ we define the corresponding (fractional) Sobolev space by

$$H^s(\Omega) := \left\{ u \in H^{\lfloor s \rfloor}(\Omega) \;\middle|\; \int_\Omega \int_\Omega \frac{|D^{\lfloor s \rfloor} u(x) - D^{\lfloor s \rfloor} u(y)|}{|x - y|^{n + (s - \lfloor s \rfloor)/2}} \mathrm{d}x \mathrm{d}y < +\infty \right\}. \tag{1.4}$$

It holds that $H^s(\Omega) \subseteq H^r(\Omega)$ for $s \geq r$.

Theorem 1.28 (Characterization of $H^k(\Omega)$, [Praet, Theorem 2.11]). *Let $n, k \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^n$ open. It holds that $C^\infty(\Omega) \cap H^k(\Omega)$ is dense in $H^k(\Omega)$.*

□

Theorem 1.29 (Existence of trace, [Praet, Theorem 2.12]). *Let $n, k \in \mathbb{N}$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary. For every $0 \leq j \leq k - 1$, there exists a bounded linear operator*

$$\mathrm{tr}_j : H^k(\Omega) \longrightarrow L^2(\partial\Omega), \text{ such that } \mathrm{tr}_j(u) = \frac{\partial^j u}{\partial \boldsymbol{n}^j} \text{ for all } u \in C^k(\overline{\Omega}). \tag{1.5}$$

□

Definition 1.30. *For the sake of simplicity, we will simply denote $\mathrm{tr}_j(u)$, defined in (1.5), by $\frac{\partial^j u}{\partial \boldsymbol{n}^j}$ for every $u \in H^k(\Omega)$ and $j = 0, \dots, k$.*

The operator

$$\mathrm{tr}^k : H^k(\Omega) \longrightarrow L^2(\partial\Omega)^k : \; u \mapsto (\mathrm{tr}_0(u), \dots, \mathrm{tr}_{k-1}(u)).$$

is called trace operator. Its importance stems from the Theorem 1.31 below.

Theorem 1.31 (Characterization of $H_0^k(\Omega)$, [Praet, Theorem 2.16])**.** *Let $n, k \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain. It holds that*

$$H_0^k(\Omega) = \left\{ u \in H^k(\Omega) \;\middle|\; \operatorname{tr}^k(u) = 0 \right\}.$$

□

Definition 1.32. *For $k \in \mathbb{N}$, $s \in [k, +\infty)$, $0 \le j \le k-1$ and a Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, we define the fractional Sobolev space*

$$H^{s-j-1/2}(\partial\Omega) := \operatorname{tr}_j(H^s(\Omega)).$$

Theorem 1.33 (Sobolev embedding theorem, [Juengel, Satz 4.9])**.** *Let $n \in \mathbb{N}$, $s \in (0, +\infty)$, and $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain. For every $m \in \mathbb{N}$ such that*

$$s - \frac{n}{2} > m,$$

the space $H^s(\Omega)$ can be continuously embedded into $C^m(\overline{\Omega})$, thus

$$H^s(\Omega) \subseteq C^m(\overline{\Omega}).$$

□

Theorem 1.34 (Integration by parts, [Praet, Corollary 2.13])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain. For every $u, v \in H^1(\Omega)$ and every $j = 1, \dots, n$, it holds that*

$$\int_\Omega \frac{\partial u}{\partial x_j} v \,\mathrm{d}x + \int_\Omega u \frac{\partial v}{\partial x_j} \,\mathrm{d}x = \int_{\partial\Omega} u \; v \; \boldsymbol{n}_j \,\mathrm{d}s.$$

As a result, the following equality holds

$$\int_\Omega v \operatorname{div}(A\nabla u) \,\mathrm{d}x = \int_{\partial\Omega} v \; A\nabla u \cdot \boldsymbol{n} \,\mathrm{d}s - \int_\Omega \nabla v \cdot A\nabla u \,\mathrm{d}x \quad \text{for all } u \in H^2(\Omega), v \in H^1(\Omega).$$

Here $A \in C^1(\Omega, \mathbb{R}^d) \cap L^\infty(\Omega)$ is an arbitrary matrix-valued function.

□

1.1.7 Inequalities

Theorem 1.35 (Cauchy–Schwarz inequality, [Kuso, Satz 13.4])**.** *For $u, v \in L^2(\mathcal{S})$ it holds that*

$$\left|\langle u, v\rangle_{L^2(\mathcal{S})}\right| \le \int_{\mathcal{S}} |u(x) \cdot v(x)| \,\mathrm{d}\mu(x) \le \|u\|_{L^2(\mathcal{S})} \|v\|_{L^2(\mathcal{S})}. \tag{1.6}$$

An application of this inequality to sums yields, that

$$\left|\sum_{j=1}^N a_j b_j\right| \le \sum_{j=1}^N |a_j b_j| \le \sqrt{\sum_{j=1}^N a_j^2}\sqrt{\sum_{j=1}^N b_j^2} \quad \text{for all } a, b \in \mathbb{R}^N. \tag{1.7}$$

□

Theorem 1.36 (Friedrichs inequality, [Praet, Corollary 2.14])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. There exists a constant $C > 0$ (only depending on the shape of Ω and not on its diameter) such that*

$$\|v\|_{L^2(\Omega)} \leq Ch_\Omega \|\nabla v\|_{L^2(\Omega)}, \text{ for all } v \in H_0^1(\Omega).$$

We define $C_P := C_P(\Omega) := Ch_\Omega$ and call C_P the Poincare constant.

□

Theorem 1.37 (Trace inequality, [Praet, Theorem 3.16])**.** *Let $T := \operatorname{conv}\{\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2\} \subseteq \mathbb{R}^2$ be a triangle. For any edge $\boldsymbol{e} \in \operatorname{edges}(T)$ and any $v \in H^1(T)$, it holds that*

$$\|v\|_{L^2(\boldsymbol{e})}^2 \leq \frac{|\boldsymbol{e}|}{|T|} \left(\|v\|_{L^2(T)}^2 + \|v \nabla v\|_{L^1(T)} \right)$$

□

2 Theory of elliptic boundary value problems and related topics

2.1 Weak solutions of the generalized Poisson problem

In this section, we briefly discuss the mathematical theory behind the problem, for which we aim to approximate a solution via the Virtual Element Method. The results of this section can be found in most books about partial differential equations. The methods employed here are therefore in no way original and one can find similar proofs in the literature (e.g. in [Juengel]).

Let us consider a generalized Poisson problem with Dirichlet boundary conditions of the following form:

$$-\mathrm{div}(A\ \nabla u) = f \quad \text{in } \Omega, \tag{2.1a}$$

$$u = g \quad \text{on } \partial\Omega. \tag{2.1b}$$

Assumption 1. *Throughout the thesis $\Omega \subseteq \mathbb{R}^n$ shall be a bounded Lipschitz domain. Furthermore, we require that there exists $\kappa > 0$ such that $A \in L^\infty(\Omega)^{n\times n}$ is symmetric and uniformly κ-elliptic, as well as that $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$.*

This leads us to the following definition, which of course only makes sense, if we assume that A maps vectors of $H^1(\Omega)$ functions onto vectors of $H^1(\Omega)$ functions (e.g. this is the case if $A \in C^1(\overline{\Omega})$).

Definition 2.1. *We call $u \in H^2(\Omega)$ a classical solution of* (2.1)*, if* (2.1a) *and* (2.1b) *hold in the sense of equality of L^2 functions.*

If, however, we want to solve problems where A does not suffice to the regularity assumptions of classical solutions, we need to find a more general concept of solutions.

Definition 2.2. *We call $u \in H^1(\Omega)$ a weak solution of* (2.1)*, if $u|_{\partial\Omega} = g$ in $H^{1/2}(\partial\Omega)$ and it holds that*

$$\langle A\ \nabla u, \nabla v\rangle_{L^2(\Omega)} = \langle f, v\rangle_{L^2(\Omega)} \text{ for all } v \in H^1_0(\Omega). \tag{2.2}$$

Before we discuss the solvability of (2.2), we want to motivate the notion of a weak solution given in Definition 2.2 by showing that it generalizes the concept of a classical solution from Definition 2.1.

Theorem 2.3. *Let $A \in C^1(\overline{\Omega})^{n\times n}$. Then, the following statements hold:*

(i). Every classical solution of (2.1) is also a weak solution.

(ii). Every weak solution of (2.1) which belongs to $H^2(\Omega)$ is also a classical solution.

Proof.
ad (i):
Let $u \in H^2(\Omega)$ be a classical solution of (2.1) and let $v \in H_0^1(\Omega)$ be an arbitrary test function. As $A \in C^1(\overline{\Omega})^{n\times n}$ and the gradient $\nabla u \in H^1(\Omega)^n$, it holds that $A\nabla u \in H^1(\Omega)^n$.

Thus we can integrate by parts (see 1.34) and we get that

$$\langle f, v\rangle_{L^2(\Omega)} = \langle -\mathrm{div}(A\ \nabla u), v\rangle_{L^2(\Omega)} = \langle A\nabla u, \nabla v\rangle_{L^2(\Omega)} + \left\langle \frac{\partial u}{\partial \mathbf{n}}, v\right\rangle_{L^2(\partial\Omega)}.$$

Since $v \in H_0^1(\Omega)$ implies that $v|_{\partial\Omega} = 0$, we derive that $\langle f, v\rangle_{L^2(\Omega)} = \langle A\nabla u, \nabla v\rangle_{L^2(\Omega)}$.

As $v \in H_0^1(\Omega)$ was arbitrary, this shows (2.2).

Furthermore, $u \in H^2(\Omega)$ satisfies that $u|_{\partial\Omega} = g$. Thus, u is a weak solution.

ad (ii):
Let $u \in H^2(\Omega) \subseteq H^1(\Omega)$ be a weak solution. Then, for every arbitrary $v \in H_0^1(\Omega)$, integration by parts and $v|_{\partial\Omega} = 0$ prove that

$$\langle f, v\rangle_{L^2(\Omega)} = \langle A\nabla u, \nabla v\rangle_{L^2(\Omega)} = \langle -\mathrm{div}(A\ \nabla u), v\rangle_{L^2(\Omega)}.$$

Due to Theorem 1.26 we can thus infer that $f = \mathrm{div}(A\ \nabla u)$. As $u|_{\partial\Omega} = g$, it holds that u is a classical solution.

□

The following lemma will be useful for proving the well-posedness of (2.1) and for deriving estimates in the following chapters.

Lemma 2.4. *Let $P \subseteq \Omega$ be an arbitrary open subset ($P = \Omega$ is possible). Under Assumption 1, $\langle A\nabla\cdot, \nabla\cdot\rangle_{L^2(P)}$ is a positive semidefinite bilinear form on $H^1(P)$ with the induced* *seminorm* $\left\|\sqrt{A}\nabla(\cdot)\right\|_{L^2(P)}$.

Furthermore, it is continuous and coercive with respect to the seminorm $\|\nabla\cdot\|_{L^2(P)}$ *and there are constants* $C_1, C_2, C_3, C_4 > 0$, *that only depend on* A, *but not on* P, *such that*

$$C_1 \|\nabla v\|_{L^2(P)} \leq C_2 \|A\nabla v\|_{L^2(P)} \leq \left\|\sqrt{A}\nabla v\right\|_{L^2(P)} \leq C_3 \|A\nabla v\|_{L^2(P)} \leq C_4 \|\nabla v\|_{L^2(P)} \tag{2.3}$$

for all $v \in H^1(P)$.

Proof. The bilinearity of $\langle A\nabla., \nabla.\rangle_{L^2(P)}$ is obvious.

Using the uniform κ-ellipticity of A, we get

$$\langle A\nabla v, \nabla v\rangle_{L^2(P)} = \int_P \nabla v^T\ A\ \nabla v\ \mathrm{d}x \geq \int_P \kappa\ |\nabla v|^2\ \mathrm{d}x = \kappa \|\nabla v\|^2_{L^2(P)}$$

for all $v \in H^1(P)$. Therefore $\langle A\nabla\cdot, \nabla\cdot\rangle_{L^2(P)}$ is coercive with respect to $\|\nabla\cdot\|_{L^2(P)}$.

Since A is bounded, the Cauchy-Schwarz inequality (1.6) yields that

$$|\langle A\nabla u, \nabla v\rangle_{L^2(P)}| \leq \int_\Omega |\nabla u^T\ A\ \nabla v|\mathrm{d}x \leq \|A\|_{L^\infty(\Omega)} \|\nabla v\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)}$$

for all $u, v \in H^1(\Omega)$. Thus $\langle A\nabla\cdot, \nabla\cdot\rangle_{L^2(P)}$ is continuous with respect to $\|\nabla\cdot\|_{L^2(P)}$.

For any $v \in H^1(P)$, the image of the induced seminorm is given by

$$\left(\langle A\nabla v, \nabla v\rangle_{L^2(P)}\right)^{1/2} = \left(\langle\sqrt{A}\nabla v, \sqrt{A}\nabla v\rangle_{L^2(P)}\right)^{1/2} = \left\|\sqrt{A}\nabla v\right\|_{L^2(P)}.$$

Using the previously proven coercivity and continuity of $\langle A\nabla\cdot, \nabla\cdot\rangle_{L^2(P)}$, we derive

$$\sqrt{\kappa}\,\|\nabla v\|_{L^2(P)} \leq \left\|\sqrt{A}\nabla v\right\|_{L^2(P)} \leq \sqrt{\|A\|_{L^\infty(\Omega)}}\,\|\nabla v\|_{L^2(P)} \quad \text{for all } v \in H^1(P).$$

With A, the squared matrix A^2 also suffices the conditions of Assumption 1. Thus all the results proven above also hold if we replace A with A^2. Therefore there are constants $\tilde{C}_1, \tilde{C}_2$ that only depend on A, such that

$$\tilde{C}_1 \|\nabla v\|_{L^2(P)} \leq \sqrt{\langle A\nabla v, A\nabla v\rangle_{L^2(P)}} = \|A\nabla v\|_{L^2(P)} \leq \tilde{C}_2 \|\nabla v\|_{L^2(P)} \quad \text{for all } v \in H^1(P).$$

Thus we can conclude that the inequality (2.3) holds for all $v \in H^1(P)$. □

In the following theorem, we prove that (2.1) is well posed in the sense that there exists a unique weak solution.

Theorem 2.5. *Under Assumption 1, there exists a unique weak solution $u \in H^1(\Omega)$ of (2.1).*

Proof. Since $g \in H^{1/2}(\partial\Omega)$, there exists $u_g \in H^1(\Omega)$ such that

$$u_g|_{\partial\Omega} = g.$$

Due to the Cauchy–Schwarz inequality (1.6), it holds that

$$\begin{aligned} |\langle f, v\rangle_{L^2(\Omega)}| &\leq \|f\|_{L^2(\Omega)} \; \|v\|_{L^2(\Omega)} && \leq \|f\|_{L^2(\Omega)} \; \|v\|_{H^1(\Omega)} \quad , \text{ and} \\ |\langle A\nabla u_g, \nabla v\rangle_{L^2(P)}| &\leq \left\|\sqrt{A}\nabla u_g\right\|_{L^2(\Omega)} \; \|v\|_{L^2(\Omega)} && \leq \left\|\sqrt{A}\nabla u_g\right\|_{L^2(\Omega)} \; \|v\|_{H^1(\Omega)} \quad , \end{aligned}$$

for all $v \in H_0^1(\Omega)$. Thus both $\langle f, \cdot\rangle_{L^2(\Omega)}$ and $\langle A\nabla u_g, \nabla\cdot\rangle_{L^2(P)}$ are continuous linear forms on $H_0^1(\Omega)$.

Due to Lemma 2.4 and the equivalence of $\|\nabla\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$, $\langle A\nabla\cdot, \nabla\cdot\rangle_{L^2(P)}$ is a continuous and coercive bilinear form on $H_0^1(\Omega)$ (it is even a scalar product).
Thus, by using the Lax–Milgram theorem (Theorem 1.10), we derive existence and uniqueness of $u_0 \in H_0^1(\Omega)$ such that

$$\langle A\nabla u_0, \nabla v\rangle_{L^2(P)} = \langle f, v\rangle_{L^2(\Omega)} - \langle A\nabla u_g, \nabla v\rangle_{L^2(P)} \quad \text{for all } v \in H_0^1(\Omega). \tag{2.4}$$

Therefore $u := u_0 + u_g$ is a weak solution of (2.1). If there was another weak solution $\tilde{u} \neq u$, $\tilde{u} - u_g \neq u - u_g = u_0$ would also satisfy (2.4), which is a contradiction to the uniqueness of u_0 guaranteed by the Lax–Milgram theorem.

□

Remark 2.6. *Theorem 2.5 also holds for more general elliptic equations of the form*

$$-\operatorname{div}(A\ \nabla u) + b \cdot \nabla u + c\ u = f, \ \textit{in } \Omega, \tag{2.5}$$

where $b \in L^\infty(\Omega)^n$ and $c \in L^\infty(\Omega)$ with $\inf_{x\in\Omega} c(x) := c_0 \geq 0$ satisfy $c_0 = 0$, if $b = 0$, or $4\ \kappa\ c_0 > \|b\|^2_{L^\infty}$, if $b \neq 0$. See [Juengel, Satz 4.20].

2.2 A general numerical framework

In general, weak solutions will not be attainable analytically. Therefore, in practice one must resort to numerical approximations. In this section, we briefly discuss a very general framework for finding appropriate approximations.
As we have seen in the proof of Theorem 2.5, the inhomogenous problem can be reduced to the homogenous problem, if one can find an $H^1(\Omega)$-extension of the boundary data. In general finding this extension is a non-trivial task. Nevertheless we will from now on restrict ourselves to the homogenous problem.

Recalling the previous section, our goal is to find a function $u \in H^1_0(\Omega)$ such that

$$\mathcal{A}(u,v) := (A\ \nabla u, \nabla v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)} \quad \text{for all } v \in H^1_0(\Omega).$$

A way to reduce the complexity of finding u is to look for a solution u_h on a smaller space $V_h \subseteq H^1_0(\Omega)$. Furthermore, we could approximate the scalar product $\mathcal{A}(\cdot,\cdot)$ by an appropriate scalar product $\mathcal{A}_h(\cdot,\cdot)$ on the space V_h and f by f_h.
Here, the parameter $h > 0$ modulates the accuracy of the scheme, in the sense that, roughly speaking, the quality of the approximation improves as h decreases.
Thus, we now look for a $u_h \in V_h$ such that

$$\mathcal{A}_h(u_h, v_h) = (f_h, v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in V_h.$$

If the space V_h is finite-dimensional with a basis $\{\xi_1, \ldots, \xi_N\}$, this problem is equivalent to the solution of a linear system

$$\mathfrak{A}x = b$$

with

$$\begin{aligned} \mathfrak{A} \in \mathbb{R}^{N\times N}, \mathfrak{A}_{ij} &:= a_h(\xi_j, \xi_i) \\ b \in \mathbb{R}^N, b_i &:= (f_h, \xi_i)_{L^2(\Omega)}. \end{aligned}$$

Then, the approximate solution $u_h \in V_h$ is given by $u_h = \sum_{i=1}^N\ x_i\ \xi_i$.

Our remaining task is to find appropriate choices of $\mathcal{A}_h, f_h$, and V_h, such that, when the parameter h tends to zero $h \to 0$, the convergence $u_h \to u$ is guaranteed. We will construct such a discretization in the next chapter.

Remark 2.7. *If there would be no approximation of the scalar product and the righthand side, i.e., $\mathcal{A}_h = \mathcal{A}$ and $f_h = f$, then the method described above would be a classical Galerkin method. We therefore call it a Galerkin-type scheme.*

Remark 2.8. *The classical lowest-order Finite Element Method (FEM) is a Galerkin method (thus $\mathcal{A}_h = \mathcal{A}, f_h = f$) with $V_h = S^1_0(\mathcal{T})$, where*

- *$\mathcal{T}$ is a regular triangulation of the domain Ω, i.e., $\mathcal{T}$ is a decomposition of the domain into triangles such that each intersection between different triangles is either empty, or a point, or a joint edge.*

- *$S^1_0(\mathcal{T})$ is the set of continuous functions that are affine on each triangle and zero on the boundary.*

The matrix B of the FEM can be calculated rather easily by using hat functions as a basis for $S_0^1(\mathcal{T})$. Then, B becomes a sparse matrix.
FEM of higher regularity, i.e., methods that use subspaces of differentiable function, or on more general meshes however require a more refined approach. Constructing them is a non-trivial task.
The method we aim to develop will be devoid of such difficulties, by avoiding an explicit representation of the discrete basis functions.

2.3 The polyharmonic equation

We want to conclude this chapter by looking at an auxiliary problem that will be useful in the next chapter. It is a generalization of the Poisson equation and was therefore warranted a section in this chapter.

Definition 2.9. *Let $\beta \in \mathbb{N}$. We define the evaluation of the polyharmonic operator (of order 2β) Δ^β at a function u, to be the β-times application of the Laplace operator to u, i.e.,*

$$\Delta^\beta u := \underbrace{\Delta \dots \Delta}_{\beta-times} u = \sum_{i_1=1}^{n} \dots \sum_{i_\beta=1}^{n} \frac{\partial^{2\beta} u}{\partial x_{i_1}^2 \dots \partial x_{i_\beta}^2}. \tag{2.6}$$

The polyharmonic equation is now given by replacing the Laplace operator in the Poisson equation by the polyharmonic operator

$$(-1)^\beta \, \Delta^\beta u = f \text{ in } \Omega. \tag{2.7}$$

Of course one needs to prescribe boundary conditions in order to make the uniqueness of solutions plausible. Throughout this section we will only look at Dirichlet boundary conditions

$$\frac{\partial^j u}{\partial \mathbf{n}^j} = q_j \text{ on } \partial\Omega, \text{ for } j = 0, \dots, \beta - 1.$$

Here, $q_j \in H^{\beta-j-1/2}(\partial\Omega)$. Note that there are also other sensible boundary conditions like Navier conditions (see [GazGruSw, equation (2.21), page 33]). We will restrict ourselves to Dirichlet conditions as they are used in the definition of Virtual Element spaces.

Now we want to find a weak formulation of our polyharmonic equation. Let $u \in H^{2\beta}(\Omega)$ be a classical solution of (2.7) and $v \in H_0^\beta(\Omega)$. Then, by integrating by parts multiple times, we get that

$$\int_\Omega (-1)^\beta f \; v \; \mathrm{d}x \;=\; \int_\Omega \Delta^\beta u \; v \; \mathrm{d}x \;=\; -\int_\Omega \nabla(\Delta^{\beta-1}u)\cdot\nabla v \; \mathrm{d}x \;+\; \underbrace{\int_{\partial\Omega} \nabla(\Delta^{\beta-1}u)\cdot\mathbf{n}\; v \; \mathrm{d}s}_{=\,\mathbf{0},\text{ as } v=0 \text{ on } \partial\Omega}$$

$$= -\int_\Omega \nabla(\Delta^{\beta-1}u)\cdot\nabla v \; \mathrm{d}x \;=\; (-1)^2\int_\Omega \Delta^{\beta-1}u\;\Delta v\;\mathrm{d}x \;-\; \underbrace{\int_{\partial\Omega} \Delta^{\beta-1}u\;\frac{\partial v}{\partial\mathbf{n}}\;\mathrm{d}s}_{=\,\mathbf{0},\text{ as } \frac{\partial v}{\partial\mathbf{n}}=0 \text{ on } \partial\Omega}$$

$$= (-1)^2\int_\Omega \Delta^{\beta-1}u\;\Delta v\;\mathrm{d}x \;=\; \dots \;=\; \begin{cases} (-1)^\beta \int_\Omega \; \Delta^{\beta/2}u\cdot\Delta^{\beta/2}v\;\mathrm{d}x & \text{for } \beta\in 2\mathbb{N}, \\ (-1)^\beta \int_\Omega \; \nabla(\Delta^{(\beta-1)/2}u)\cdot\nabla(\Delta^{(\beta-1)/2}v)\;\mathrm{d}x & \text{else.}\end{cases}$$

If we define the Laplacian to the power $k+\frac{1}{2}$ for $k\in\mathbb{N}_0$ by

$$\Delta^{k+1/2}u := \nabla\Delta^k u$$

we can state the following definition.

Definition 2.10. *$u\in H^\beta(\Omega)$ is a weak solution of the polyharmonic equation with Dirichlet boundary data*

$$(-1)^\beta\;\Delta^\beta u = f \quad \text{in } \Omega, \tag{2.8a}$$

$$\frac{\partial^j u}{\partial \boldsymbol{n}^j} = q_j \;\; \text{on } \partial\Omega, \text{ for } j=0,\dots,\beta-1, \tag{2.8b}$$

if (2.8b) is satisfied in $H^{\beta-j-1/2}(\partial\Omega)$ and

$$\int_\Omega f\;v\;\mathrm{d}x \;=\; \int_\Omega \Delta^{\beta/2}u\cdot\Delta^{\beta/2}v\;\mathrm{d}x \quad \text{for all } v\in H_0^\beta(\Omega). \tag{2.9}$$

Note that this is well defined as $\Delta^{\beta/2}$ is a differential operator of β-th order. Before we prove the well-posedness of the weak problem, we will prove some auxiliary results.

Lemma 2.11. *Let $\Omega\subseteq\mathbb{R}^n$ be a domain with Lipschitz boundary, $k\in\mathbb{N}$, and $u\in H_0^{2k}(\Omega)$. It holds that*

$$\left\|D^{2k}u\right\|_{L^2(\Omega)} = \left\|\Delta^k u\right\|_{L^2(\Omega)}.$$

Proof. Since $C_{00}^\infty(\Omega)$ is dense in $H_0^{2k}(\Omega)$, it is sufficient to prove this equality for all $u\in C_{00}^\infty(\Omega)$.

Due to the compact support of u, identifying u with its trivial extension in $\mathbb{R}^n$, it holds that $\left\|\Delta^k u\right\|^2_{L^2(\Omega)} = \left\|\Delta^k u\right\|^2_{L^2(\mathbb{R}^n)}$ and therefore

$$\left\|\Delta^k u\right\|^2_{L^2(\Omega)} = \int_{\mathbb{R}^n} \left(\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \frac{\partial^{2k} u}{\partial x_{i_1}^2 \dots \partial x_{i_k}^2} \right) \left(\sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \frac{\partial^{2k} u}{\partial x_{j_1}^2 \dots \partial x_{j_k}^2} \right) \mathrm{d}x.$$

This can be written as

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \int_{\mathbb{R}^n} \frac{\partial^{2k} u}{\partial x_{i_1}^2 \dots \partial x_{i_k}^2} \frac{\partial^{2k} u}{\partial x_{j_1}^2 \dots \partial x_{j_k}^2} \mathrm{d}x.$$

Due to the Fubini theorem (Theorem 1.23), we can write this integral over $\mathbb{R}^n$ as a series of integrals over $\mathbb{R}$ and choose the order of integration. Thus we get

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\partial^{2k} u}{\partial x_{i_1}^2 \dots \partial x_{i_k}^2} \frac{\partial^{2k} u}{\partial x_{j_1}^2 \dots \partial x_{j_k}^2} \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

By using one-dimensional integration by parts with respect to the variable x_{i_1} as well as the compact support of u and its derivatives, we derive

$$-\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\partial^{2k-1} u}{\partial x_{i_1} \partial x_{i_2}^2 \dots \partial x_{i_k}^2} \frac{\partial^{2k+1} u}{\partial x_{i_1} \partial x_{j_1}^2 \dots \partial x_{j_k}^2} \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Another integration by parts, this time with respect to the variable x_{j_1}, gives us

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\partial^{2k} u}{\partial x_{i_1} \partial x_{j_1} \partial x_{i_2}^2 \dots \partial x_{i_k}^2} \frac{\partial^{2k} u}{\partial x_{i_1} \partial x_{j_1} \partial x_{j_2}^2 \dots \partial x_{j_k}^2} \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

By repeating this procedure another $k-1$ times, we get

$$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left(\frac{\partial^{2k} u}{\partial x_{i_1} \partial x_{j_1} \dots \partial x_{i_k} \partial x_{j_k}} \right)^2 \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

This is just $\left\|D^{2k} u\right\|^2_{L^2(\Omega)}$. Thus we conclude that

$$\left\|\Delta^k u\right\|^2_{L^2(\Omega)} = \left\|D^{2k} u\right\|^2_{L^2(\Omega)}.$$

□

Theorem 2.12 (Generalized Friedrichs inequality). *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain, $\beta \in \mathbb{N}$, and $u \in H_0^\beta(\Omega)$. Furthermore, let C_P be the Poincaré constant of this domain. Then, the following statements (i)–(iii) are satisfied:*

(i) It holds that.

$$\left\|D^\beta u\right\|_{L^2(\Omega)} \geq C_P^{-1}\left\|D^{\beta-1} u\right\|_{L^2(\Omega)} \geq \ldots \geq C_P^{-\beta}\|u\|_{L^2(\Omega)}. \tag{2.10}$$

(ii) If $\beta = 2k$ is even, then (2.10) holds for $\|\Delta^j u\|_{L^2(\Omega)}$ instead of $\|D^{2j} u\|_{L^2(\Omega)}$, for $j = 1, \ldots, k$, i.e., it holds that

$$\left\|\Delta^k u\right\|_{L^2(\Omega)} \geq C_P^{-1}\left\|D^{2k-1} u\right\|_{L^2(\Omega)} \geq C_P^{-2}\left\|\Delta^{k-1} u\right\|_{L^2(\Omega)} \geq \ldots \geq C_P^{-\beta}\|u\|_{L^2(\Omega)}.$$

(iii) It holds that

$$\left\|D^\beta u\right\|_{L^2(\Omega)} \leq \|u\|_{H^\beta(\Omega)} \leq \eta\left\|D^\beta u\right\|_{L^2(\Omega)} \tag{2.11}$$

with

$$\eta := \sqrt{\frac{C_p^{2(\beta+1)} - 1}{C_p^2 - 1}}. \tag{2.12}$$

Proof.
ad (i): We prove the first point by induction with regard to $\beta \in \mathbb{N}$.

Base case $\beta = 1$: In this case equality (2.10) is the standard Friedrichs inequality; see Theorem 1.36.

Inductive step $\beta \to \beta + 1$: Assume that (2.10) holds for all $v \in H_0^j(\Omega)$, $j = 1, \ldots, \beta$. For any $u \in H_0^{\beta+1}(\Omega)$ all derivatives

$$\frac{\partial^\beta u}{\partial x_{i_1} \ldots \partial x_{i_\beta}} \in H_0^1(\Omega)$$

suffice to the Friedrichs inequality (Theorem 1.36)

$$\left\|\frac{\partial^\beta u}{\partial x_{i_1} \ldots \partial x_{i_\beta}}\right\|_{L^2(\Omega)} \leq C_P\left\|\nabla\left(\frac{\partial^\beta u}{\partial x_{i_1} \ldots \partial x_{i_\beta}}\right)\right\|_{L^2(\Omega)}.$$

Thus, we derive

$$\begin{aligned}\left\|D^{\beta+1}u\right\|^2_{L^2(\Omega)} &= \sum_{i_1=1}^{n}\cdots\sum_{i_{\beta+1}=1}^{n}\int_\Omega \left(\frac{\partial^{\beta+1}u}{\partial x_{i_1}\dots\partial x_{i_{\beta+1}}}\right)^2 \,\mathrm{d}x \\ &= \sum_{i_1=1}^{n}\cdots\sum_{i_\beta=1}^{n}\underbrace{\sum_{i_{\beta+1}=1}^{n}\int_\Omega \left(\frac{\partial}{\partial x_{i_{\beta+1}}}\frac{\partial^\beta u}{\partial x_{i_1}\dots\partial x_{i_\beta}}\right)^2 \,\mathrm{d}x}_{=\left\|\nabla\left(\frac{\partial^\beta u}{\partial x_{i_1}\dots\partial x_{i_\beta}}\right)\right\|^2_{L^2(\Omega)} \geq C_P^{-2}\left\|\frac{\partial^\beta u}{\partial x_{i_1}\dots\partial x_{i_\beta}}\right\|^2_{L^2(\Omega)}} \\ &\geq C_P^{-2}\sum_{i_1=1}^{n}\cdots\sum_{i_\beta=1}^{n}\left\|\frac{\partial^\beta u}{\partial x_{i_1}\dots\partial x_{i_\beta}}\right\|^2_{L^2(\Omega)} = C_P^{-2}\left\|D^\beta u\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Finally we get $\left\|D^{\beta+1}u\right\|_{L^2(\Omega)} \geq C_P^{-1}\left\|D^\beta u\right\|_{L^2(\Omega)}$ and due to our induction hypothesis, (2.10) holds.

ad (ii): This follows directly from using Lemma 2.11 on (2.10).

ad (iii): The inequality (2.11) holds, as

$$\begin{aligned}\left\|D^\beta u\right\|_{L^2(\Omega)} \leq \|u\|_{H^\beta(\Omega)} &= \sqrt{\sum_{j=0}^{\beta}\|D^j u\|^2_{L^2(\Omega)}} \\ &\leq \sqrt{\sum_{j=0}^{\beta} C_P^{2j}}\left\|D^\beta u\right\|_{L^2(\Omega)} = \sqrt{\frac{C_p^{2(\beta+1)}-1}{C_p^2-1}}\left\|D^\beta u\right\|_{L^2(\Omega)}.\end{aligned}$$

□

Now we have the tools to prove the existence of weak solutions.

Theorem 2.13. *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain, $f \in L^2(\Omega)$ and $q_j \in H^{\beta-j-1/2}(\partial\Omega)$ for $j = 0, \dots, \beta-1$. Then there is a unique weak solution $u \in H^\beta(\Omega)$ of (2.8).*

Proof. Just as in the proof of Theorem 2.5 we will show the existence and uniqueness of the weak solution by using the Lax-Milgram theorem (Theorem 1.10).

To do this, we first define the bilinear form $a : H_0^\beta(\Omega) \times H_0^\beta(\Omega) \to \mathbb{R}$ by

$$a(u, v) = \langle \Delta^{\beta/2}u, \Delta^{\beta/2}v\rangle_{L^2(\Omega)}.$$

Due to the Cauchy–Schwarz inequality $a(\cdot,\cdot)$ is continuous, as

$$|a(u,v)| = |\langle \Delta^{\beta/2}u, \Delta^{\beta/2}v\rangle_{L^2(\Omega)}| \leq \left\|\Delta^{\beta/2}u\right\|_{L^2(\Omega)}\left\|\Delta^{\beta/2}v\right\|_{L^2(\Omega)} \leq \|u\|_{H^\beta(\Omega)}\|v\|_{H^\beta(\Omega)}.$$

The coercivity of $a(\cdot,\cdot)$ is a result of the generalized Friedrichs inequality. Due to (2.10) and the second point (ii) we derive

$$|a(u,u)| = \left\|\Delta^{\beta/2}u\right\|_{L^2(\Omega)}^2 \geq C_P^{-2}\ \left\|\Delta^{(\beta-1)/2}u\right\|_{L^2(\Omega)}^2 \geq \dots \geq C_P^{-\beta}\ \|u\|_{L^2(\Omega)}^2$$

and thus

$$|a(u,u)| \geq \eta^{-2}\ \|u\|_{H^\beta(\Omega)}^2 ,$$

with η defined by (2.12).

By definition there exists a $u_q \in H^\beta(\Omega)$ such that $\frac{\partial^j u_q}{\partial \mathbf{n}^j} = q_j$ for all $j = 0,\dots,\beta-1$.

As $a(\cdot,\cdot)$ is continuous and coercive, the Lax-Milgram theorem (Theorem 1.10) can be applied to guarantee the existence of a unique u, such that

$$a(u,v) = \langle f, v\rangle_{L^2(\Omega)} - a(u_q, v) \quad \text{for all } v \in H_0^\beta(\Omega),$$

since $v \mapsto \langle f, v\rangle_{L^2(\Omega)}$ is a continuous linear form on $H_0^\beta(\Omega)$.

Just as in Theorem 2.5 one proves that this u must be the unique weak solution of (2.8). □

We cite the following theorem, that shall prove to be useful later on.

Theorem 2.14. *Assume $\Omega \subset \mathbb{R}^2$ is a polygon. Let $\tilde\beta \geq \beta$, $f \in C^\infty(\Omega)$ and $q_j \in H^{\tilde\beta-j-1/2}(\partial\Omega)$, for $j = 0,\dots,\beta-1$. Then the weak solution u is an element of $H^{\tilde\beta}(\Omega)$.*

Proof. In [GazGruSw, Theorem 2.20] this theorem is stated under the assumption of sufficiently smooth boundary and for integers $k \geq 2\tilde\beta$.
In [Grisvard, Theorem 7.3.2.1, page 335] this was proved for polygons and the biharmonic equation ($\beta = 2$), the proof can be generalized. □

Of course in general one will be given certain boundary conditions q_j, $j = 0,\dots,\beta-1$ without knowing whether they are sufficiently regular to admit the application of Theorem 2.13 for proofing the well posedness of the problem.
Thus one should develop a more useful characterization of $\gamma^\beta(H^\beta(\Omega)) = \Pi_{j=0}^{\beta-1} H^{\beta-j-1/2}(\partial\Omega)$.

We will therefore restrict ourselves to polygons and cite parts of the theory about polygonal trace lifting developed in [BerDaMa].

First we need some definitions.

Definition 2.15. *Let $P \subseteq \mathbb{R}^2$. For every vertex $\boldsymbol{x}_i \in \text{vertices}(P)$, we define*

$$\boldsymbol{e}^+(\boldsymbol{x}_i) := \overline{\boldsymbol{x}_i\boldsymbol{x}_{i+1}},\ \ \boldsymbol{e}^-(\boldsymbol{x}_i) := \overline{\boldsymbol{x}_{i-1}\boldsymbol{x}_i}$$

Furthermore we define for every edge $\boldsymbol{e} \in \mathrm{edges}(P)$ and all $n, k \in \mathbb{N}$ the space of differential operators

$$\mathcal{E}_{n,k}(\boldsymbol{e}) := \left\{ \sum_{j=0}^{\min\{n,k-1\}} c_j \frac{\partial^j}{\partial \boldsymbol{n}_{\boldsymbol{e}}^j} \frac{\partial^{n-j}}{\partial \tau_{\boldsymbol{e}}^{n-j}} \;\middle|\; c_j \in \mathbb{R},\ j = 0, \dots, \min\{n, k-1\} \right\},$$

and the projection

$$\left[\pi_{\boldsymbol{e}} \sum_{j=0}^{\min\{n,k-1\}} c_j \frac{\partial^j}{\partial \boldsymbol{n}_{\boldsymbol{e}}^j} \frac{\partial^{n-j}}{\partial \tau_{\boldsymbol{e}}^{n-j}} \right] = \left(c_j \frac{\partial^{n-j}}{\partial \tau_{\boldsymbol{e}}^{n-j}} \right)_{j=0,\dots,\min\{n,k-1\}}$$

This means that for a tuple of functions $(g_0, \dots, g_{k-1})$ the image under the projection is

$$\left[\pi_{\boldsymbol{e}} \sum_{j=0}^{\min\{n,k-1\}} c_j \frac{\partial^j}{\partial \boldsymbol{n}_{\boldsymbol{e}}^j} \frac{\partial^{n-j}}{\partial \tau_{\boldsymbol{e}}^{n-j}} \right] (g_0, \dots, g_{k-1}) = \left(c_j \frac{\partial^{n-j} g_j}{\partial \tau_{\boldsymbol{e}}^{n-j}} \right)_{j=0,\dots,\min\{n,k-1\}}$$

Remark 2.16. *In the following $H^{k+1/2}(\boldsymbol{e})$ can be viewed as a one dimensional fractional Sobolev space $H^{k+1/2}((0, |\boldsymbol{e}|))$ defined in (1.4). Note that for every $l \in \mathbb{N}$ the space of polynoms $\mathbb{P}_l(\boldsymbol{e}) \subseteq H^{k+1/2}(\boldsymbol{e})$.*

Definition 2.17. *Let $P \subseteq \mathbb{R}^2$. We define for every $\beta \in \mathbb{R}$, $m \in \mathbb{N}$ with $\beta \geq \frac{1}{2}$ and $0 \leq k - 1 \leq \beta - \frac{1}{2}$, the space*

$$\mathbb{W}^{\beta,k}(\partial P) := \left\{ (g_0, \dots, g_{k-1}) \;\middle|\; g_j|_{\boldsymbol{e}} \in H^{\beta-j-1/2}(\boldsymbol{e}),\ \textit{for } \boldsymbol{e} \in \mathrm{edges}(P),\ j = 0, \dots, k-1 \right\}.$$

We conclude this chapter with the following theorem, which will be useful in the next chapter.

Theorem 2.18. *Let $P \subseteq \mathbb{R}^2$ be a polygon, $\beta \in \mathbb{R}$, $k \in \mathbb{N}$, with $0 \leq k - 1 \leq \beta - \frac{1}{2}$. Furthermore, let $G := (g_0, \dots, g_{k-1}) \in \mathbb{W}^{\beta,k}(\partial P)$. There exists a $u \in H^\beta(P)$ with*

$$\frac{\partial^j u}{\partial \boldsymbol{n}^j} = g_j, \ \textit{for } j = 0, \dots, k-1, \tag{2.13}$$

if and only if the following two conditions hold for every vertex $\boldsymbol{x} \in \mathrm{vertices}(P)$

- *For all $n \in \mathbb{N}$, with $0 \leq n < \beta - 1$ and every $\mathcal{L}^+ \in \mathcal{E}_{n,k}(\boldsymbol{e}^+(\boldsymbol{x}))$ and $\mathcal{L}^- \in \mathcal{E}_{n,k}(\boldsymbol{e}^-(\boldsymbol{x}))$ with*

  $$\mathcal{L}^+ + \mathcal{L}^- = 0$$

  it holds that

  $$[\pi_{\boldsymbol{e}^-(\boldsymbol{x})} \mathcal{L}^-](G)(\boldsymbol{x}) + [\pi_{\boldsymbol{e}^+(\boldsymbol{x})} \mathcal{L}^+](G)(\boldsymbol{x}) = 0 \tag{2.14}$$

- *If $\beta - 1 \in \mathbb{N}$, then for $n = \beta - 1$ and all $\mathcal{L}^+ \in \mathcal{E}_{n,k}(\boldsymbol{e}^+(\boldsymbol{x}))$ and $\mathcal{L}^- \in \mathcal{E}_{n,k}(\boldsymbol{e}^-(\boldsymbol{x}))$ with*

$$\mathcal{L}^+ + \mathcal{L}^- = 0$$

it holds that for sufficiently small $h_{\boldsymbol{x}} > 0$

$$\int_0^{h_x} \frac{\left|[\pi_{\boldsymbol{e}^-(\boldsymbol{x})}\mathcal{L}^-](G)(\boldsymbol{x} + s\tau_{\boldsymbol{e}^-(\boldsymbol{x})}) + [\pi_{\boldsymbol{e}^+(\boldsymbol{x})}\mathcal{L}^+](G)(\boldsymbol{x} + s\tau_{\boldsymbol{e}^+(\boldsymbol{x})})\right|^2}{s} \,\mathrm{d}s < +\infty \tag{2.15}$$

Proof. We refer to [BerDaMa, Theorem 5.7] .

□

Corollary 2.19. *Let $P \subseteq \mathbb{R}^2$ be a polygon $\beta, k \in \mathbb{N}$, with $0 \leq k - 1 \leq \beta - \frac{1}{2}$.*

- *For a tuple $(g_0, \ldots, g_{k-1})$ of piecewise polynomials, i.e. $g_l|_{\boldsymbol{e}} \in \mathbb{P}_m(\boldsymbol{e})$ for $l = 0, \ldots, k-1$ and $\boldsymbol{e} \in \mathrm{edges}(P)$, $(g_0, \ldots, g_{k-1}) \in \mathbb{W}^{\beta,k}(\partial P)$ is guaranteed. Furthermore the conditions (2.14) and (2.15) are equivalent for all $n \in \mathbb{N}$.*

- *For every $\boldsymbol{x} \in \mathrm{vertices}(P)$ the number of conditions that determine, whether a tuple $(g_0, \ldots, g_{k-1}) \in \mathbb{W}^{\beta,k}(\partial P)$ suffices (2.13), is given by*

$$\sum_{n=0}^{\beta-1} \dim \left[\mathcal{E}_{n,k}(\boldsymbol{e}^-(\boldsymbol{x})) \bigcap \mathcal{E}_{n,k}(\boldsymbol{e}^+(\boldsymbol{x}))\right].$$

The dimension of the intersection is equal to

$$\dim \left[\mathcal{E}_{n,k}(\boldsymbol{e}^-(\boldsymbol{x})) \bigcap \mathcal{E}_{n,k}(\boldsymbol{e}^+(\boldsymbol{x}))\right] = \begin{cases} n+1 & , \text{for } n \leq k-1 \\ 2k-1-n & , \text{for } k-1 < n \leq 2(k-1) \\ 0 & , \text{for } n > 2(k-1) \end{cases}$$

Proof. The first point can be checked immediately. For the proof of the second point we refer to [BerDaMa, Lemma 5.4].

□

3 Development of the Virtual Element Method

In this section, we introduce the VEM by specifying the discrete space V_h and a (possible choice for the) discrete bilinear form $\mathcal{A}_h$ (see Section 2.2). Furthermore, we define the loading term, construct the stiffness matrix and finally look at discretizations of locally reduced regularity.

From now on, we restrict ourselves to the two dimensional Poisson problem with homogenous Dirichlet data, i.e., $\Omega \subseteq \mathbb{R}^2$ and $g = 0$ in (2.1).

Like the FEM, the VEM is a Galerkin-type scheme that works by splitting up the domain in various subdomains and then using functions with a particular form on those subdomains to approximate the exact solution.

Unlike in classical FEM-schemes, as we will see later, VEM uses more abstract functions, whose concrete form is not explicitly needed. This allows for more general forms of subdomains to be used and it will provide an easier path (compared to the FEM) to finding approximations of a certain regularity.

3.1 The mesh

Assumption 2. *To formally define VEM, we assume that there is a family of decompositions of Ω into smaller polygons $(\Omega_h)_{h>0}$ such that the index h is the maximal diameter of each decomposition, i.e., $h = \max_{P\in\Omega_h} h_P$ and, for every $h > 0$, it holds that for $P, \tilde{P} \in \Omega_h,\ P \neq \tilde{P}$ the intersection $\tilde{P} \cap P$ is either empty or a collection of joint edges and vertices, and*

$$\overline{\Omega} = \bigcup_{P\in\Omega_h} P.$$

Each decomposition Ω_h is called a mesh.

Remark 3.1. *Note that we allow the mesh Ω_h to have (quasi) hanging nodes, i.e., there may be a polygon $P \in \Omega_h$ such that for $d := \#\mathrm{vertices}(P)$ and $j \in \{0, \dots, d-2\}$ it holds that $\boldsymbol{x}_{j+1} \in \overline{\boldsymbol{x}_j, \boldsymbol{x}_{j+2}}$. Hereby, $\mathrm{vertices}(P) = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{d-1}\}$ denotes an enumeration of the vertices of P (see Definition 1.6).*

Definition 3.2. *Let*

$$\mathrm{edges}(\Omega_h) := \bigcup_{P\in\Omega_h} \mathrm{edges}(P).$$

We call an edge $\boldsymbol{e} \in \mathrm{edges}(\Omega_h)$ *an inner edge, if* $\boldsymbol{e} \cap \Omega \neq \emptyset$*. For any inner edge* $\boldsymbol{e}$*, there exist exactly two polygons* $P, \tilde{P}$ *such that* $\boldsymbol{e} \in \mathrm{edges}(P) \cap \mathrm{edges}(\tilde{P})$*. We set*

$$P_1(\boldsymbol{e}) := P, \;\; P_2(\boldsymbol{e}) := \tilde{P}.$$

Definition 3.3. *Let* $P \in \Omega_h$*, then we define its neighborhood by*

$$\mathrm{neighbors}(P) := \left\{ \tilde{P} \in \Omega_h \;\middle|\; \mathrm{vertices}(P) \cap \mathrm{vertices}(\tilde{P}) \neq \emptyset \right\}.$$

Definition 3.4. *A domain* $M \subseteq \mathbb{R}^n, n \in \mathbb{N}$ *is star-shaped with respect to a set* $B \subseteq \mathbb{R}^n$*, if*

$$\mathrm{conv}\left[\{\boldsymbol{x}\} \cup B\right] \subseteq M \text{ for all } \boldsymbol{x} \in M.$$

Remark 3.5. *Note that* M *being star-shaped with respect to* B *implies that* $B \subseteq M$*. Furthermore, every convex set is star-shaped with respect to any of its subsets.*

We make the following regularity assumption regarding the decomposition Ω_h.

Assumption 3. *There exists a constant* $1 \geq \gamma > 0$ *such that, for all* $h > 0$ *and all* $P \in \Omega_h$ *there exists a ball* B_P *of radius* $r \geq \gamma h_P$*, such that* P *is star-shaped with respect to* B_P*.*
There exists a constant $1 \geq \gamma' > 0$ *such that, for all* $h > 0$*, all* $P \in \Omega_h$*, and all distinct vertices* $\boldsymbol{x} \neq \boldsymbol{y}$ *of* P*, it holds that* $|\boldsymbol{x} - \boldsymbol{y}| \geq \gamma' h_P$*, if* $\boldsymbol{x} \neq \boldsymbol{y}$*.*

Remark 3.6. *Assumption 3 will not be needed for the definition of the VEM spaces and the construction of the discrete bilinear form.*
We will use the assumption for the A PRIORI *and the* A POSTERIORI *analysis.*

The following results will be useful for proving error estimates in the later sections.

Lemma 3.7. *Under Assumption 3, there exists a constant* $C > 0$*, independent of* h*,such that*

$$\#\mathrm{neighbors}(P) \leq \#\mathrm{vertices}(P) \leq C \text{ for all } P \in \Omega_h.$$

Proof. By definition, it holds that #neighbors(P) $\leq$ #vertices(P).

Let B_P be the ball defined in Assumption 3 and $\mathbf{x} \in B_P$ be the center of B_P. Since h_P is the diameter of P, it holds that $|\mathbf{x} - \mathbf{y}| \leq h_P$ for every $\mathbf{y} \in \text{vertices}(P)$. Thus, it is clear that

$$B_{\gamma' h_P/3}(\mathbf{y}) \subseteq B_{(1+\gamma'/3)h_P}(\mathbf{x}) \text{ for all } \mathbf{y} \in \text{vertices}(P).$$

Due to Assumption 3, it holds that

$$B_{\gamma' h_P/3}(\mathbf{y}) \cap B_{\gamma' h_P/3}(\mathbf{z}) = \emptyset \text{ for all } \mathbf{y}, \mathbf{z} \in \text{vertices}(P) \text{ with } \mathbf{y} \neq \mathbf{z}.$$

Thus, due to the translation invariance of the Lebesgue measure, we derive that

$$\#\text{vertices}(P)\ \left|B_{\gamma' h_P/3}(\mathbf{x})\right| = \sum_{\mathbf{y}\in\text{vertices}} \left|B_{\gamma' h_P/3}(\mathbf{y})\right| = \left|\bigcup_{\mathbf{y}\in\text{vertices}} B_{\gamma' h_P/3}(\mathbf{y})\right| \leq \left|B_{(1+\gamma'/3)h_P}(\mathbf{x})\right|.$$

Therefore, we conclude that

$$\#\text{vertices}(P) \leq \frac{\left|B_{(1+\gamma'/3)h_P}(\mathbf{x})\right|}{\left|B_{\gamma' h_P/3}(\mathbf{y})\right|} = \frac{\pi(1+\gamma'/3)^2 h_P^2}{\pi(\gamma')^2 h_P^2/3^2} = \left(\frac{3+\gamma'}{\gamma'}\right)^2 := C.$$

This concludes the proof. ∎

Theorem 3.8 ([Verfuerth1, page 715-716]). *Let $\Omega \subseteq \mathbb{R}^n, n \geq 2$, be a bounded domain with diameter h_Ω. Let $m \in \mathbb{N}$. If Ω is star-shaped with respect to a point $\boldsymbol{x} \in \mathbb{R}^n$, then there exists a constant $c_{m,j} > 0$ for every $0 \leq j \leq m$, such that*

$$\sup_{u\in H^{m+1}(\Omega)} \inf_{p\in\mathbb{P}_m(\Omega)} \frac{\|D^j(u-p)\|_{L^2(\Omega)}}{\|D^{m+1}u\|_{L^2(\Omega)}} \leq c_{m,j}\ h_\Omega^{m+1-j}.$$

Here the constant $c_{m,j}$ can be bounded by

$$c_{m,j} \leq \xi \sqrt{\binom{n+j-1}{j} \frac{(m+1-j)!}{\left\lceil \frac{m+1-j}{n} \right\rceil !}},$$

with

$$\xi := \sqrt{\max\left\{\frac{4}{\pi^2}K_1(\kappa) + K_2(\kappa), K_3(\kappa)\right\}^{m+1-j}},$$

and

$$\kappa := \frac{\max_{\boldsymbol{y}\in\partial\Omega}\ |\boldsymbol{x}-\boldsymbol{y}|}{\min_{\boldsymbol{y}\in\partial\Omega}\ |\boldsymbol{x}-\boldsymbol{y}|}$$

and

$$K_1(z) := 4z^{n-2} - 3z^{-2}$$
$$K_2(z) := \frac{4z^{n-2} - 4z^{-2}}{n(n+2)}$$
$$K_3(z) := \begin{cases} \log(z) - \frac{1}{2} + \frac{z^{-2}}{2} & \text{if } n = 2, \\ \frac{2z^{n-2}}{n(n-2)} - \frac{1}{n-2} + \frac{n^{-2}}{n} & \text{if } n \geq 3. \end{cases}$$

If Ω is convex, ξ can be set to $\xi := \pi^{j-(m+1)}$. This is indeed a tighter bound, as $\kappa \geq 1$ and therefore $K_1(\kappa) > 1, K_2(\kappa) > 0$.

□

Definition 3.9. *Let $m \in \mathbb{N}_0$ and $P \subseteq \mathbb{R}^n$. We denote the L^2-orthogonal projection onto $\mathbb{P}_m(P)$ by $\pi_m^P : L^2(P) \to \mathbb{P}_m(P)$.*

Lemma 3.10. *Let $\Omega \subseteq \mathbb{R}^2$ be a domain with a polygonal decomposition Ω_h, satisfying Assumptions 2 and 3. Let $m \geq 0$ and $0 \leq k \leq m+1$ be natural numbers. Then, there exists a constant $C_{m,k} > 0$, which depends only on γ (see Assumption 3), such that, for all polygons $P \in \Omega_h$, it holds that*

$$\|v - \pi_m^P(v)\|_{L^2(P)} \leq C_{m,k}\, h_P^k\, \|D^k v\|_{L^2(P)} \quad \text{for all } v \in H^{m+1}(P). \tag{3.1}$$

Proof. We split the proof into two parts. First we prove the statement for $k = m+1$ and $k = 0$, then for general $0 \leq k \leq m+1$. Let $P \in \Omega_h$ be arbitrary.

Part (i):
For $k = 0$ this is a direct consequence of the projection properties of π_m^P, as

$$\|v - \pi_m^P(v)\|_{L^2(P)} \leq \|v\|_{L^2(P)}.$$

Now let $k = m+1$. Since π_m^P is the orthogonal projection onto $\mathbb{P}_m(P)$, we get that

$$\|v - \pi_m^P(v)\|_{L^2(P)} = \inf_{p \in \mathbb{P}_m(P)} \|v - p\|_{L^2(P)} \leq \sup_{u \in H^{m+1}(\Omega)} \inf_{p \in \mathbb{P}_m(P)} \frac{\|u - p\|_{L^2(P)}}{\|D^{m+1}u\|_{L^2(P)}} \|D^{m+1}v\|_{L^2(P)}.$$

Due to Assumption 3, P is star-shaped with respect to a ball $B_P \subseteq P$. In particular, P is star-shaped with respect to the center of the ball, which we denote by $\mathbf{x}$. Thus, according to Theorem 3.8, we can dominate the difference by

$$\|v - \pi_m^P(v)\|_{L^2(P)} \leq c_{m,0}\, h_P^{m+1} \|D^{m+1}v\|_{L^2(P)} \leq \xi \sqrt{\frac{(m+1)!}{\lceil \frac{m+1}{n} \rceil !}} h_P^{m+1} \|D^{m+1}v\|_{L^2(P)}.$$

Since the radius of B_P is larger than or equal to γh_P (again due to Assumption 3), it holds that

$$\min_{\mathbf{y}\in\partial\Omega} |\mathbf{x}-\mathbf{y}| \geq \gamma h_P.$$

Thus, we derive that

$$\kappa := \frac{\max_{\mathbf{y}\in\partial\Omega} |\mathbf{x}-\mathbf{y}|}{\min_{\mathbf{y}\in\partial\Omega} |\mathbf{x}-\mathbf{y}|} \leq \frac{h_P}{\gamma h_P} = \frac{1}{\gamma}.$$

Since K_1, K_2, K_3 are continuous functions and $\kappa \in [1, \frac{1}{\gamma}]$, we can dominate the term

$$\xi := \sqrt{\max\left\{\frac{4}{\pi^2}K_1(\kappa) + K_2(\kappa), K_3(\kappa)\right\}^{m+1}},$$

by

$$\xi \leq \max_{z\in[1,\frac{1}{\gamma}]} \sqrt{\max\left\{\frac{4}{\pi^2}K_1(z) + K_2(z), K_3(z)\right\}^{m+1}} < \infty.$$

We define

$$C_{m,m+1} := \max_{z\in[1,\frac{1}{\gamma}]} \sqrt{\max\left\{\frac{4}{\pi^2}K_1(z) + K_2(z), K_3(z)\right\}^{m+1}} \sqrt{\frac{(m+1)!}{\lceil\frac{m+1}{n}\rceil!}},$$

which only depends on γ and satisfies (3.1).

Part (ii):
Let $0 < k < m+1$ be an arbitrary natural number. By using the triangle inequality and $\pi_m^P \pi_{k-1}^P = \pi_{k-1}^P$, we get

$$\begin{aligned}\|v - \pi_m^P(v)\|_{L^2(P)} &\leq \|v - \pi_{k-1}^P(v)\|_{L^2(P)} + \|\pi_{k-1}^P(v) - \pi_m^P(v)\|_{L^2(P)} \\ &= \|v - \pi_{k-1}^P(v)\|_{L^2(P)} + \|\pi_m^P(\pi_{k-1}^P(v)) - \pi_m^P(v)\|_{L^2(P)} \leq 2\|v - \pi_{k-1}^P(v)\|_{L^2(P)}.\end{aligned}$$

Using the results of part (i), we conclude that

$$\|v - \pi_m^P(v)\|_{L^2(P)} \leq 2\|v - \pi_{k-1}^P(v)\|_{L^2(P)} \leq \underbrace{2C_{k-1,k}}_{=:C_{m,k}} h_P^k \|D^k v\|_{L^2(P)}.$$

This concludes the proof. □

Furthermore, we cite the following lemma from [BdV M 14, Lemma 4.3], [CaGePrSu, Theorem 1].

Lemma 3.11. *Let $1+\alpha \leq s \leq m$ and $P \in \Omega_h$. There exists a constant $C > 0$, independent of h and P, such that*

$$\left\|\nabla(v - \pi_m^P(v))\right\|_{L^2(P)} \leq C h_P^s \left\|D^{s+1} v\right\|_{L^2(P)} \text{ for all } v \in H^{s+1}(P).$$

Proof. One can prove this via defining a polygonal projection similar to the one defined in Definition 3.55 below and by using Theorem 3.8. □

The last Lemma of this section will be useful when we prove attributes of A POSTERIORI error estimators.

Lemma 3.12. *Let $\Omega \subseteq \mathbb{R}^2$ be a domain with a polygonal decomposition Ω_h satisfying Assumptions 2 and 3. Let $P \in \Omega_h$ and $\mathbf{e} \in \operatorname{edges}(P)$. Then, there exists a triangle $T \subseteq P$ such that $\mathbf{e}$ is an edge of the triangle and that*

$$|T| \geq \frac{\gamma' \gamma h_P^2}{2}.$$

Proof. Let $\mathbf{x}_1 \in B_P$ be the center of B_P. Furthermore, let $\mathbf{x}_2, \mathbf{x}_3$ be the endpoints of the edge $\mathbf{e}$, i.e., $\mathbf{e} = \overline{\mathbf{x}_2, \mathbf{x}_3}$. We define $T := \operatorname{conv}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. Since P is star shaped with respect to $\mathbf{x}_1$, it holds that $T \subseteq P$.

Due to Assumption 3, the length of the edge $\mathbf{e}$ is at least $\gamma' h_P$. As the radius of B_P is greater than or equal to γh_P, the height of T perpendicular to $\mathbf{e}$ has length greater than or equal to γh_P.

Thus, we get that

$$|T| \geq \frac{1}{2}\, \gamma' h_P\, \gamma h_P = \frac{\gamma' \gamma h_P^2}{2}.$$

□

Definition 3.13. *To simplify the notation, we will use the symbol $\lesssim$ to denote less than or equal to, up to a multiplicative constant that is independent of h, i.e., $a \lesssim b$ means that there exists a $C > 0$, independent of h, such that $a \leq Cb$.*

3.2 The VEM space

Now we construct the VEM space, first locally on a single polygon P, then globally. To this end, we introduce the two parameters on which the VEM depends.

Definition 3.14 (VEM parameters)**.** *Let $\alpha \in \mathbb{N}_0$ and $m \in \mathbb{N}$ with $m \geq \alpha + 1$.*

1. *We call α the regularity index. It indicates the smoothness of functions in the discrete space, i.e., all discrete functions will be in $C^\alpha(\overline{\Omega})$.*

2. *We call m the consistency index. It determines the order of piecewise polynomials that can be perfectly approximated by the VEM and will later give an upper bound for the order of convergence of our approximation.*

In order to define the local VEM spaces, we need the following auxiliary definitions.

Definition 3.15. *Let $j = 0, \dots \alpha$, then we define*

$$\alpha_j := \max\{2(\alpha - j) + 1\ ,\ m - j\} \geq 1.$$

Remark 3.16. *Note that, for all $j = 0, \dots, \alpha$, it holds that $\alpha_j - (\alpha - j) \geq 1$. Therefore, it holds that*

$$\alpha + 1 \leq \sum_{j=0}^{\alpha} (\alpha_j - (\alpha - j)) = \sum_{j=0}^{\alpha} \alpha_j - \sum_{j=0}^{\alpha} (\alpha - j) = \sum_{j=0}^{\alpha} \alpha_j - \sum_{j=0}^{\alpha} j = \sum_{j=0}^{\alpha} (\alpha_j - j)\,.$$

Definition 3.17. *For every polygon $P \in \Omega_h$ and any integer $s \geq 0$, we define the space*

$$\mathbb{B}_s(\partial P) := \left\{ v \in L^2(\partial P)\ \mid\ v|_{\boldsymbol{e}} \in \mathbb{P}_s(\boldsymbol{e}) \text{ for all } \boldsymbol{e} \in \operatorname{edges}(P) \right\}.$$

Furthermore, we define

$$\mathbb{B}^{\alpha,m}(\partial P) := \left\{ (v_0, \dots, v_\alpha) \in \prod_{j=0}^{\alpha} \mathbb{B}_{\alpha_j}(\partial P) \ \middle|\ \exists v \in H^{\alpha+1}(P) :\ \forall j = 0, \dots, \alpha : \frac{\partial^j v}{\partial \boldsymbol{n}^j} = v_j \right\}.$$

Lemma 3.18. *For any $P \in \Omega_h$, the dimension of the boundary space $\mathbb{B}^{\alpha,m}(\partial P)$ is equal to*

$$\dim \mathbb{B}^{\alpha,m}(\partial P) = \#\text{vertices}(P) \sum_{j=0}^{\alpha} (\alpha_j - j)\,.$$

Proof. $\mathbb{B}^{\alpha,m}(\partial P)$ is the space of all those elements of $\prod_{j=0}^{\alpha} \mathbb{B}_{\alpha_j}(\partial P)$ that satisfy the conditions in Theorem 2.18. Thus, if we set N to be the number of those conditions, it holds that

$$\dim \mathbb{B}^{\alpha,m}(\partial P) = \dim \left[\prod_{j=0}^{\alpha} \mathbb{B}_{\alpha_j}(\partial P) \right] - N = \sum_{j=0}^{\alpha} [\#\text{edges}(P)(\alpha_j + 1)] - N.$$

Due to Corollary 2.19, the total number of conditions is equal to

$$N = \#\text{vertices}(P) \sum_{j=0}^{\alpha} (j+1).$$

Since $\#\text{edges}(P) = \#\text{vertices}(P)$, we conclude that

$$\dim \mathbb{B}^{\alpha,m}(\partial P) = \#\text{vertices}(P) \sum_{j=0}^{\alpha} (\alpha_j + 1 - (j+1)) = \#\text{vertices}(P) \sum_{j=0}^{\alpha} (\alpha_j - j)\,.$$

This concludes the proof. □

Now we have all the ingredients to define the local VEM space.

Definition 3.19 (Local VEM space)**.** *Let $P \in \Omega_h$ be an arbitrary polygon. We define*

$$V_{h|P} := \left\{ v \in H^{\alpha+1}(P) \;\middle|\; \Delta^{\alpha+1} v \in \mathbb{P}_{m-2}(P) \text{ and } \left(\frac{\partial^j v}{\partial \boldsymbol{n}^j}\right)_{j=0,\dots,\alpha} \in \mathbb{B}^{\alpha,m}(\partial P) \right\}. \tag{3.2}$$

By $\Delta^{\alpha+1} v \in \mathbb{P}_{m-2}$, we mean that there exsists a polynom $p \in \mathbb{P}_{m-2}(P)$ such that $\Delta^{\alpha+1} v = p$ holds in a weak sense (see 2.9).

Remark 3.20. *It is trivial to prove that the local VEM space is a linear subspace of $H^{\alpha+1}(P)$.*

Lemma 3.21. *Let $P \in \Omega_h$ be a polygon. The dimension of the local element space is given by*

$$\dim V_{h|P} = \frac{m(m-1)}{2} + \#\text{vertices}(P) \sum_{j=0}^{\alpha} (\alpha_j - j)\,.$$

Proof. Due to Theorem 2.13, there exists a unique $u \in H^{\alpha+1}(P)$ for every $p \in \mathbb{P}_{m-2}$ and every $(q_0, \dots, q_\alpha) \in \mathbb{B}^{\alpha,m}(\partial P)$ such that

$$(-1)^{\alpha+1}\ \Delta^{\alpha+1} u = p \text{ in } \Omega,$$
$$\frac{\partial^j u}{\partial \mathbf{n}^j} = q_j \text{ on } \partial\Omega, \text{ for } j = 0, \dots, \alpha.$$

Therefore, the map $\phi : V_{h|P} \to \mathbb{P}_{m-2}(P) \times \mathbb{B}^{\alpha,m}(\partial P)$, defined by

$$\phi(u) = \left((-1)^{\alpha+1}\ \Delta^{\alpha+1} u,\, u|_{\partial P}\,, \frac{\partial u}{\partial \mathbf{n}}, \dots, \frac{\partial^\alpha u}{\partial \mathbf{n}^\alpha} \right),$$

is bijective. It is obvious that ϕ is linear and thus an isomorphism. Thus, according to Lemma 3.18, we get

$$\begin{aligned}\dim V_{h|P} &= \dim\left[\mathbb{P}_{m-2}(P) \times \mathbb{B}^{\alpha,m}(\partial P)\right] = \dim \mathbb{P}_{m-2}(P) + \dim \mathbb{B}^{\alpha,m}(\partial P) \\ &= \frac{m(m-1)}{2} + \#\text{vertices}(P) \sum_{j=0}^{\alpha} (\alpha_j - j)\,.\end{aligned}$$

This concludes the proof. □

As a consequence of Lemma 3.21 the space $V_{h|P}$ is not trivial.

The following Lemma is a consequence of the fact that the conditions for trace lifting in Theorem 2.18 are identical for β and $\lfloor \beta \rfloor$ for piecewise polynomial data.

Lemma 3.22. *It holds that $V_{h|P} \subseteq H^{\alpha+1+1/2}(P)$ and hence also $V_{h|P} \subseteq C^{\alpha}(\overline{P})$.*

Proof. Let $v \in V_{h|P}$. By definition v is the weak solution of a polyharmonic equation with Dirichlet boundary data

$$\begin{aligned}(-1)^{\alpha+1}\ \Delta^{\alpha+1} v &= p \text{ in } \Omega,\\ \frac{\partial^j v}{\partial \mathbf{n}^j} &= q_j \text{ on } \partial\Omega, \text{ for } j = 0, \ldots, \alpha,\end{aligned}$$

for some $p \in \mathbb{P}_{m-2}(P)$ and $(q_0, \ldots, q_\alpha) \in \mathbb{B}^{\alpha,m}(\partial P)$.

Since $(q_0, \ldots, q_\alpha) \in \mathbb{B}^{\alpha,m}(\partial P)$, condition (2.14) is satisfied for all integers $0 \leq n < \alpha$ and, since $\alpha \in \mathbb{N}$, condition (2.15) is satisfied for $n = \alpha$.
As $(q_0, \ldots, q_\alpha)$ is a tuple of piecewise polynomials, Corollary 2.19 gives us that condition (2.14) is equivalent to condition (2.15).
Thus, $(q_0, \ldots, q_\alpha)$ satisfies condition (2.14) for all integers $0 \leq n \leq \alpha$ and by that of course for all integers $0 \leq n < \alpha + \frac{1}{2}$.

Therefore, Theorem 2.18 yields $w \in H^{\alpha+1+1/2}(P)$ such that

$$\frac{\partial^j w}{\partial \mathbf{n}^j} = q_j \text{ for } j = 0, \ldots, \alpha.$$

Thus, the boundary data satisfy

$$q_j \in H^{\alpha+1+1/2-j}(\partial P), \text{ for } j = 0, \ldots, \alpha.$$

Since $p \in C^\infty(P)$, we get by Theorem 2.14 that $v \in H^{\alpha+1+1/2}(P)$.

Since $\alpha + 1 + \frac{1}{2} - \frac{2}{2} = \alpha + \frac{1}{2} > \alpha$, the Sobolev embedding theorem (Theorem 1.33) ensures $v \in C^\alpha(\overline{P})$.

□

Lemma 3.23. *Let $V_{h|P}$ be defined as in (3.2). Then $\mathbb{P}_m(P) \subseteq V_{h|P}$.*

Proof. Let $p \in \mathbb{P}_m(P)$ be arbitrary. Since $p \in C^\infty(P)$, the weak formulation of $\Delta^{\alpha+1} p$ is equal to the evaluation of the $\alpha + 1$ power of the Laplacian. As taking a derivative of a polynomial reduces its degree by one, it holds that

$$\Delta p = \frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2} \in \mathbb{P}_{m-2}(P).$$

Thus we derive that $\Delta^{\alpha+1} p \in \mathbb{P}_{m-2}(P)$.

For any edge $\mathbf{e}$ and any $j = 0, \ldots, \alpha$, it holds that $\left.\frac{\partial^j p}{\partial \mathbf{n}^j}\right|_{\mathbf{e}} \in \mathbb{P}_{m-j}(\mathbf{e})$. Since $\alpha_j = \max\{2(\alpha - j) + 1, m - j\} \geq m - j$, and therefore $\left.\frac{\partial^j p}{\partial \mathbf{n}^j}\right|_{\mathbf{e}} \in \mathbb{P}_{m-j}(\mathbf{e}) \subseteq \mathbb{P}_{\alpha_j}(\mathbf{e})$.

As $\left.\frac{\partial^j p}{\partial \mathbf{n}^j}\right|_{\mathbf{e}}$ is bounded for every edge $\mathbf{e}$, all normal derivatives $\frac{\partial^j p}{\partial \mathbf{n}^j}$ are bounded and thus $\frac{\partial^j p}{\partial \mathbf{n}^j} \in L^2(\partial P)$. Since $p \in H^{\alpha+1}(P)$, we also get

$$\left(p|_{\mathbf{e}}, \ldots, \left.\frac{\partial^\alpha p}{\partial \mathbf{n}^\alpha}\right|_{\mathbf{e}} \right) \in \mathbb{B}^{\alpha,m}(\partial P).$$

As all conditions are fulfilled, we see that $p \in V_{h|P}$. Since p was arbitrary we can conclude that $\mathbb{P}_m(P) \subseteq V_{h|P}$.

□

Remark 3.24. *Due to Lemma 3.23, one can view the local VEM space $V_{h|P}$ as the space of polynomials $\mathbb{P}_m(P)$, plus some additional unknown functions.*
All functions in $V_{h|P}$ are polynomials on the boundary and therefore known on the boundary. Inside the polygon P, we only know the polynomials $\mathbb{P}_m(P)$. This will motivate the degrees of freedom defined in the next chapter.
Furthermore, we want to point out that, since each local VEM space at least contains $\mathbb{P}_m(P)$ (which are the functions that make up local Finite Element spaces), one would guess that the approximation properties of VEM are at least the same as of FEM.

Now the global VEM space V_h can be defined as the set of all $H^{\alpha+1}$-functions whose restriction to each polygon P belongs to the local space defined above, i.e.,

$$V_h := \left\{ v \in H^{\alpha+1}(\Omega) \;\middle|\; \forall P \in \Omega_h : \; v|_P \in V_{h|P}, \; v|_{\partial\Omega} = 0 \right\}. \tag{3.3}$$

Here the boundary condition $v|_{\partial\Omega} = 0$ is imposed, as our goal is to approximate a weak solution of the Poisson problem with homogenous Dirichlet boundary conditions.

Remark 3.25. *Note that if we define the global VEM space without boundary conditions by*

$$W_h := \left\{ v \in H^{\alpha+1}(\Omega) \;\middle|\; \forall P \in \Omega_h : \; v|_P \in V_{h|P} \right\},$$

then $V_h = W_h \cap H_0^1(\Omega) \subseteq H_0^1(\Omega)$. Thus the VEM space is a subspace of $H_0^1(\Omega)$.

Theorem 3.26 (Regularity of V_h)**.** *It holds that $V_h \subseteq C^\alpha(\overline{\Omega})$. Conversely any function $v \in C^\alpha(\overline{\Omega})$ such that $v|_P \in V_{h|P}$ for all $P \in \Omega_h$ is an element of V_h.*

Proof. By definition $V_h \subseteq H^{\alpha+1}(\Omega)$ and thus due to Sobolev embedding (Theorem 1.33), it holds that $V_h \subseteq C^{\alpha-1}(\overline{\Omega})$ (for $\alpha \geq 1$).
Furthermore, Lemma 3.22 gives us local regularity, i.e., $V_h \subseteq C^\alpha(\overline{P})$ for every $P \in \Omega_h$.

Thus it is sufficient to prove that, for every $v \in V_h$, the derivative $D^\alpha v$, which is defined locally on every $P \in \Omega_h$, is continuous across the edges, i.e., for every $P_1, P_2 \in \Omega_h$ and every $\mathbf{x} \in P_1 \cap P_2$, it holds that

$$\lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in P_1}} D^\alpha v(\mathbf{y}) = \lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in P_2}} D^\alpha v(\mathbf{y}) \tag{3.4}$$

Assume that for some $v \in V_h$ (3.4) does not hold. Then, there is a derivative of degree alpha $\frac{\partial^\alpha}{\partial x_{j_1}\ldots\partial x_{j_\alpha}}$ such that

$$\lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in P_1}} \frac{\partial^\alpha v}{\partial x_{j_1}\ldots\partial x_{j_\alpha}}(\mathbf{y}) \neq \lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in P_2}} \frac{\partial^\alpha v}{\partial x_{j_1}\ldots\partial x_{j_\alpha}}(\mathbf{y}).$$

Define $g : \Omega \to \mathbb{R}$ by

$$g(\mathbf{y}) = \frac{\partial^\alpha v}{\partial x_{j_1}\ldots\partial x_{j_\alpha}}(\mathbf{y}) \text{ for } \mathbf{y} \in P$$

for every $P \in \Omega_h$. Without loss of generality, we can assume that

$$\begin{aligned} &\lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in P_1}} g(\mathbf{y}) > \lim_{\substack{\mathbf{y}\to\mathbf{x}\\ \mathbf{y}\in P_2}} g(\mathbf{y}) \text{ and that}\\ &\mathbf{n}_1^{P_1}(\mathbf{x}) > 0. \end{aligned} \tag{3.5}$$

We can assume that the $\mathbf{x}$ in (3.5) lies in the interior of an edge that is shared by P_1, P_2, i.e., $\mathbf{x} \notin \partial\Omega$ and there exists $\mathbf{e_x} \in \text{edges}(P_1) \bigcap \text{edges}(P_2)$ with $\mathbf{e_x} \in \mathbf{e} \setminus \text{vertices}(\mathbf{e_x})$. Because if such a point would not exist, g would indeed be continuous.

Since $v \in H^{\alpha+1}(\Omega)$, we get that $g \in H^1(\Omega)$. The weak gradient of g is an $L^2(\Omega)$-function and has to coincide with the classical gradient on every polygon $P \in \Omega_h$ (because g is piecewise in $C^1(P)$).

Let $\phi \in H_0^1(\Omega)$, then by integrating by parts twice we get that

$$\begin{aligned} \int_\Omega g\ \nabla\phi\ \mathrm{d}x &= \sum_{P\in\Omega_h} \int_P g\ \nabla\phi\ \mathrm{d}x = \sum_{P\in\Omega_h} \left(\int_{\partial P} g\ \phi\ \mathbf{n}\ \mathrm{d}s - \int_P \nabla g\ \phi\ \mathrm{d}x \right) \\ &= \sum_{P\in\Omega_h} \int_{\partial P} g\ \phi\ \mathbf{n}\ \mathrm{d}s - \int_\Omega \nabla g\ \phi\ \mathrm{d}x \\ &= \sum_{P\in\Omega_h} \int_{\partial P} g\ \phi\ \mathbf{n}\ \mathrm{d}s - \underbrace{\int_{\partial\Omega} g\ \phi\ \mathbf{n}\ \mathrm{d}x}_{=0} + \int_\Omega g\ \nabla\phi\ \mathrm{d}x \\ &= \sum_{P\in\Omega_h} \int_{\partial P} g\ \phi\ \mathbf{n}\ \mathrm{d}s + \int_\Omega g\ \nabla\phi\ \mathrm{d}x. \end{aligned}$$

Thus, we derive that

$$\sum_{P\in\Omega_h}\int_{\partial P} g\ \phi\ \mathbf{n}\ \mathrm{d}s = 0 \text{ for all } \phi \in H_0^1(\Omega).$$

Since $\mathbf{x} \notin \partial\Omega$, we get that $\mathbf{e_x}$ is an inner edge. As $\mathbf{x}$ is in the interior of $\mathbf{e_x}$ and because $\mathbf{e_x} \in \text{edges}(P)$ for $P \in \Omega_h$ if and only if $P = P_1(\mathbf{e_x}), P_2(\mathbf{e_x})$, we can find $\delta > 0$ such that

$$B_\delta(\mathbf{x}) \cap \left(\bigcup_{P\in\Omega_h} \partial P\right) = B_\delta(\mathbf{x}) \cap \mathbf{e_x}.$$

By making δ even smaller (if necessary), we can assume (due to (3.5)) that there exists $\epsilon > 0$ such that

$$\left(\lim_{\substack{\mathbf{y}\to\mathbf{z}\\ \mathbf{y}\in P_1}} g(\mathbf{y}) - \lim_{\substack{\mathbf{y}\to\mathbf{z}\\ \mathbf{y}\in P_2}} g(\mathbf{y})\right) \mathbf{n}_1^{P_1}(\mathbf{z}) > \epsilon \text{ for all } \mathbf{z} \in B_{\delta/2}(\mathbf{x}) \cap \mathbf{e_x}.$$

There exists $\phi \in C_{00}^\infty(\Omega) \subseteq H_0^1(\Omega)$ such that

$$\phi \geq 0,\ \operatorname{supp} \phi \subseteq B_\delta(\mathbf{x}),\ \text{and } \phi = 1 \text{ on } B_{\delta/2}(\mathbf{x})$$

See [Kalt, example 15.5.4] for a construction.

Finally, we get the contradiction

$$\begin{aligned}
0 &= \sum_{P\in\Omega_h}\int_{\partial P} g\ \phi\ \mathbf{n}_1\ \mathrm{d}s = \sum_{P\in\Omega_h}\int_{\partial P\cap B_\delta(\mathbf{x})} g\ \phi\ \mathbf{n}_1\ \mathrm{d}s \\
&= \int_{\mathbf{e_x}\cap B_\delta(\mathbf{x})} g\ \phi\ \mathbf{n}_1^{P_1}\ \mathrm{d}s + \int_{\mathbf{e_x}\cap B_\delta(\mathbf{x})} g\ \phi\ \mathbf{n}_1^{P_2}\ \mathrm{d}s \\
&= \int_{\mathbf{e_x}\cap B_\delta(\mathbf{x})} \underbrace{\left(\lim_{\substack{\mathbf{y}\to\mathbf{s}\\ \mathbf{y}\in P_1}} g(\mathbf{y}) - \lim_{\substack{\mathbf{y}\to\mathbf{s}\\ \mathbf{y}\in P_2}} g(\mathbf{y})\right) \phi\ \mathbf{n}_1^{P_1}}_{\geq 0}\ \mathrm{d}s \\
&\geq \int_{\mathbf{e_x}\cap B_{\delta/2}(\mathbf{x})} \underbrace{\left(\lim_{\substack{\mathbf{y}\to\mathbf{s}\\ \mathbf{y}\in P_1}} g(\mathbf{y}) - \lim_{\substack{\mathbf{y}\to\mathbf{s}\\ \mathbf{y}\in P_2}} g(\mathbf{y})\right) \phi\ \mathbf{n}_1^{P_1}}_{\geq \epsilon}\ \mathrm{d}s > 0.
\end{aligned}$$

Thus, $D^\alpha v$ has to be continuous across the edges for every $v \in V_h$ and therefore we conclude that $V_h \subseteq C^\alpha(\overline{\Omega})$.

Now, let $v \in C^\alpha(\overline{\Omega})$ such that $v|_P \in V_{h|P}$. To prove that $v \in V_h$, it is sufficient to show that $D^{\alpha+1}v \in L^2(\Omega)$. We will show this by proving that $D^{\alpha+1}(v)\big|_P = D^{\alpha+1}(v|_P)$ for every $P \in \Omega_h$.

Let $g := \frac{\partial^\alpha v}{\partial x_{j_1} \dots \partial x_{j_\alpha}}$ be an arbitrary derivative of $D^\alpha v$ and $\phi \in C_{00}^\infty(\Omega)$, then due to the continuity of g and because $\phi = 0$, on $\partial\Omega$ it holds that

$$\sum_{P \in \Omega_h} \int_{\partial P} g \ \phi \ \mathbf{n} \ \mathrm{d}s = 0.$$

Thus integration by parts gives us

$$\begin{aligned}\int_\Omega g \nabla \phi \ \mathrm{d}x &= \sum_{P \in \Omega_h} \int_P g \nabla \phi \ \mathrm{d}x = \sum_{P \in \Omega_h} \left(\int_{\partial P} g \ \phi \ \mathbf{n} \ \mathrm{d}s - \int_P \nabla g \phi \ \mathrm{d}x \right) \\ &= \underbrace{\sum_{P \in \Omega_h} \int_{\partial P} g \ \phi \ \mathbf{n} \ \mathrm{d}s}_{=0} - \sum_{P \in \Omega_h} \int_P \nabla g \phi \ \mathrm{d}x = - \sum_{P \in \Omega_h} \int_P \nabla g \phi \ \mathrm{d}x,\end{aligned}$$

and therefore it is indeed true, that $D^{\alpha+1}(v)\big|_P = D^{\alpha+1}\left(v|_P\right)$ for every $P \in \Omega_h$, which proves $v \in H^{\alpha+1}(\Omega)$ and by that $v \in V_h$.

□

3.3 Degrees of freedom

The degrees of freedom of the VEM space are parameters, through which one can determine any function of that space uniquely. More formally, we define:

Definition 3.27 (Degrees of freedom)**.** *A finite family of functionals $(\psi_i)_{i=1,\dots,k}$ on a vector space V is called a set of degrees of freedom, if the linear map $\Psi : V \to \mathbb{R}^k$ defined by*

$$\Psi(v)_i = \psi_i(v), \ \ i = 1, \dots, k$$

is bijective. This bijectivity property is called unisolvence.

Remark 3.28. *Obviously the existence of degrees of freedom as in Definition 3.27 implies that the vector space V is finite dimensional. An equivalent definition of degrees of freedom is to define them as a basis of the (topological) dual space. This definition also extends to infinite dimensional vector spaces.*

Lemma 3.29. *Let V be a vector space and $(\psi_i)_{i=1,\dots,k}$ be degrees of freedom. Then, there exist $\epsilon_1, \dots, \epsilon_k \in V$ such that*

$$\psi_i(\epsilon_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

$\epsilon_1, \dots, \epsilon_k$ form a basis of V and, for every $x \in \mathbb{R}^k$, the unique element $v \in V$ satisfying $\Psi(v) = x$, is given by

$$v = \sum_{j=1}^{k} x_j \epsilon_j.$$

Proof. Let $j = 1, \dots, k$ and define $e_k \in \mathbb{R}^k$ by

$$(e_j)_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}.$$

By definition (see Definition 3.27) Ψ is an isomorphism. Therefore $\epsilon_j := \Psi^{-1}(e_j) \in V$ is well defined.

By definition, it holds that

$$\psi_i(\epsilon_j) = \Psi(\epsilon_j)_i = \Psi(\Psi^{-1}(e_j))_i = (e_j)_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}.$$

Let $x \in \mathbb{R}^k$, due to the linearity of Ψ, it holds that

$$\Psi\left(\sum_{j=1}^{k} x_j \epsilon_j\right)_i = \sum_{j=1}^{k} x_j \Psi(\epsilon_j)_i = x_i.$$

The uniqueness of $v := \sum_{j=1}^{k} x_j \epsilon_j$ is a consequence of the bijectivity of Ψ.

Let $y \in \mathbb{R}^k$ with

$$0 = \sum_{j=1}^{k} y_j \epsilon_j.$$

Then, it holds that

$$0 = \Psi\left(\sum_{j=1}^{k} y_j \epsilon_j\right)_i = y_i \text{ for all } i = 1, \dots, k.$$

Thus $\epsilon_1, \dots, \epsilon_k$ are linearly independent and, since $k = \dim \mathbb{R}^k = \dim V$, they form a basis of V.

□

Definition 3.30. *The basis $\epsilon_1, \dots, \epsilon_k \in V$, defined in Lemma 3.29, is called the canonical basis corresponding to the degrees of freedom $(\psi_i)_{i=1,\dots,k}$.*

The degrees of freedom are the central ingredient of VEM, as the basis functions of the discrete space are never used directly. Instead, one uses the degrees of freedom to calculate the stiffness matrix or approximation errors.

Like in the previous section, we first define the degrees of freedom locally on a single polygon P, before we move to the definition of the global degrees of freedom. Throughout this section the decomposition Ω_h is fixed.

Definition 3.31. *Let $\boldsymbol{x} \in \Omega$, such that there exists a polygon $P \in \Omega_h$ such that $\boldsymbol{x} \in \partial P$. We set*

$$h_{\boldsymbol{x}} := \max_{P : \boldsymbol{x} \in \partial P} h_P.$$

Definition 3.32 (Local degrees of freedom)**.** *In VEM we distinguish between 3 types of degrees of freedom:*

1. ***Vertex degrees of freedom*** $\mathcal{V}_P^h$ *are (scaled) evaluations of a function and its derivatives at the vertices of the polygon. More precisely, the vertex degrees of freedom consist of all derivatives*

$$\mathcal{V}_P^h := \left\{ \, v \mapsto h_{\boldsymbol{x}}^j D^j v(\boldsymbol{x}) \;\middle|\; j = 0, \ldots, \alpha, \;\; \boldsymbol{x} \in \text{vertices}(P) \, \right\}. \tag{3.6}$$

   Note that here we slightly abuse the definition of degrees of freedom, as the functions defined above map into a vector space instead of a scalar field. In fact each evaluation of each derivative up to the α-th order on each vertex is a degree of freedom.
   We point out that, since $V_{h|P} \subseteq H^{\alpha+1}(\Omega)$ with $D^\alpha v \in C^0(\partial\Omega)$ for all $v \in V_{h|P}$, the vertex degrees of freedom are well defined linear functionals.
   Due to the Schwarz Theorem, a derivative of higher order remains the same if the order of its partial derivatives is changed. Thus, for $j = 0, \ldots, \alpha$, $D^j v(\boldsymbol{x})$ provides exactly $j+1$ degrees of freedom. In total, there are $\sum_{i=1}^{\alpha+1} i = \frac{(\alpha+1)\,(\alpha+2)}{2}$ degrees of freedom per vertex.

2. ***Edge degrees of freedom*** $\mathcal{E}_P^h$ *are (scaled) evaluations of a function and its normal derivatives at nodes, that are chosen on each edge of the polygon.*
   Given a set of $\mathcal{N}_j^{\alpha,m} := \max\{m - (\alpha+1) - (\alpha - j), 0\}$ distinct nodes $(\boldsymbol{x}_i^j)_{i=1,\ldots,\mathcal{N}_j^{\alpha,m}}$ on an edge $\boldsymbol{e}$, we define the associated edge degrees of freedom as

$$\mathcal{E}_P^h(\boldsymbol{e}) := \left\{ \; v \mapsto h_{\boldsymbol{x}_i^j}^j \frac{\partial^j v(\boldsymbol{x}_i^j)}{\partial \boldsymbol{n}^j} \;\middle|\; j = 0, \ldots, \alpha \quad i = 1, \ldots, \mathcal{N}_j^{\alpha,m} \; \right\}$$

and the total set of edge degrees of freedom on the polygon P by

$$\mathcal{E}_P^h := \bigcup_{\boldsymbol{e}\in \text{edges}(P)} \mathcal{E}_P^h(\boldsymbol{e}). \tag{3.7}$$

Again, the continuity of functions in the VEM space guarantees, that these linear functionals are well defined on $V_{h|P}$.
In total, there are

$$\begin{aligned}
\sum_{j=0}^{\alpha} \mathcal{N}_j^{\alpha,m} &= \sum_{j=0}^{\alpha} \mathcal{N}_{\alpha-j}^{\alpha,m} = \sum_{j=0}^{\alpha} \max\{m-(\alpha+1)-j,0\} = \sum_{j=0}^{\min\{m-(\alpha+1),\alpha\}} (m-(\alpha+1)-j)\\
&= (m-(\alpha+1))(\min\{m-(\alpha+1),\alpha\}+1) - \sum_{j=0}^{\min\{m-(\alpha+1),\alpha\}} j\\
&= (m-(\alpha+1))(\min\{m-(\alpha+1),\alpha\}+1) - \dots\\
&\dots - \frac{\min\{m-(\alpha+1),\alpha\}(\min\{m-(\alpha+1),\alpha\}+1)}{2}\\
&= (\min\{m-(\alpha+1),\alpha\}+1)\left(m-(\alpha+1)-\frac{\min\{m-(\alpha+1),\alpha\}}{2}\right)\\
&= \min\{m-\alpha,\alpha+1\}\frac{2(m-(\alpha+1))+\max\{(\alpha+1)-m,-\alpha\}}{2}\\
&= \frac{\min\{m-\alpha,\alpha+1\}\max\{m-(\alpha+1),2m-3\alpha-2\}}{2}
\end{aligned}$$

degrees of freedom per edge on the polygon P. Note that in [BdV M 14, page 764] a different formula for the total number of degrees of freedom per edge is derived.

Remark 3.33. *For the case of minimal consistency $m = \alpha + 1$, there are no edge degrees of freedom.*

Remark 3.34. *Note that*

$$\mathcal{N}_j^{\alpha,m} + 2(\alpha-j)+1 = \max\{m-(\alpha+1)-(\alpha-j),0\} + 2(\alpha-j)+1 = \alpha_j.$$

Thus, it holds that

$$\mathcal{N}_j^{\alpha,m} + (\alpha-j)+1 = \alpha_j - (\alpha-j).$$

3. ***Internal degrees of freedom*** *$\mathcal{P}_P^h$ are integral operators working on the inside of the polygon.*

Let $\boldsymbol{x}_P$ be the barycenter of the polygon P, then we define the centralised and scaled monomial basis of $\mathbb{P}_{m-2}(P)$ by

$$\mathcal{M}_{m-2} := \left\{ \boldsymbol{x} \mapsto \left(\frac{\boldsymbol{x} - \boldsymbol{x}_P}{h_P} \right)^{\boldsymbol{s}} \;\middle|\; \boldsymbol{s} \in \mathbb{N}^2 \text{ such that } |\boldsymbol{s}| = s_1 + s_2 \leq m-2 \right\}. \tag{3.8}$$

Here, we use the notation $\boldsymbol{x}^{\boldsymbol{s}}$ for the monomial $x_1^{s_1} x_2^{s_2}$. We define the internal degrees of freedom by

$$\mathcal{P}_P^h := \left\{ v \mapsto \frac{1}{|P|} \int_P q(\boldsymbol{x})\; v(\boldsymbol{x})\; \mathrm{d}\boldsymbol{x} \;\middle|\; q \in \mathcal{M}_{m-2}. \right\} \tag{3.9}$$

The edge and vertex degrees of freedom completely determine the functions and their normal derivatives on the boundary, as Lemma 3.35 shows. This coincides with the fact that the functions, which make up the local VEM spaces, are completely known on the boundary (see Remark 3.24).
On the inside of each polygon P the VEM spaces contain the polynomials $\mathbb{P}_m(P)$ and some additional, unknown functions. As we will show in Lemma 3.37, the internal degrees of freedom determine the L^2-orthogonal projection onto $\mathbb{P}_{m-2}(P)$.

Lemma 3.35. *For any function $v \in V_{h|P}$, the restriction to the boundary and its normal derivatives $\frac{\partial^j v}{\partial \boldsymbol{n}^j}$, $j = 0, \ldots, \alpha$ are completely determined by its vertex and edge degrees of freedom $\psi(v)$, $\psi \in \mathcal{V}_P^h \bigcup \mathcal{E}_P^h$.*

Proof. Let $\mathbf{e}$ be an edge of ∂P and $j = 0, \ldots, \alpha$ be arbitrary. Since $\frac{\partial^j v}{\partial \mathbf{n}^j} \in \mathbb{B}_{\alpha_j}(\partial P)$, the restriction to the edge $\left.\frac{\partial^j v}{\partial \mathbf{n}^j}\right|_{\mathbf{e}} \in \mathbb{P}_{\alpha_j}(\mathbf{e})$, is a polynomial of degree at most $\alpha_j = \max\{2(\alpha - j) + 1, m - j\}$.

The edge degrees of freedom $\mathcal{E}_P^h$ contain the scaled evaluation $h^j_{\mathbf{x}_i^j} \frac{\partial^j v(\mathbf{x}_i^j)}{\partial \mathbf{n}^j} \in \mathbb{P}_{\alpha_j}(\mathbf{e})$ at $\mathcal{N}_j^{\alpha,m} = \max(m - (\alpha+1) - (\alpha - j), 0)$ distinct nodes $(\mathbf{x}_i^j)_{i=1,\ldots,\mathcal{N}_j^{\alpha,m}}$.
The multiplication by $h^{-j}_{\mathbf{x}_i^j}$ gives us the evaluations of the normal derivatives.

Let $\mathbf{x}_1, \mathbf{x}_2$ be the two vertices that are connected by the edge $\mathbf{e}$. The vertex degrees of freedom contain the scaled evaluations of the function and its derivatives up to order α at those vertices $h^j_{\mathbf{x}_i} D^j v(x_i), j = 0, \ldots, \alpha, i = 1, 2$.
Since we can interpret $D^{j+k} v(x_i)$ as a vector representation of the $j+k$-th Frechet derivative we can obtain $\frac{\partial^{j+k}}{\partial \mathbf{n}^j \partial \tau_{\mathbf{e}}^k}(x_i)$ by multiplying j times the scaled normal vector $h^{-1}_{\mathbf{x}_i} \mathbf{n}$ and k times the scaled tangential vector $h^{-1}_{\mathbf{x}_i} \tau_{\mathbf{e}}$ onto $h^{j+k}_{\mathbf{x}_i} D^{j+k} v(x_i)$.

As $\frac{\partial^{j+k} v}{\partial \mathbf{n}^j \partial \tau_{\mathbf{e}}^k}$ is the k-th derivative of the one dimensional polynomial $\left.\frac{\partial^j v}{\partial \mathbf{n}^j}\right|_{\mathbf{e}} \in \mathbb{P}_{\alpha_j}(\mathbf{e})$, the vertex degrees of freedom determine the value of $\frac{\partial^j v}{\partial \mathbf{n}^j}$ and its $\alpha - j$ derivatives $\frac{\partial^{j+k} v}{\partial \mathbf{n}^j \partial \tau_{\mathbf{e}}^k}$, $k = 1, \ldots \alpha - j$ at two distinct points. Therefore we attain $2(\alpha - j) + 2$ datapoints for polynomial interpolation.

Together with the $\mathcal{N}_j^{\alpha,m}$ evaluations of $\frac{\partial^j v}{\partial \mathbf{n}^j}$ provided by the edge degrees of freedom, we have $2(\alpha-j)+2+\mathcal{N}_j^{\alpha,m} = \alpha_j+1$ datapoints for polynomial interpolation. Due to the Hermite interpolation theorem 1.17, $\left.\frac{\partial^j v}{\partial \mathbf{n}^j}\right|_{\mathbf{e}}$ is uniquely determined by these interpolation points.

Since $j = 0,\dots,\alpha$ and the edge $\mathbf{e}$ were arbitrary, all derivatives $\frac{\partial^j v}{\partial \mathbf{n}^j}$ are completely determined by the vertex and edge degrees of freedom.

□

Corollary 3.36. *For any $v \in V_{h|P}$ the derivatives $\left.D^j v\right|_{\partial P}, j = 0,\dots,\alpha$ are computable on the boundary using only the edge and vertex degrees of freedom $\mathcal{V}_P^h \bigcup \mathcal{E}_P^h$. Furthermore they are (componentwise) polynomials on every edge, i.e., for every $\mathbf{e} \in \operatorname{edges}(P)$, it holds that $\left.D^j v\right|_{\mathbf{e}} \in \mathbb{P}_{\alpha_j}(\mathbf{e})^{j+1}$ for $j = 0,\dots,\alpha$.*

Proof. The case $j = 0$ is a trivial consequence of Lemma 3.35, as $\left.D^j v\right|_{\partial P} = v|_{\partial P} = \frac{\partial^0 v}{\partial \mathbf{n}^0}$.

We will only conduct the proof for $j = 1$ (assuming of course that $\alpha > 0$) as it is the only case that will be relevant for us. For $j > 1$ the proof is almost identical, but requires uglier notation.

Let $v \in V_{h|P}$ and $\mathbf{e} \in \operatorname{edges}(P)$. Due to Lemma 3.35, both $v|_{\partial P}$ and $\frac{\partial v}{\partial \mathbf{n}}$ are computable using the edge and vertex degrees of freedom.

We can compute $\left.\frac{\partial v}{\partial \tau}\right|_{\mathbf{e}}$ using only $v|_{\mathbf{e}}$. Since $v|_{\mathbf{e}} \in \mathbb{P}_{\alpha_0}(\mathbf{e})$ it holds that

$$\left.\frac{\partial v}{\partial \tau}\right|_{\mathbf{e}} \in \mathbb{P}_{\alpha_0-1}(\mathbf{e}) \subseteq \mathbb{P}_{\alpha_1}(\mathbf{e}).$$

Thus both $\left.\frac{\partial v}{\partial \mathbf{n}}\right|_{\mathbf{e}}, \left.\frac{\partial v}{\partial \tau}\right|_{\mathbf{e}} \in \mathbb{P}_{\alpha_1}(\mathbf{e})$ are computable using only the edge and vertex degrees of freedom.

Let $(\mathbf{n_e}\ \tau_\mathbf{e}) \in \mathbb{R}^{2\times 2}$ be the matrix such that the first column is the vector $\mathbf{n_e}$ and the second column is the vector $\tau_\mathbf{e}$. Since $\mathbf{n_e}, \tau_\mathbf{e}$ form an orthonormal basis, this matrix is orthogonal. Thus, we get that

$$\begin{aligned}\nabla v|_{\mathbf{e}} &= \mathrm{I}_2\ \nabla v|_{\mathbf{e}} = (\mathbf{n_e}\ \tau_\mathbf{e})^{\mathrm{T}} (\mathbf{n_e}\ \tau_\mathbf{e})\ \nabla v|_{\mathbf{e}} = (\mathbf{n_e}\ \tau_\mathbf{e})^{\mathrm{T}} (\mathbf{n_e} \cdot \nabla v|_{\mathbf{e}}\ \ \tau_\mathbf{e} \cdot \nabla v|_{\mathbf{e}}) \\ &= (\mathbf{n_e}\ \tau_\mathbf{e})^{\mathrm{T}} \left(\left.\frac{\partial v}{\partial \mathbf{n}}\right|_{\mathbf{e}}\ \ \left.\frac{\partial v}{\partial \tau}\right|_{\mathbf{e}} \right).\end{aligned}$$

Thus, we can conclude that, for every $\mathbf{e} \in \operatorname{edges}(P)$, $\nabla v|_{\mathbf{e}} \in \mathbb{P}_{\alpha_1}(\mathbf{e})^2$ is computable using only the edge and vertex degrees of freedom.

□

Lemma 3.37. *For any function $v \in V_{h|P}$ the projection onto $\mathbb{P}_{m-2}(P)$, $\pi_k^P v$, is completely determined by its internal degrees of freedom $\mathcal{P}_P^h$.*

Proof. $\mathbb{P}_{m-2}(P)$ is a $\frac{(m-1)m}{2}$ dimensional subspace of $L^2(P)$. Due to the theorem of Gram-Schmidt (Theorem 1.13), there exists an orthonormal basis $b_1, \dots, b_{m(m-1)/2} \in \mathbb{P}_{m-2}(P)$.

Due to Lemma 1.12, the projection can be written as

$$\pi_k^P v = \sum_{i=1}^{m(m-1)/2} \langle v, b_i \rangle_{L^2(P)} \, b_i.$$

Since the monomials $\mathcal{M}_{m-2} =: \{w_i, \; i = 1, \dots, \frac{m(m-1)}{2}\}$ defined in (3.8) form a basis of $\mathbb{P}_{m-2}(P)$, there exist scalars $\beta_i^j, \; i, j = 1, \dots, \frac{m(m-1)}{2}$ such that

$$b_j = \sum_{i=1}^{m(m-1)/2} \beta_i^j w_i \text{ for } j = 1, \dots, \frac{m(m-1)}{2}.$$

These scalars can be computed from the monomials by using the formulas given in the theorem of Gram-Schmidt.

Thus, we derive that

$$\pi_k^P v = \sum_{j=1}^{m(m-1)/2} \sum_{i=1}^{m(m-1)/2} \beta_i^j \langle v, w_i \rangle_{L^2(P)} \, b_j. \tag{3.10}$$

By the definition of the internal degrees of freedom $\mathcal{P}_P^h$ (3.9), there exists a $\psi \in \mathcal{P}_P^h$ for every $i = 1, \dots, m(m-1)/2$ such that

$$\langle v, w_i \rangle_{L^2(P)} = |P| \psi(v).$$

Thus, the term (3.10) can be calculated by using only the internal degrees of freedom $\mathcal{P}_P^h$.

□

Definition 3.38. *The local degrees of freedom are given by*

$$\mathrm{DOF}_h(P) := \mathcal{V}_P^h \bigcup \mathcal{E}_P^h \bigcup \mathcal{P}_P^h.$$

Their total number is denoted by

$$\mathcal{N}_P^{\alpha,m} := \#\mathrm{DOF}_h(P). \tag{3.11}$$

Theorem 3.39. *The number of local degrees of freedom $\mathcal{N}_P^{\alpha,m}$ is equal to the dimension of the local VEM space $V_{h|P}$ and is given by*

$$\mathcal{N}_P^{\alpha,m} = \dim V_{h|P} = \frac{m(m-1)}{2} + \#\mathrm{vertices}(P) \; \mathrm{boundary\_dim},$$

with

$$\mathrm{boundary\_dim} := \frac{(\alpha+1)(\alpha+2)}{2} + \frac{\min\{m-\alpha, \alpha+1\} \max\{m-(\alpha+1), 2m-3\alpha-2\}}{2}.$$

Proof. We proof this equality in two steps. First we derive the formula for $\mathcal{N}_P^{\alpha,m}$, then we show the equality $\mathcal{N}_P^{\alpha,m} = \dim V_{h|P}$.

step (i): formula for $\mathcal{N}_P^{\alpha,m}$.
By definition, the number of degrees of freedom is equal to

$$\begin{aligned}\mathcal{N}_P^{\alpha,m} = \underbrace{\#\mathcal{M}_{m-2}}_{=\#\mathcal{P}_P^h} + \underbrace{\#\text{vertices}(P)\frac{(\alpha+1)(\alpha+2)}{2}}_{=\#\mathcal{V}_P^h} \\ + \underbrace{\#\text{edges}(P)\frac{\min\{m-\alpha,\alpha+1\}\max\{m-(\alpha+1),2m-3\alpha-2\}}{2}}_{=\#\mathcal{E}_P^h}\,.\end{aligned}$$

Since $\#\text{edges}(P) = \#\text{vertices}(P)$ and $\#\mathcal{M}_{m-2} = \frac{m(m-1)}{2}$, this reduces to

$$\mathcal{N}_P^{\alpha,m} = \frac{m(m-1)}{2} + \#\text{vertices}(P)\ \text{boundary\_dim}.$$

step (ii): equality of $\mathcal{N}_P^{\alpha,m}$ and $\dim V_{h|P}$.
According to Lemma 3.21, the dimension of the local space is given by

$$\dim V_{h|P} = \frac{m(m-1)}{2} + \#\text{vertices}(P)\sum_{j=0}^{\alpha}(\alpha_j - j)\,.$$

Thus showing the equality $\mathcal{N}_P^{\alpha,m} = \dim V_{h|P}$ reduces to showing that

$$\sum_{j=0}^{\alpha}(\alpha_j - j) = \text{boundary\_dim}.$$

By using the summation formula, the formula for the total number of edge degrees of freedom per edge and, Remark 3.34, we finally get

$$\begin{aligned}\text{boundary\_dim} &= \frac{(\alpha+1)(\alpha+2)}{2} + \frac{\min\{m-\alpha,\alpha+1\}\max\{m-(\alpha+1),2m-3\alpha-2\}}{2} \\ &= \sum_{j=0}^{\alpha}\left(j+1+\mathcal{N}_j^{\alpha,m}\right) = \sum_{j=0}^{\alpha}\left((\alpha-j)+1+\mathcal{N}_j^{\alpha,m}\right) = \sum_{j=0}^{\alpha}(\alpha_j-(\alpha-j)) \\ &= \sum_{j=0}^{\alpha}\alpha_j - \sum_{j=0}^{\alpha}(\alpha-j) = \sum_{j=0}^{\alpha}\alpha_j - \sum_{j=0}^{\alpha} j = \sum_{j=0}^{\alpha}(\alpha_j-j)\,.\end{aligned}$$

□

The set of functionals defined in Definition 3.38 are indeed degrees of freedom on the local VEM space $V_{h|P}$ as Theorem 3.40 shows.

Theorem 3.40. *The set of linear functionals* $\mathrm{DOF}_h(P)$ *is unisolvent (in the sense of Definition 3.27) on the local VEM space* $V_{h,|P}$

Proof. Let $\Psi : V_{h|P} \to \mathbb{R}^{\mathcal{N}_P^{\alpha,m}}$ be defined by $\Psi(v) = (\psi(v))_{\psi \in \mathrm{DOF}_h(P)}$. To prove unisolvence we have to show that this linear map is bijective.

Since Ψ is linear and $\mathcal{N}_P^{\alpha,m} = \dim V_{h|P}$, it is sufficient to prove that Ψ is injective.

Let $v \in \ker \Psi$. Due to Lemma 3.35 and Lemma 3.37, we get that

$$\frac{\partial^j v}{\partial \mathbf{n}^j} = 0 \text{ on } \partial P, \text{ for } j = 0, \dots, \alpha$$
$$\pi_{m-2}^P v = 0 \text{ on } P.$$

Define the operator $T : \mathbb{P}_{m-2}(P) \to H_0^{\alpha+1}(P)$ by $T(q) = u$, where u is the weak solution of the polyharmonic equation (see (2.9))

$$(-1)^{\alpha+1} \Delta^{\alpha+1} u = q \text{ in } P \tag{3.12a}$$
$$\frac{\partial^j v}{\partial \mathbf{n}^j} = 0 \text{ on } \partial P, \text{ for } j = 0, \dots, \alpha. \tag{3.12b}$$

This map is well defined and, due to the linearity of the polyharmonic problem, is linear. We want to show that $T(0) = v$, which proves the injectivity of Ψ.

We define the linear map $R : \mathbb{P}_{m-2}(P) \to \mathbb{P}_{m-2}(P)$ by

$$R(q) = \pi_{m-2}^P(T(q)) = \pi_{m-2}^P(u).$$

If $q \in \mathbb{P}_{m-2}(P)$ with $R(q) = 0$, then

$$0 = \langle R(q), q \rangle_{L^2(P)} = \langle \pi_{m-2}^P(T(q)), q \rangle_{L^2(P)}. \tag{3.13}$$

As π_{m-2}^P is the L^2-orthogonal projection onto $\mathbb{P}_{m-2}(P)$, (3.13) simplifies to

$$\langle T(q), q \rangle_{L^2(P)} = 0. \tag{3.14}$$

Since $T(q)$ is the weak solution of (3.12), the term (3.14) can be written as

$$\langle \Delta^{(\alpha+1)/2} q, \Delta^{(\alpha+1)/2} q \rangle_{L^2(P)} = 0.$$

We derive the following inequality from the gerneralized Friedrichs inequality (2.10)

$$0 = \langle R(q), q \rangle_{L^2(P)} = \langle \Delta^{(\alpha+1)/2} q, \Delta^{(\alpha+1)/2} q \rangle_{L^2(P)} \geq C_P^{-\alpha-1} \|q\|_{L^2(P)}$$

Thus $q = 0$ and therefore $\ker R = \{0\}$.

Since

$$0 = \pi^P_{m-2}(v) = R((-1)^{\alpha+1}\Delta^{\alpha+1}v)$$

we derive that $(-1)^{\alpha+1}\Delta^{\alpha+1}v = 0$, which means that $T(0) = v$ and therefore $v = 0$.

Thus Ψ is injective.

□

Of course the domain of the degrees of freedom can be extended to include any sufficiently regular function. This is useful for the definition of interpolants.

Definition 3.41. *Let $g \in C^\alpha(\Omega)$, then we define its local interpolant g^{I}_P by*

$$g^{\mathrm{I}}_P = \sum_{\psi \in \mathrm{DOF}_h(P)} \psi(g)\epsilon^P_\psi$$

where $\epsilon^P_\psi, \psi \in \mathrm{DOF}_h(P)$ is the canonical basis corresponding to the local degrees of freedom (see Definition 3.30).

Furthermore we define its global interpolant g^{I} by

$$g^{\mathrm{I}}\big|_P = g^{\mathrm{I}}_P \text{ for all } P \in \Omega_h.$$

Remark 3.42. *Note that the interpolant is well defined for all $g \in H^{\alpha+2}(\Omega)$.*

We cite the following Lemma from [BdV M 14, Lemma 4.3]. It will be useful for the A PRIORI analysis.

Lemma 3.43. *Let $1 + \alpha \leq s \leq m$, $P \in \Omega_h$. There exists a constant $C > 0$ independent of h (and P), such that*

$$\left\|\nabla(v - v^{\mathrm{I}})\right\|_{L^2(P)} \leq Ch^s_P \left\|D^{s+1}v\right\|_{L^2(P)} \text{ for all } v \in H^{s+1}(P).$$

The global degrees of freedom are given by the union of all local degrees of freedom, except those local degrees of freedom that are trivial, which are the evaluations of the functions on the boundary.
Again we distinguish between vertex, edge and internal degrees of freedom.

1. **Vertex degrees of freedom** are

$$\mathcal{V}^h := \bigcup_{P \in \Omega_h} \mathcal{V}_P^h \setminus \{v \mapsto v(\mathbf{x}),\ \mathbf{x} \in \partial\Omega \text{ is a vertex}\}. \tag{3.15}$$

2. **Edge degrees of freedom** are given by a recursive construction. Let $\mathcal{N}(h) := \#\Omega_h$ be the number of polygons and $\{P_1, \ldots, P_{\mathcal{N}(h)}\} = \Omega_h$ be an enumeration of the mesh. We define

$$\mathcal{E}_0^h := \emptyset \text{ and}$$
$$\mathcal{E}_k^h := \mathcal{E}_{k-1}^h \bigcup \left(\mathcal{E}_{P_k}^h \setminus \{\psi \in \mathcal{E}_{P_k}^h \mid -\psi \in \mathcal{E}_{k-1}^h\} \right) \text{ for } k = 1, \ldots, \mathcal{N}(h)$$

The global edge degrees of freedom are given by

$$\mathcal{E}^h := \mathcal{E}_{\mathcal{N}(h)}^h \setminus \{v \mapsto v(\mathbf{x}),\ \mathbf{x} \in \mathbf{e} \subseteq \partial\Omega,\ \mathbf{e} \text{ is an edge}\}. \tag{3.16}$$

Remark 3.44. *The reason we construct the global edge degrees of freedom recursively is that for $P, \tilde{P} \in \Omega_h$ and $\boldsymbol{e} \in \operatorname{edges}(P) \cap \operatorname{edges}(\tilde{P})$ it holds that*

$$\boldsymbol{n}^P = -\boldsymbol{n}^{\tilde{P}} \text{ on } \boldsymbol{e},$$

and by that

$$\frac{\partial^j}{\partial \left(\boldsymbol{n}^P\right)^j} = (-1)^j \frac{\partial^j}{\partial \left(\boldsymbol{n}^{\tilde{P}}\right)^j}$$

Thus simply forming the union over all local edge degrees of freedom $\mathcal{E}_P^h$ would introduce linear dependence and by that redundancy to our global degrees of freedom.

3. **Internal degrees of freedom** are

$$\mathcal{P}^h := \bigcup_{P \in \Omega_h} \mathcal{P}_P^h. \tag{3.17}$$

The global degrees of freedom are given by

$$\mathrm{DOF}_h := \mathcal{V}^h \bigcup \mathcal{E}^h \bigcup \mathcal{P}^h \tag{3.18}$$

Theorem 3.45 (Unisolvence of the global degrees of freedom)**.** *The global degrees of freedom DOF_h, defined by (3.18) are unisolvent on the global VEM space V_h, defined by (3.3).*

Proof. By definition we have to show that the map Ψ defined in Definition 3.27 is bijective.

Ψ is well defined, because every $v \in V_h$ is an element of $C^\alpha(\overline{\Omega})$ (see Theorem 3.26) and by that the evaluation of the same local degrees of freedom for two different polygons have to match, i.e., for $P, \tilde{P} \in \Omega_h$

$$\psi(v|_P) = \psi(v|_{\tilde{P}}) \text{ for } \psi \in \mathrm{DOF}_h(P) \cap \mathrm{DOF}_h(\tilde{P}),$$
$$\psi(v|_P) = -\psi(v|_{\tilde{P}}) \text{ for } \psi \in \mathrm{DOF}_h(P) \cap -\mathrm{DOF}_h(\tilde{P})$$

Let $v \in V_h$ with $\Psi(v) = 0$, then by definition it holds that $\psi(v) = 0$, for all $P \in \Omega_h$ and $\psi \in \mathrm{DOF}_h(P)$. Since the local degrees of freedom $\mathrm{DOF}_h(P)$ are unisolvent this means, that $v|_P = 0$ for every $P \in \Omega_h$, and thus $v = 0$ on Ω. Thus Ψ is injective.

Let $x_\psi \in \mathbb{R}$, $\psi \in \mathrm{DOF}_h$, then for every $P \in \Omega_h$ there is a unique $v_P \in V_{h|P}$ such that for every $\psi \in \mathrm{DOF}_h(P)$ it holds

$$\psi(v_P) = \begin{cases} a_\psi & \text{for } \psi \in \mathrm{DOF}_h, \\ -a_\psi & \text{for } -\psi \in \mathrm{DOF}_h, \\ 0 & \text{else.} \end{cases}$$

The last case differentiation comes from the fact that DOF_h does not include evaluations at the boundary $\partial\Omega$.

Define v by $v|_P := v_P$ for all $P \in \Omega_h$. Since the evaluations of the local degrees of freedom match for different polygons and the edge and vertex degrees of freedom determine the normal derivatives on the boundary of the polygons, one can easily derive that $v \in C^\alpha(\overline{\Omega})$.

Due to Theorem 3.26, this implies $v \in V_h$. Thus we have found $v \in V_h$ such that

$$\psi(v) = a_\psi \text{ for all } \psi \in \mathrm{DOF}_h.$$

Thus, Ψ is surjective and by that the global degrees of freedom are indeed unisolvent.

□

3.4 Construction of the discrete bilinear form $\mathcal{A}_h$

Now that we have defined the VEM space and its degrees of freedom, we construct an approximation $\mathcal{A}_h$ to the scalar product $a : V_h \times V_h \to \mathbb{R}$, with
$a(u, v) := \langle A\nabla u, \nabla v\rangle_{L^2(\Omega)}$.
We accomplish this by splitting up $\mathcal{A}_h$ into local bilinear forms $\mathcal{A}_{h,P}$, for $P \in \Omega_h$, operating on $V_{h|P}$. This means that we define

$$\mathcal{A}_h(w, v) := \sum_{P\in\Omega_h} \mathcal{A}_{h|P}(w, v) \text{ for all } w, v \in V_h. \tag{3.19}$$

Here each $\mathcal{A}_{h,P}$ is a bilinear form on $V_{h|P}$ satisfying following the three conditions:

1. **Symmetry:** For all $w_h, v_h \in V_{h|P}$ it holds that

$$\mathcal{A}_{h|P}(w_h, v_h) = \mathcal{A}_{h|P}(v_h, w_h).$$

2. **Consistency:** For every polynom $p \in \mathbb{P}_m(P)$ and all $v_h \in V_{h|P}$, it holds that

$$\mathcal{A}_{h|P}(p, v_h) = \langle \pi_{m-1}^P(A\nabla p), \nabla v_h \rangle_{L^2(P)}. \tag{3.20}$$

Remark 3.46. *Note that if A is constant on P, the consistency condition (3.20) reduces to*

$$\mathcal{A}_{h|P}(p, v_h) = \langle A\nabla p, \nabla v_h \rangle_{L^2(P)}.$$

3. **Stability:** There exist $a^*, a_* > 0$, both independent of the maximal diameter h and the polygon P, such that, for all $v_h \in V_{h|P}$, it holds that

$$a_* \langle A\nabla v_h, \nabla v_h \rangle_{L^2(P)} \le \mathcal{A}_{h|P}(v_h, v_h) \le a^* \langle A\nabla v_h, \nabla v_h \rangle_{L^2(P)}. \tag{3.21}$$

Before we discuss the construction of such an $\mathcal{A}_h$, we delve into its properties.

3.4.1 Properties of $\mathcal{A}_h$

Theorem 3.47. *For any given $v_h \in V_{h|P}$ and $p \in \mathbb{P}_m(P)$, $\mathcal{A}_{h|P}(p, v_h)$ can be computed exactly by the local degrees of freedom $\mathrm{DOF}_h(P)$, i.e., there exist computable scalar values $(\alpha_\Psi)_{\Psi \in \mathrm{DOF}_h(P)}$ such that*

$$\mathcal{A}_{h|P}(p, v_h) = \sum_{\Psi \in \mathrm{DOF}_h(P)} \alpha_\Psi \; \Psi(v_h).$$

Proof. Using the consistency (3.20) and integrating by parts, we get that

$$\begin{aligned} \mathcal{A}_{h|P}(p, v_h) &= \langle \pi_{m-1}^P(A\nabla p), \nabla v_h \rangle_{L^2(P)} \\ &= \langle \pi_{m-1}^P(A\nabla p) \cdot \mathbf{n}, v_h \rangle_{L^2(\partial P)} - \langle \mathrm{div}(\pi_{m-1}^P(A\nabla p)), v_h \rangle_{L^2(P)}. \end{aligned} \tag{3.22}$$

By splitting up the boundary of ∂P into its edges, this can be written as

$$\sum_{\mathbf{e} \in \mathrm{edges}(P)} \langle \pi_{m-1}^P(A\nabla p) \cdot \mathbf{n}, v_h \rangle_{L^2(\mathbf{e})} - \langle \mathrm{div}(\pi_{m-1}^P(A\nabla p), v_h \rangle_{L^2(P)}.$$

Let $\mathbf{e}$ be an arbitrary edge of P. According to the proof of Lemma 3.35, $v_h|_{\mathbf{e}}$ is the Hermite interpolant to certain local vertex and edge degrees of freedom. Thus, there are computable scalar values (corresponding to the quadrature using the Hermite interpolant) $\beta_\Psi^{\mathbf{e}}$, $\Psi \in \mathcal{V}_P^h \bigcup \mathcal{E}_P^h$, such that

$$\langle \pi^P_{m-1}(A\nabla p)\cdot \mathbf{n}, v_h\rangle_{L^2(\mathbf{e})} = \int_{\mathbf{e}} \pi^P_{m-1}(A\nabla p)\cdot \mathbf{n}\ v_h\ \mathrm{d}s = \sum_{\Psi\in\mathcal{V}^h_P\bigcup\mathcal{E}^h_P} \beta^{\mathbf{e}}_\Psi \Psi(v_h).$$

If we define

$$\alpha_\Psi := \sum_{\mathbf{e}\in\mathrm{edges}(P)} \beta^{\mathbf{e}}_\Psi,$$

the first term in (3.22) can be written as

$$\langle \pi^P_{m-1}(A\nabla p)\cdot \mathbf{n}, v_h\rangle_{L^2(\partial P)} = \sum_{\Psi\in\mathcal{V}^h_P\bigcup\mathcal{E}^h_P} \alpha_\Psi \Psi(v_h).$$

As $\mathrm{div}(\pi^P_{m-1}(A\nabla p)) \in \mathbb{P}_{m-2}(P)$, we can write it as

$$\mathrm{div}(\pi^P_{m-1}(A\nabla p)) = \sum_{q\in\mathcal{M}_{m-2}} \beta_q\ q.$$

As each monomial $q \in \mathcal{M}$ corresponds to an internal degree of freedom $\Psi \in \mathcal{P}^h_P$ (see (3.9)), we derive

$$-\langle \mathrm{div}(\pi^P_{m-1}(A\nabla p)), v_h\rangle_{L^2(P)} = \sum_{q\in\mathcal{M}_{m-2}} -\beta_q\ \langle q, v_h\rangle_{L^2(P)} = \sum_{\Psi\in\mathcal{P}^h_P} -\beta_\Psi |P|\Psi(v_h).$$

If we define $\alpha_\Psi := -\beta_\Psi|P|$ for $\Psi \in \mathcal{P}^h_P$, (3.22) reduces to

$$\mathcal{A}_{h|P}(p, v_h) = \sum_{\Psi\in\mathcal{V}^h_P\bigcup\mathcal{E}^h_P} \alpha_\Psi \Psi(v_h) + \sum_{\Psi\in\mathcal{P}^h_P} \alpha_\Psi \Psi(v_h) = \sum_{\Psi\in\mathrm{DOF}_h(P)} \alpha_\Psi \Psi(v_h).$$

□

Lemma 3.48. *Let $p, q \in \mathbb{P}_m(P) \subseteq V_{h|P}$, then*

$$\mathcal{A}_{h|P}(p, q) = \langle A\nabla p, \nabla q\rangle_{L^2(P)}.$$

Thus, for two piecewise polynomial functions v, w, $\mathcal{A}_h$ corresponds to the scalar product $\mathcal{A} = \langle \nabla\cdot, \nabla\cdot\rangle_{L^2(\Omega)}$, i.e.,

$$\mathcal{A}_h(v, w) = \mathcal{A}(v, w).$$

Proof. As every component of ∇q is an element of $\mathbb{P}_{m-1}(P)$, we get that

$$\langle A\nabla p, \nabla q\rangle_{L^2(P)} = \langle \pi^P_{m-1}(A\nabla p), \nabla q\rangle_{L^2(P)}.$$

Due to the consistency property (3.20), this is just $\mathcal{A}_{h|P}(p, q)$.

□

Lemma 3.49. *There exists a constant $C = C(\alpha^*, \|A\|_\infty) > 0$ independent of h and P, such that*

$$|\mathcal{A}_{h|P}(v_h, w_h)| \leq C \|\nabla v_h\|_{L^2(P)} \|\nabla w_h\|_{L^2(P)} \tag{3.23}$$

Proof. Let $t := \frac{\mathcal{A}_{h|P}(v_h,v_h)}{\mathcal{A}_{h|P}(v_h,w_h)}$. Due to the symmetry and the stability (3.21), we get that

$$\begin{aligned}
0 &\leq \alpha_* \langle A\nabla(v_h - tw_h), \nabla(v_h - tw_h)\rangle_{L^2(P)} \leq \mathcal{A}_{h|P}(v_h - tw_h, v_h - tw_h) \\
&= \mathcal{A}_{h|P}(v_h, v_h) - 2t\mathcal{A}_{h|P}(v_h, w_h) + t^2 \mathcal{A}_{h|P}(w_h, w_h) \\
&= \mathcal{A}_{h|P}(v_h, v_h) - 2\frac{\mathcal{A}_{h|P}(v_h, v_h)}{\mathcal{A}_{h|P}(v_h, w_h)}\mathcal{A}_{h|P}(v_h, w_h) + \frac{\mathcal{A}_{h|P}(v_h, v_h)^2}{\mathcal{A}_{h|P}(v_h, w_h)^2}\mathcal{A}_{h|P}(w_h, w_h) \\
&= -\mathcal{A}_{h|P}(v_h, v_h) + \frac{\mathcal{A}_{h|P}(v_h, v_h)^2}{\mathcal{A}_{h|P}(v_h, w_h)^2}\mathcal{A}_{h|P}(w_h, w_h).
\end{aligned}$$

Thus we derive

$$\mathcal{A}_{h|P}(v_h, v_h) \leq \frac{\mathcal{A}_{h|P}(v_h, v_h)^2}{\mathcal{A}_{h|P}(v_h, w_h)^2}\mathcal{A}_{h|P}(w_h, w_h),$$

which is equivalent to

$$\mathcal{A}_{h|P}(v_h, w_h)^2 \leq \mathcal{A}_{h|P}(v_h, v_h)\mathcal{A}_{h|P}(w_h, w_h). \tag{3.24}$$

By using the stability condition and taking the square root, (3.24) transforms to

$$\begin{aligned}
|\mathcal{A}_{h|P}(v_h, w_h)| &\leq \sqrt{\mathcal{A}_{h|P}(v_h, v_h)\mathcal{A}_{h|P}(w_h, w_h)} \leq \sqrt{\alpha^* \langle A\nabla v_h, \nabla v_h\rangle_{L^2(P)} \langle A\nabla w_h, \nabla w_h\rangle_{L^2(P)}} \\
&\leq \sqrt{\alpha^*}\|A\|_\infty \|\nabla v_h\|_{L^2(P)} \|\nabla w_h\|_{L^2(P)}.
\end{aligned}$$

Both α^* and $\|A\|_\infty$ are independant of h and P.

□

Theorem 3.50. *$\mathcal{A}_h$ is a symmetric, continuous and coercive bilinear form, with respect to both $\|.\|_{H^1(\Omega)}$ and the norm induced by $\mathcal{A}$. It is thus a scalar product.*
Both the coercivity constant and the continuity constant do not depend on h.

Proof. As the sum of local bilinear forms, $\mathcal{A}_h$ clearly is a bilinear form. The symmetry follows directly from the symmetry of those local bilinear forms.

Let $v_h, w_h \in V_h$ be arbitrary, then by Lemma 3.49 it holds that

$$|\mathcal{A}(v_h, w_h)| = \left| \sum_{P\in\Omega_h} \mathcal{A}_{h|P}(v_h, w_h) \right| \leq \sum_{P\in\Omega_h} |\mathcal{A}_{h|P}(v_h, w_h)| \leq \sum_{P\in\Omega_h} C \|\nabla v_h\|_{L^2(P)} \|\nabla w_h\|_{L^2(P)}.$$

Note that C is independent of h. By using the Cauchy Schwarz inequality for sums (1.7), we get

$$C\sum_{P\in\Omega_h}\|\nabla v_h\|_{L^2(P)}\|\nabla w_h\|_{L^2(P)} \leq C\sqrt{\sum_{P\in\Omega_h}\|\nabla v_h\|_{L^2(P)}^2}\sqrt{\sum_{P\in\Omega_h}\|\nabla w_h\|_{L^2(P)}^2}$$
$$= C\|\nabla v_h\|_{L^2(\Omega)}\|\nabla w_h\|_{L^2(\Omega)}.$$

This gives us

$$|\mathcal{A}(v_h,w_h)| \leq C\|\nabla v_h\|_{L^2(\Omega)}\|\nabla w_h\|_{L^2(\Omega)} = C\sqrt{\mathcal{A}(v_h,v_h)}\sqrt{\mathcal{A}(w_h,w_h)} \leq C\|v_h\|_{H^1(\Omega)}\|w_h\|_{H^1(\Omega)}.$$

Thus $\mathcal{A}_h$ is continuous with respect to both $\|.\|_{H^1(\Omega)}$ and the norm induced by $\mathcal{A}$.

For any $v_h \in V_h$, the stability condition (3.21) and the uniform κ-ellipticity of A gives us

$$\mathcal{A}(v_h,v_h) = \sum_{P\in\Omega_h}\mathcal{A}_{h|P}(v_h,v_h) \geq \sum_{P\in\Omega_h}\alpha_*\langle Av_h, v_h\rangle_{L^2(P)}$$
$$= \alpha_*\langle A\nabla v_h,\nabla v_h\rangle_{L^2(\Omega)} \geq \alpha_*\kappa\|\nabla v_h\|_{L^2(\Omega)}^2 = \alpha_*\kappa\ \mathcal{A}(v_h,v_h).$$

Thus $\mathcal{A}_h$ is coercive with respect to $\mathcal{A}$ and due to the Friedrichs inequality (Theorem 1.36) also with respect to $\|.\|_{H^1(\Omega)}$. Neither α_*, κ nor the Poincare constant depend on h. □

Corollary 3.51. *For any continuous linear form $F_h : V_h \to \mathbb{R}$ there is a unique $u_h \in V_h$, such that*

$$\mathcal{A}_h(u_h,v_h) = F_h(v_h) \text{ for all } v_h \in V_h.$$

Proof. This is a consequence of Theorem 3.50 and the Lax–Milgram theorem (Theorem 1.10) □

3.4.2 The projection Π_m^P

Definition 3.52. *Let $P \in \Omega_h$. We define the average vertex value of a continuous function $g \in C(\overline{P})$ by*

$$\overline{g}^P := \frac{1}{\#\text{vertices}(P)}\sum_{\boldsymbol{x}\in\text{vertices}(P)} g(\boldsymbol{x}). \tag{3.25}$$

We cite the following Lemma from [BBCMMR, page 212].

Lemma 3.53. *Assume $\alpha = 0$. Then there exists a constant independent of h such that*

$$\left\|v_h - \overline{v_h}^P\right\|_{L^2(P)} \leq Ch_P\|\nabla v_h\|_{L^2(P)} \text{ for all } v_h \in V_h.$$

Next we want to define a special polynomial projection. For this we need the following lemma.

Lemma 3.54. *For any $v_h \in V_{h|P}$ there is a unique $p \in \mathbb{P}_m(P)$ such that*

$$\begin{aligned} \langle A\nabla p, \nabla q\rangle_{L^2(P)} &= \langle \pi^P_{m-1}(A\nabla q), \nabla v_h\rangle_{L^2(P)} \text{ for all } q \in \mathbb{P}_m(P) \\ \overline{p}^P &= \overline{v_h}^P. \end{aligned} \tag{3.26}$$

Proof. Let $v_h \in V_{h|P}$ be arbitrary. Of course it is enough to only consider monomials $q \in \mathcal{M}_m$ (see (3.8)) in the first equation of (3.26).
Thus the first equation of (3.26) reduces to

$$\langle A\nabla p, \nabla q\rangle_{L^2(P)} = \langle \pi^P_{m-1}(A\nabla q), \nabla v_h\rangle_{L^2(P)} = \langle A\nabla q, \nabla \pi^P_{m-1}(v_h)\rangle_{L^2(P)} \text{ for } q \in \mathcal{M}_m.$$

This gives us $\dim(\mathcal{M}_m) - 1$ linear equations for p, as it is zero on both sides for $q = 1$.

Thus, together with the equation for $\overline{p}^P$, (3.26) consists of $\dim \mathcal{M}_m$ linear equations for the same number of unknown coefficients and is therefore a quadratic system.

This means, that the existence of solutions is equivalent to their uniqueness and therefore we are left to prove, that $p = 0$ is the only trivial solution.

Let $v_h = 0$ and $p \in \mathbb{P}_m(P)$ be a solution of (3.26), i.e.

$$\begin{aligned} \langle A\nabla p, \nabla q\rangle_{L^2(P)} &= 0 \text{ for all } q \in \mathbb{P}_m(P) \\ \overline{p}^P &= 0. \end{aligned}$$

Then by setting $q = p$ and by using to the κ-ellipticity of A, we get

$$0 = \langle A\nabla p, \nabla p\rangle_{L^2(P)} \geq \kappa \left\|\nabla p\right\|_{L^2(P)}.$$

This implies $\nabla p = 0$, which in turn implies that p is constant. But since $\overline{p}^P = 0$, we get that $p = 0$.

Thus 0 is the only trivial solution to (3.26) and therefore there exists a unique solution to every right hand side.

□

Definition 3.55. *We define the map $\Pi^P_m : V_{h|P} \to \mathbb{P}_m(P)$ by $\Pi^P_m(v_h) := p$ for $v_h \in V_{h|P}$, where $p \in \mathbb{P}_m(P)$ is the unique solution of (3.26).*

Lemma 3.56. *$\Pi^P_m : V_{h|P} \to \mathbb{P}_m(P)$ is a linear projection onto $\mathbb{P}_m(P)$, i.e. it is a linear operator that satisfies*

$$\Pi^P_m \circ \Pi^P_m = \Pi^P_m. \tag{3.27}$$

Proof. As $\Pi_m^P(v_h)$ is the solution of a finite dimensional linear equation, one can easily verify that Π_m^P is indeed a linear operator.

Let $v_h \in \mathbb{P}_m(P)$ be an arbitrary polynomial. We set $p := v_h$. It is clear that p is the solution of (3.26) and thus, $\Pi_m^P(v_h) = v_h$.

Since $\Pi_m^P(v_h) \in \mathbb{P}_m(P)$ for arbitrary $v_h \in V_{h|P}$, this implies (3.27).

□

Remark 3.57. *Let $v_h, w_h \in V_{h|P}$, then by definition it holds that*

$$\langle A\nabla\Pi_m^P v_h, \nabla\Pi_m^P w_h\rangle_{L^2(P)} = \langle \pi_{m-1}^P(A\nabla\Pi_m^P w_h), \nabla v_h\rangle_{L^2(P)}, \tag{3.28}$$

which in general is not the same as

$$\langle A\nabla\Pi_m^P w_h, \nabla v_h\rangle_{L^2(P)}. \tag{3.29}$$

Thus there might be $v_h, w_h \in V_{h|P}$ such that $\langle A\nabla\Pi_m^P w_h, \nabla(v_h - \Pi_m^P)\rangle_{L^2(P)} \neq 0$ and therefore Π_m^P is not necessarily an $\langle A\nabla.,\nabla.\rangle_{L^2(P)}$-orthogonal projection.

If however A is constant on P, then $A\nabla\Pi_m^P w_h \in \mathbb{P}_{m-1}(P)$, which means that (3.28) is equal to (3.29) and thus Π_m^P is an $\langle A\nabla.,\nabla.\rangle_{L^2(P)}$-orthogonal projection.

Assumption 4. *For the rest of this section we shall assume, that A is piecewise constant, i.e. $A|_P \in \mathbb{P}_0(P)$ for all $P \in \Omega_h$.*
Thus due to Remark 3.57 Π_m^P is a orthogonal projection.

Before we finish this section with the construction of a suitable local bilinear form $\mathcal{A}_{h|P}$, we want to show the following approximation property of the projection Π_m^P.

Lemma 3.58. *Let $s \in \mathbb{N}$. There exists a constant $C > 0$ independent of h such that*

$$\left\|\nabla v - \nabla\Pi_m^P(v)\right\|_{L^2(P)} \leq Ch^s \left\|D^{s+1}v\right\|_{L^2(P)} \text{ for all } v \in H^{s+1}(P).$$

Proof. Let $v \in H^{s+1}(P)$. By using the triangle inequality, the projection property of Π_m^P and the uniform κ-ellipticity of A we derive

$$\begin{aligned}
\left\|\nabla v - \nabla\Pi_m^P(v)\right\|_{L^2(P)} &\leq \left\|\nabla\left(v - \pi_m^P(v)\right)\right\|_{L^2(P)} + \left\|\nabla\left(\pi_m^P(v) - \Pi_m^P(v)\right)\right\|_{L^2(P)} \\
&= \left\|\nabla\left(v - \pi_m^P(v)\right)\right\|_{L^2(P)} + \left\|\nabla\Pi_m^P\left(\pi_m^P(v) - v\right)\right\|_{L^2(P)} \\
&\lesssim \left\|\nabla\left(v - \pi_m^P(v)\right)\right\|_{L^2(P)} + \left\|\sqrt{A}\nabla\Pi_m^P\left(\pi_m^P(v) - v\right)\right\|_{L^2(P)}.
\end{aligned}$$

Since Π_m^P is an $\langle A\nabla ., \nabla . \rangle_{L^2(P)}$-orthogonal projection, we get

$$\begin{aligned}\left\| \sqrt{A}\nabla\Pi_m^P\left(\pi_m^P(v) - v\right)\right\|_{L^2(P)} &= \sqrt{\langle A\nabla\Pi_m^P(\pi_m^P(v) - v), \nabla\Pi_m^P(\pi_m^P(v) - v)\rangle_{L^2(P)}} \\ &\leq \sqrt{\langle A\nabla(\pi_m^P(v) - v), \nabla(\pi_m^P(v) - v)\rangle_{L^2(P)}} \\ &\lesssim \left\|\nabla\left(\pi_m^P(v) - v\right)\right\|_{L^2(P)}\end{aligned}$$

Therefore we derive that

$$\left\|\nabla v - \nabla\Pi_m^P(v)\right\|_{L^2(P)} \lesssim \left\|\nabla\left(v - \pi_m^P(v)\right)\right\|_{L^2(P)},$$

and by using Lemma 3.11, we can conclude

$$\left\|\nabla v - \nabla\Pi_m^P(v)\right\|_{L^2(P)} \lesssim h^s \left\|D^{s+1}v\right\|_{L^2(P)}.$$

□

3.4.3 Construction of $\mathcal{A}_{h|P}$

Now we are able to define the local bilinear forms $\mathcal{A}_{h|P}$. We split them up into two parts. One ensures the consistency (3.20) and the other one guarantees the stability (3.21).
The part that guarantees stability only needs to satisfy a weakened stability property itself and therefore we have some freedom on how to choose it. We thus keep it general before specifying a concrete choice later on.

Definition 3.59. *Let $\mathcal{S}^P$ be a scalar product on the local VEM space $V_{h|P}$ such that there exist constants $c_0, c_1 > 0$, independent of h and P that satisfy*

$$c_0\langle A\nabla v_h, \nabla v_h\rangle_{L^2(P)} \leq \mathcal{S}^P(v_h, v_h) \leq c_1\langle A\nabla v_h, \nabla v_h\rangle_{L^2(P)} \text{ for all } v_h \in \ker\Pi_m^P. \tag{3.30}$$

Then we define for $v_h, w_h \in V_{h|P}$

$$\mathcal{A}_{h|P}(v_h, w_h) := \langle A\nabla\Pi_m^P(v_h), \nabla\Pi_m^P(w_h)\rangle_{L^2(P)} \; + \; \mathcal{S}^P(v_h - \Pi_m^P(v_h), w_h - \Pi_m^P(w_h)). \tag{3.31}$$

Theorem 3.60. *$\mathcal{A}_{h|P}$ is a symmetric bilinear form that satisfies the stability (3.21) and consistency (3.20) property.*

Proof. The bilinearity and symmetry of $\mathcal{A}_{h|P}$ are direct consequences of the bilinearity and symmetry of both $\langle A\nabla\cdot, \nabla\cdot\rangle_{L^2(P)}$ and $\mathcal{S}^P(\cdot,\cdot)$.

Let $p \in \mathbb{P}_m(P)$ and $v_h \in V_{h|P}$ be arbitrary. Since Π_m^P is a projection onto $\mathbb{P}_m(P)$, it holds that

$$p = \Pi_m^P(p),$$

and thus we get that

$$\mathcal{S}^P(p - \Pi_m^P(p), v_h - \Pi_m^P(v_h)) = 0.$$

Thus by using the projection property of Π_m^P and (3.26), we get

$$\begin{aligned}\mathcal{A}_{h|P}(p, v_h) &= \langle A\nabla\Pi_m^P(p), \nabla\Pi_m^P(v_h)\rangle_{L^2(P)} = \langle A\nabla p, \nabla\Pi_m^P(v_h)\rangle_{L^2(P)} \\ &= \langle A\nabla\Pi_m^P(v_h), \nabla p\rangle_{L^2(P)} = \langle \pi_{m-1}^P(A\nabla p), \nabla v_h\rangle_{L^2(P)}.\end{aligned}$$

Therefore $\mathcal{A}_{h|P}$ satisfies the consistency property (3.20).

We look at

$$\mathcal{A}_{h|P}(v_h, v_h) = \langle A\nabla\Pi_m^P(v_h), \nabla\Pi_m^P(v_h)\rangle_{L^2(P)} \ + \ \mathcal{S}^P(v_h - \Pi_m^P(v_h), v_h - \Pi_m^P(v_h)).$$

It is clear that $v_h - \Pi_m^P(v_h) \in \ker \Pi_m^P$ and due to (3.30) we get

$$\begin{aligned}&\mathcal{A}_{h|P}(v_h, v_h) \leq \langle A\nabla\Pi_m^P(v_h), \nabla\Pi_m^P(v_h)\rangle_{L^2(P)} \ + \ c_1\langle A\nabla(v_h - \Pi_m^P(v_h)), \nabla(v_h - \Pi_m^P(v_h))\rangle_{L^2(P)} \\ &\leq \max\{1, c_1\} \ (\langle A\nabla\Pi_m^P(v_h), \nabla\Pi_m^P(v_h)\rangle_{L^2(P)} \ + \ \langle A\nabla(v_h - \Pi_m^P(v_h)), \nabla(v_h - \Pi_m^P(v_h))\rangle_{L^2(P)}).\end{aligned}$$

Due to Assumption 4 and Remark 3.57, Π_m^P is an $\langle A\nabla\cdot, \nabla\cdot\rangle_{L^2(P)}$-orthogonal projection and thus

$$\langle A\nabla\Pi_m^P(v_h), \nabla\Pi_m^P(v_h)\rangle_{L^2(P)} \ + \ \langle A\nabla(v_h - \Pi_m^P(v_h)), \nabla(v_h - \Pi_m^P(v_h))\rangle_{L^2(P)} = \langle A\nabla v_h, \nabla v_h\rangle_{L^2(P)},$$

and therefore it holds that

$$\mathcal{A}_{h|P}(v_h, v_h) \leq \max\{1, c_1\}\langle A\nabla v_h, \nabla v_h\rangle_{L^2(P)}$$

By the same arguments one obtains the lower estimate

$$\begin{aligned}&\mathcal{A}_{h|P}(v_h, v_h) \geq \langle A\nabla\Pi_m^P(v_h), \nabla\Pi_m^P(v_h)\rangle_{L^2(P)} \ + \ c_0\langle A\nabla(v_h - \Pi_m^P(v_h)), \nabla(v_h - \Pi_m^P(v_h))\rangle_{L^2(P)} \\ &\geq \min\{1, c_0\} \ (\langle A\nabla\Pi_m^P(v_h), \nabla\Pi_m^P(v_h)\rangle_{L^2(P)} \ + \ \langle A\nabla(v_h - \Pi_m^P(v_h)), \nabla(v_h - \Pi_m^P(v_h))\rangle_{L^2(P)}) \\ &\qquad = \min\{1, c_0\}\langle A\nabla v_h, \nabla v_h\rangle_{L^2(P)}\end{aligned}$$

Since both c_0 and c_1 are independent of h and P, the local bilinear form $\mathcal{A}_{h|P}$ satisfies the stability condition (3.21).

□

A particular choice for $\mathcal{S}^P$ in Definition 3.59 is

$$\mathcal{S}^P(v_h, w_h) := \sum_{\psi \in \mathrm{DOF}_h(P)} \psi(v_h)\psi(w_h). \tag{3.32}$$

We refer to [BdV M 14, page 769] and [BdVLoRu, Lemma 3.1] for a dissusion of the stability property required by Definition 3.59.

Definition 3.61. *Thus a possible choice for the local discrete bilinear form is given by*

$$\mathcal{A}_{h|P}(v_h, w_h) := \left\langle A\nabla\Pi_m^P(v_h), \nabla\Pi_m^P(w_h)\right\rangle_{L^2(P)} + \sum_{\psi \in \mathrm{DOF}_h(P)} \psi\left(v_h - \Pi_m^P(v_h)\right)\psi\left(w_h - \Pi_m^P(w_h)\right)$$

3.5 Construction of the loading term and the discrete equation

Let $f \in L^2(\Omega)$ be the right hand side of (2.1). We now define the discrete loading term f_h.

Definition 3.62. *For the definition of the loading term, we have to distinguish two cases.*

1. $\underline{\mathbf{m=1}}$ *We define f_h elementwise by*

$$f_h|_P = \pi_0^P(f) \text{ on } P,$$

for each $P \in \Omega_h$.

Furthermore we define the corresponding linear functional $F_h : V_h \to \mathbb{R}$ by

$$F_h(v_h) := \sum_{P \in \Omega_h} \langle f_h, \overline{v_h}^P\rangle_{L^2(P)} = \sum_{P \in \Omega_h} |P|\pi_0^P(f)\overline{v_h}^P,$$

where $\overline{v_h}^P$ is the average vertex value defined in (3.25).

2. $\underline{\mathbf{m \geq 2}}$ *Again we define f_h elementwise by*

$$f_h|_P = \pi_{m-2}^P(f) \text{ on } P,$$

for each $P \in \Omega_h$.

The corresponding linear functional $F_h : V_h \to \mathbb{R}$ is defined by

$$F_h(v_h) := \sum_{P \in \Omega_h} \langle f_h, v_h\rangle_{L^2(P)} = \sum_{P \in \Omega_h} \langle f, \pi_{m-2}^P(v_h)\rangle_{L^2(P)}.$$

Lemma 3.63. *F_h is a continuous linear function on V_h and thus there exists a unique $u_h \in V_h$ such that*

$$\mathcal{A}_h(u_h, v_h) = F_h(v_h) \text{ for all } v_h \in V_h. \tag{3.33}$$

Proof. The linearity of F_h can easily be seen from the definition. Since F_h is a linear map on the finite dimensional Hilbert space V_h, it is automatically also continuous. Thus due to Corollary 3.51 there exists a unique solution u_h.

□

The following theorem will be useful for deriving A PRIORIestimates in the next chapter.

Theorem 3.64. *Let s be a natural number such that $1+\alpha \leq s \leq m$ and assume that $f|_P \in H^{s-1}(P)$ for all $P \in \Omega_h$.*
Then there exists a constant $C > 0$ independent of h such that

$$\left|F_h(v_h) - \langle f, v_h\rangle_{L^2(\Omega)}\right| \leq Ch^s \sqrt{\sum_{P\in\Omega_h} \|D^{s-1}f\|^2_{L^2(P)}}\, \|\nabla v_h\|_{L^2(\Omega)} \text{ for all } v_h \in V_h.$$

If $f \in H^{s-1}(\Omega)$, this reduces to

$$\left|F_h(v_h) - \langle f, v_h\rangle_{L^2(\Omega)}\right| \leq Ch^s \left\|D^{s-1}f\right\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)} \text{ for all } v_h \in V_h.$$

Proof. We have to consider the two different definitions of F_h. Thus we look at two cases.

<u>**First case: m = 1**</u> (this implies $\alpha = 0$, $s = 1$)

As $\overline{v_h}^P \in \mathbb{P}_0(P)$, we get that

$$\langle f_h, \overline{v_h}^P\rangle_{L^2(P)} = \langle f, \overline{v_h}^P\rangle_{L^2(P)} + \underbrace{\langle \pi_0^P(f) - f, \overline{v_h}^P\rangle_{L^2(P)}}_{=0} = \langle f, \overline{v_h}^P\rangle_{L^2(P)}.$$

By using the triangle inequality and the Cauchy–Schwarz inequality (1.6) for the $L^2(P)$-scalar product, we derive

$$\begin{aligned}\left|F_h(v_h) - \langle f, v_h\rangle_{L^2(\Omega)}\right| &= \left|\sum_{P\in\Omega_h} \langle f_h, \overline{v_h}^P\rangle_{L^2(P)} - \sum_{P\in\Omega_h} \langle f, v_h\rangle_{L^2(P)}\right| \\ &\leq \sum_{P\in\Omega_h} \left|\langle f, \overline{v_h}^P - v_h\rangle_{L^2(P)}\right| \leq \sum_{P\in\Omega_h} \|f\|_{L^2(P)} \left\|\overline{v_h}^P - v_h\right\|_{L^2(P)}.\end{aligned}$$

Finally using Lemma 3.53 and the Cauchy–Schwarz inequality for sums (1.7), we get

$$\begin{aligned}\left|F_h(v_h) - \langle f, v_h\rangle_{L^2(\Omega)}\right| &\leq \sum_{P\in\Omega_h} \|f\|_{L^2(P)} \left\|\overline{v_h}^P - v_h\right\|_{L^2(P)} \leq \sum_{P\in\Omega_h} Ch_P \|f\|_{L^2(P)} \|\nabla v_h\|_{L^2(P)} \\ &\leq Ch \sqrt{\sum_{P\in\Omega_h} \|f\|^2_{L^2(P)}} \sqrt{\sum_{P\in\Omega_h} \|\nabla v_h\|^2_{L^2(P)}} = Ch \|f\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)}.\end{aligned}$$

C is independent of h.

Second case: $\mathbf{m \geq 2}$

By definition it holds that

$$\left|F_h(v_h) - \langle f, v_h\rangle_{L^2(\Omega)}\right| = \left|\sum_{P\in\Omega_h} \langle f_h, v_h\rangle_{L^2(P)} - \sum_{P\in\Omega_h} \langle f, v_h\rangle_{L^2(P)}\right| = \left|\sum_{P\in\Omega_h} \langle \pi^P_{m-2}(f) - f, v_h\rangle_{L^2(P)}\right|$$
$$\leq \sum_{P\in\Omega_h} \left|\langle \pi^P_{m-2}(f) - f, v_h\rangle_{L^2(P)}\right|.$$

Since $\pi_0^P(v_h) \in \mathbb{P}_0(P) \subseteq \mathbb{P}_{m-2}(P)$, it holds that

$$\langle \pi^P_{m-2}(f) - f, \pi_0^P(v_h)\rangle_{L^2(P)} = 0.$$

Thus we get that

$$\left|F_h(v_h) - \langle f, v_h\rangle_{L^2(\Omega)}\right| \leq \sum_{P\in\Omega_h} \left|\langle \pi^P_{m-2}(f) - f, v_h\rangle_{L^2(P)}\right| = \sum_{P\in\Omega_h} \left|\langle \pi^P_{m-2}(f) - f, v_h - \pi_0^P(v_h)\rangle_{L^2(P)}\right|.$$

By using first the Cauchy–Schwarz inequality (1.6) on the $L^2(P)$-scalar product, then the inequality for the projection error (3.1) and finally the Cauchy–Schwarz inequality for sums (1.7), we get that

$$\left|F_h(v_h) - \langle f, v_h\rangle_{L^2(\Omega)}\right| \leq \sum_{P\in\Omega_h} \left|\langle \pi^P_{m-2}(f) - f, v_h - \pi_0^P(v_h)\rangle_{L^2(P)}\right|$$
$$\leq \sum_{P\in\Omega_h} \left\|\pi^P_{m-2}(f) - f\right\|_{L^2(P)} \left\|v_h - \pi_0^P(v_h)\right\|_{L^2(P)} \leq \sum_{P\in\Omega_h} Ch^{s-1} \left\|D^{s-1} f\right\|_{L^2(P)} h \left\|\nabla v_h\right\|_{L^2(P)}$$
$$\leq Ch^s \sqrt{\sum_{P\in\Omega_h} \|D^{s-1} f\|^2_{L^2(P)}} \sqrt{\sum_{P\in\Omega_h} \|\nabla v_h\|^2_{L^2(P)}} = Ch^s \sqrt{\sum_{P\in\Omega_h} \|D^{s-1} f\|^2_{L^2(P)}} \|\nabla v_h\|^2_{L^2(\Omega)}.$$

C is independent of h.

□

3.6 The stiffness matrix

Just as before we construct the stiffness matrix first locally on a single polygon and then globally.

3.6.1 The local stiffness matrix

Let $P \in \Omega_h$ be fixed throughout this subsection.

Definition 3.65. *We define* $\mathcal{M} := \frac{(m+1)(m+2)}{2} = \dim(\mathbb{P}_m(P))$.

Throughout this subsection we enumerate the $\mathcal{M}$ *polynomials in* $\mathcal{M}_m$ *(see Definition 3.8) and denote them by* $p_1^P, \dots, p_{\mathcal{M}}^P$. *Hereby* p_1^P *shall denote the constant* 1.

Furthermore we enumerate the local degrees of freedom $\mathrm{DOF}_h(P)$ *in such a way, that the first* $\#\mathrm{vertices}(P)$ *degrees of freedom are the function evaluations at the vertices, i.e., that* $\left\{\psi_1^P, \dots, \psi_{\mathcal{N}_P^{\alpha,m}}^P\right\} = \mathrm{DOF}_h(P)$ *with*

$$\psi_j^P(v) = v(\boldsymbol{x}_{j-1}) \text{ for all } v \in V_{h|P},\ j = 1, \dots, \#\mathrm{vertices}(P).$$

The canonical basis of $V_{h|P}$ *(see Definition 3.30) shall be denoted by* $\epsilon_1^P, \dots, \epsilon_{\mathcal{N}_P^{\alpha,m}}^P$.

Definition 3.66. *We define the local stiffness matrix* $\mathfrak{A}_P$ *by*

$$(\mathfrak{A}_P)_{ij} := \mathcal{A}_{h|P}(\epsilon_i^P, \epsilon_j^P) \text{ for } i, j = 1, \dots, \mathcal{N}_P^{\alpha,m}.$$

Although $\mathcal{A}_{h|P}(\epsilon_i^P, p)$ is computable for every $i = 1, \dots, \mathcal{N}_P^{\alpha,m}$ and $p \in \mathbb{P}_m(P)$ (see Theorem 3.47), showing the computability of the local stiffness matrix is a non trivial task. In the following we define several computable auxiliary matrices and derive a formula for calculating the stiffness matrix by them.

Definition 3.67. *The matrix* $N_P \in \mathbb{R}^{\mathcal{N}_P^{\alpha,m} \times \mathcal{M}}$ *is defined by*

$$(N_P)_{ij} := \psi_i^P(p_j) \text{ for } i = 1, \dots, \mathcal{N}_P^{\alpha,m},\ j = 1, \dots, \mathcal{M}.$$

We define $R_P \in \mathbb{R}^{\mathcal{N}_P^{\alpha,m} \times \mathcal{M}}$ *by*

$$(R_P)_{ij} := \langle A \nabla \Pi_m^P(\epsilon_i^P), \nabla p_j \rangle_{L^2(P)} \text{ for } i = 1, \dots, \mathcal{N}_P^{\alpha,m},\ j = 1, \dots, \mathcal{M}.$$

Furthermore we define $K_P := N_P^{\mathrm{T}} R_P \in \mathbb{R}^{\mathcal{M} \times \mathcal{M}}$ *and* $\widehat{K_P} \in \mathbb{R}^{(\mathcal{M}-1) \times (\mathcal{M}-1)}$ *by*

$$\left(\widehat{K_P}\right)_{ij} := \langle A \nabla p_{i+1}, \nabla p_{j+1} \rangle_{L^2(P)} \text{ for } i, j = 1, \dots, \mathcal{M} - 1$$

Lemma 3.68. *The following statements hold* *(i)* $\mathfrak{A}_P N_P = R_P$.

(ii) *The matrix* K_P *can be written as*

$$(K_P)_{ij} := \langle A\nabla p_i, \nabla p_j\rangle_{L^2(P)} \text{ for } i,j = 1,\dots,\mathcal{M},$$

and is therefore equal to

$$K_P = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{K_P} \end{pmatrix}.$$

(iii) *The matrix* $\widehat{K_P}$ *is symmetric and positive definite and thus* K_P *is symmetric and positive semidefinite.*

(iv) $\widehat{K_P}$ *is regular.*

Proof.
ad (i): Let $1 \le i \le \mathcal{N}_P^{\alpha,m}$ and $1 \le j \le \mathcal{M}$ be arbitrary indices. By first using Lemma 3.29, then the consistency of the bilinear form $\mathcal{A}_{h|P}$ (3.20) and finally the definition of Π_m^P (3.26), we get

$$\begin{aligned}(M_P N_P)_{ij} &= \sum_{k=1}^{\mathcal{N}_P^{\alpha,m}} \mathcal{A}_{h|P}\left(\epsilon_i^P, \epsilon_k^P\right)\psi_k(p_j) = \mathcal{A}_{h|P}\left(\epsilon_i^P, \sum_{k=1}^{\mathcal{N}_P^{\alpha,m}} \psi_k(p_j)\epsilon_k^P\right) = \mathcal{A}_{h|P}\left(\epsilon_i^P, p_j\right) \\ &= \langle \pi_{m-1}^P(A\nabla p_j), \nabla \epsilon_i^P\rangle_{L^2(P)} = \langle A\nabla \Pi_m^P(\epsilon_i^P), \nabla p_j\rangle_{L^2(P)} = (R_P)_{ij}\,.\end{aligned}$$

Thus we derive $M_P N_P = R_P$.

ad (ii): Let $1 \le i \le \mathcal{N}_P^{\alpha,m}$ and $1 \le j \le \mathcal{M}$ be arbitrary indices. By using Lemma 3.29 and the projection property of Π_m^P (3.27)

$$\begin{aligned}(K_P)_{ij} &= \left(N_P^{\mathrm{T}} R_P\right)_{ij} = \sum_{k=1}^{\mathcal{N}_P^{\alpha,m}} \left(N_P^{\mathrm{T}}\right)_{ik} (R_P)_{kj} = \sum_{k=1}^{\mathcal{N}_P^{\alpha,m}} (N_P)_{ki} (R_P)_{kj} \\ &= \sum_{k=1}^{\mathcal{N}_P^{\alpha,m}} \psi_k^P(p_i)\langle A\nabla\Pi_m^P(\epsilon_k^P), \nabla p_j\rangle_{L^2(P)} = \langle A\nabla\Pi_m^P \underbrace{\left(\sum_{k=1}^{\mathcal{N}_P^{\alpha,m}} \psi_k^P(p_i)\epsilon_k^P\right)}_{=p_i}, \nabla p_j\rangle_{L^2(P)} \\ &= \langle A\nabla \underbrace{\Pi_m^P(p_i)}_{=p_i}, \nabla p_j\rangle_{L^2(P)} = \langle A\nabla p_i, \nabla p_j\rangle_{L^2(P)}.\end{aligned}$$

ad (iii): Let $0 \neq x \in \mathbb{R}^{\mathcal{M}-1}$. Define

$$p_x := \sum_{j=1}^{\mathcal{M}-1} x_j p_{j+1}.$$

As p_x is not constant it holds that $\|\nabla p_x\|_{L^2(P)} > 0$ and thus

$$x^{\mathrm{T}}\widehat{K_P}x = \langle A\nabla p_x, \nabla p_x\rangle_{L^2(P)} \gtrsim \|\nabla p_x\|^2_{L^2(P)} > 0.$$

Thus $\widehat{K_P}$ is positive definite. Due to (ii) this also implies that K_P is positive semidfefinite. The symmetry of both $\widehat{K_P}$ and K_P is clear.

ad (iv): This is a direct consequence of the positive definiteness of $\widehat{K_P}$.

□

Definition 3.69. *Due to the regularity of $\widehat{K_P}$ we can define the pseudoinverse of K_P by*

$$K_P^{\dagger} := \begin{pmatrix} 0 & 0 \\ 0 & \widehat{K_P}^{-1} \end{pmatrix}.$$

Lemma 3.70. *For all indices $i, j = 1, \dots, \mathcal{N}_P^{\alpha,m}$ it holds that*

$$\left(R_P K_P^{\dagger} R_P^{\mathrm{T}}\right)_{ij} = \left\langle A\nabla\Pi_m^P(\epsilon_i^P), \nabla\Pi_m^P(\epsilon_j^P)\right\rangle_{L^2(P)}.$$

Proof. Let $k \in \left\{1, \dots, \mathcal{N}_P^{\alpha,m}\right\}$ and let $e_k \in \mathbb{R}^{\mathcal{N}_P^{\alpha,m}}$ be the vector, such that for all $i = 1, \dots, \mathcal{N}_P^{\alpha,m}$

$$(e_k)_i = \begin{cases} 1 \text{ if } k = i, \\ 0 \text{ if } k \neq i. \end{cases}$$

Then for $i = 1, \dots, \mathcal{M}$ it holds that

$$\left(R_P^{\mathrm{T}} e_k\right)_i = \sum_{j=1}^{\mathcal{N}_P^{\alpha,m}} \left(R_P^{\mathrm{T}}\right)_{ij} (e_k)_j = \sum_{j=1}^{\mathcal{N}_P^{\alpha,m}} (R_P)_{ji} (e_k)_j = (R_P)_{ki} = \langle A\nabla\Pi_m^P(\epsilon_k^P), \nabla p_i\rangle_{L^2(P)}.$$

Note that as a consequence of $\nabla p_1 = 0$, it holds that $\left(R_P^{\mathrm{T}} e_k\right)_1 = 0$.

Let $\widehat{\nu} \in \mathbb{R}^{\mathcal{M}-1}$ be define by

$$K_P^{\dagger} R_P^{\mathrm{T}} e_k =: \begin{pmatrix} 0 \\ \widehat{\nu} \end{pmatrix}.$$

Then due to the definition of Π_m^P and because

$$K_P K_P^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I}_{\mathcal{M}-1} \end{pmatrix},$$

the vector $(0\ \widehat{\nu})^{\mathrm{T}}$ is the solution to the linear equation

$$K_P \begin{pmatrix} 0 \\ \widehat{\nu} \end{pmatrix} = \begin{pmatrix} 0 \\ \langle A\nabla \Pi_m^P(\epsilon_k^P), \nabla p_2 \rangle_{L^2(P)} \\ \vdots \\ \langle A\nabla \Pi_m^P(\epsilon_k^P), \nabla p_{\mathcal{M}} \rangle_{L^2(P)} \end{pmatrix} = \begin{pmatrix} \langle \pi_{m-1}^P(A\nabla p_1), \nabla \epsilon_k^P \rangle_{L^2(P)} \\ \langle \pi_{m-1}^P(A\nabla p_2), \nabla \epsilon_k^P \rangle_{L^2(P)} \\ \vdots \\ \langle \pi_{m-1}^P(A\nabla p_{\mathcal{M}}), \nabla \epsilon_k^P \rangle_{L^2(P)} \end{pmatrix}. \tag{3.34}$$

Define

$$v := \sum_{j=1}^{\mathcal{M}-1} \widehat{\nu}_j p_{j+1} \in \mathbb{P}_m(P).$$

Furthermore we define the vector

$$\nu := \begin{pmatrix} \overline{\epsilon_k^P}^P - \overline{v}^P \\ \widehat{\nu} \end{pmatrix}$$

and the according polynom

$$w := \sum_{j=1}^{\mathcal{M}} \nu_j p_j = v + \overline{\epsilon_k^P}^P - \overline{v}^P \in \mathbb{P}_m(P).$$

Since the first row of K_P and R_P is zero, it holds that

$$K_P \nu = K_P \begin{pmatrix} 0 \\ \widehat{\nu} \end{pmatrix} \text{ and } R_P \nu = R_P \begin{pmatrix} 0 \\ \widehat{\nu} \end{pmatrix} \tag{3.35}$$

Then due to Lemma 3.68 (ii) and (3.34), we get

$$\langle A\nabla p_i, \nabla w \rangle_{L^2(P)} = (K_P \nu)_i = \left(K_P \begin{pmatrix} 0 \\ \widehat{\nu} \end{pmatrix} \right)_i = \langle \pi_{m-1}^P(A\nabla p_i), \nabla \epsilon_k^P \rangle_{L^2(P)} \text{ for all } i = 1, \dots, \mathcal{M}.$$

As the polynoms $p_1, \dots, p_{\mathcal{M}}$ form a basis of $\mathbb{P}_m(P)$, this means that

$$\langle A\nabla p, \nabla w \rangle_{L^2(P)} = \langle \pi_{m-1}^P(A\nabla p), \nabla \epsilon_k^P \rangle_{L^2(P)} \text{ for all } p \in \mathbb{P}_m(P). \tag{3.36}$$

Furthermore it holds that

$$\overline{w}^P = \overline{v + \overline{\epsilon_k^P}^P - \overline{v}^P}^P = \overline{v}^P + \overline{\epsilon_k^P}^P - \overline{v}^P = \overline{\epsilon_k^P}^P \tag{3.37}$$

By the definition of Π_m^P (see Definition 3.55), equations (3.36) and (3.37) tell us that $\Pi_m^P(\epsilon_k^P) = w$.

By using (3.35) and the definition of w we can conclude

$$\begin{aligned}
\left(R_P K_P^\dagger R_P^\mathrm{T}\right)_{ik} &= \left(R_P K_P^\dagger R_P^\mathrm{T} e_k\right)_i = \left(R_P \begin{pmatrix} 0 \\ \widehat{\nu} \end{pmatrix}\right)_i = (R_P \nu)_i = \sum_{j=1}^{\mathcal{M}} (R_P)_{ij} \nu_j \\
&= \sum_{j=1}^{\mathcal{M}} \nu_j \langle A \nabla \Pi_m^P(\epsilon_i^P), \nabla p_j \rangle_{L^2(P)} = \langle A \nabla \Pi_m^P(\epsilon_i^P), \nabla \underbrace{(\sum_{j=1}^{\mathcal{M}} \nu_j p_j)}_{= w = \Pi_m^P(\epsilon_k^P)} \rangle_{L^2(P)} \\
&= \langle A \nabla \Pi_m^P(\epsilon_i^P), \nabla \Pi_m^P(\epsilon_k^P) \rangle_{L^2(P)}.
\end{aligned}$$

□

Definition 3.71. *Let $\xi \in \mathbb{R}^{\mathcal{N}_P^{\alpha,m}}$ be defined by*

$$\xi_i = \begin{cases} \frac{1}{\#\text{vertices}(P)} & \text{if } i \leq \#\text{vertices}(P), \\ 0 & \text{else.} \end{cases}$$

Furthermore we define the matrix $\Pi_P \in \mathbb{R}^{\mathcal{M} \times (\mathcal{N}_P^{\alpha,m} + \mathcal{M})}$ by

$$\Pi_P := \left(\begin{array}{c|c|ccc} \xi^\mathrm{T} & 0 & -\overline{p_2}^P & \dots & -\overline{p_\mathcal{M}}^P \\ \hline 0 & 0 & & \mathrm{I}_{\mathcal{M}-1} & \end{array}\right) \begin{pmatrix} \mathrm{I}_{\mathcal{N}_P^{\alpha,m}} \\ K_P^\dagger R_P^\mathrm{T} \end{pmatrix}$$

Lemma 3.72. *Π_P is a matrix representation of Π_m^P, i.e., that for every $v := \sum_{j=1}^{\mathcal{N}_P^{\alpha,m}} x_j \epsilon_j^P$, if $\Pi_P x = y$, then $\sum_{j=1}^{\mathcal{M}} y_j p_j = \Pi_m^P(v)$.*

Proof. Let $k \in \{1, \dots, \mathcal{N}_P^{\alpha,m}\}$ be arbitrary and let $e_k \in \mathbb{R}^{\mathcal{N}_P^{\alpha,m}}, \widehat{\nu} \in \mathbb{R}^{\mathcal{M}-1}, \nu \in \mathbb{R}^{\mathcal{M}}$ be defined as in the proof of Lemma 3.70.

We define $y := \Pi_P e_k$. Then by definition it holds that

$$y = \left(\begin{array}{c|c|ccc} \xi^{\mathrm{T}} & 0 & -\overline{p_2}^P & \dots & -\overline{p_{\mathcal{M}}}^P \\ \hline 0 & 0 & & \mathrm{I}_{\mathcal{M}-1} & \end{array}\right) \begin{pmatrix} \mathrm{I}_{\mathcal{N}_P^{\alpha,m}} \\ K_P^{\dagger} R_P^{\mathrm{T}} \end{pmatrix} e_k = \left(\begin{array}{c|c|ccc} \xi^{\mathrm{T}} & 0 & -\overline{p_2}^P & \dots & -\overline{p_{\mathcal{M}}}^P \\ \hline 0 & 0 & & \mathrm{I}_{\mathcal{M}-1} & \end{array}\right) \begin{pmatrix} e_k \\ 0 \\ \widehat{\nu} \end{pmatrix}$$

$$= \begin{pmatrix} \xi_k - \sum_{j=1}^{\mathcal{M}-1} \widehat{\nu}_j \, \overline{p_{j+1}}^P \\ \widehat{\nu} \end{pmatrix}.$$

As

$$\xi_k = \frac{1}{\#\mathrm{vertices}(P)} \sum_{j=1}^{\#\mathrm{vertices}(P)} \psi_j^P(\epsilon_k^P) = \frac{1}{\#\mathrm{vertices}(P)} \sum_{\mathbf{x} \in \mathrm{vertices}(P)} \epsilon_k^P(\mathbf{x}) = \overline{\epsilon_k^P}^P,$$

we get that $y = \nu$.

We have already shown in Lemma 3.70, that

$$\sum_{j=1}^{\mathcal{M}} \nu_j p_j = \Pi_m^P(\epsilon_k^P).$$

Since $\left\{\epsilon_1^P, \dots, \epsilon_{\mathcal{N}_P^{\alpha,m}}^P\right\}$ forms a basis of $V_{h|P}$, Π_P is a matrix representation of Π_m^P.

□

Corollary 3.73. *For all $i, j = 1, \dots, \mathcal{N}_P^{\alpha,m}$ it holds that*

$$\left((\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P\Pi_P)^{\mathrm{T}}(\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P\Pi_P)\right)_{ij} = \sum_{k=1}^{\mathcal{N}_P^{\alpha,m}} \psi_k^P\left(\epsilon_i^P - \Pi_m^P(\epsilon_i^P)\right) \psi_k^P\left(\epsilon_j^P - \Pi_m^P(\epsilon_j^P)\right).$$

Proof. Let $k \in \{1, \dots, \mathcal{N}_P^{\alpha,m}\}$ and $e_k \in \mathbb{R}^{\mathcal{N}_P^{\alpha,m}}, \nu \in \mathbb{R}^{\mathcal{M}}$ be defined as in the proof of Lemma 3.70.

Due to Lemma 3.72 we get that

$$N_P\Pi_P e_k = N_P\nu = \psi_l^P\left(\sum_{r=1}^{\mathcal{M}} \nu_r p_r\right)_{l=1,\dots,\mathcal{N}_P^{\alpha,m}} = \psi_l^P\left(\Pi_m^P(\epsilon_j^P)\right)_{l=1,\dots,\mathcal{N}_P^{\alpha,m}}.$$

Thus we obtain for $i, j = 1, \dots, \mathcal{N}_P^{\alpha,m}$

$$\begin{aligned}\left((\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P\Pi_P)^{\mathrm{T}}(\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P\Pi_P)\right)_{ij} &= e_i^{\mathrm{T}}(\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P\Pi_P)^{\mathrm{T}}(\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P\Pi_P)e_j \\ &= \sum_{k=1}^{\mathcal{N}_P^{\alpha,m}} \psi_k^P\left(\epsilon_i^P - \Pi_m^P(\epsilon_i^P)\right) \psi_k^P\left(\epsilon_j^P - \Pi_m^P(\epsilon_j^P)\right).\end{aligned}$$

□

Theorem 3.74. *If the local bilinear form $\mathcal{A}_{h|P}$ is given by (??), then the local stiffness matrix can be written as*

$$\mathfrak{A}_P = R_P K_P^{\dagger} R_P^{\mathrm{T}} + (\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P \Pi_P)^{\mathrm{T}} (\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P \Pi_P).$$

Proof. Let $i, j = 1, \ldots, \mathcal{N}_P^{\alpha,m}$, then due to Corollary 3.73 and Lemma 3.70 we get

$$\begin{aligned}(\mathfrak{A}_P)_{ij} &= \mathcal{A}_{h|P}(\epsilon_i^P, \epsilon_j^P) \\ &= \langle A \nabla \Pi_m^P(\epsilon_i^P), \nabla \Pi_m^P(\epsilon_j^P) \rangle_{L^2(P)} + \sum_{k=1}^{\mathcal{N}_P^{\alpha,m}} \psi_k^P \left(\epsilon_i^P - \Pi_m^P(\epsilon_i^P)\right) \psi_k^P \left(\epsilon_j^P - \Pi_m^P(\epsilon_j^P)\right) \\ &= \left(R_P K_P^{\dagger} R_P^{\mathrm{T}}\right)_{ij} + \left((\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P \Pi_P)^{\mathrm{T}} (\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P \Pi_P)\right)_{ij}\end{aligned}$$

□

Remark 3.75. *Our local stiffness matrix deviates sligthly from the one described in [BdV M 14, equation (4.25)].*

3.6.2 The global stiffness matrix

Let $\mathcal{N}^{\alpha,m} := \#V_h$ be the total number of global degrees of freedom. We enumerate the global degrees of freedom $\mathrm{DOF}_h = \{\psi_1, \ldots, \psi_{\mathcal{N}^{\alpha,m}}\}$. Let $\epsilon_1, \ldots, \epsilon_{\mathcal{N}^{\alpha,m}} \in V_h$ be the corresponding canonical basis (see Definition 3.30).

Definition 3.76. *The global stiffness matrix $\mathfrak{A} \in \mathbb{R}^{\mathcal{N}^{\alpha,m} \times \mathcal{N}^{\alpha,m}}$ is defined by*

$$(\mathfrak{A})_{ij} := \mathcal{A}_h(\epsilon_i, \epsilon_j) \text{ for } i, j = 1, \ldots, \mathcal{N}^{\alpha,m}.$$

$\mathfrak{A}$ can easily be assembled using the local stiffness matrices, $\mathfrak{A}_P$ for $P \in \Omega_h$, constructed in subsection 3.6.1.

Since the global degrees of freedom are given by the union of all local degrees of freedom minus the evaluations at the boundary $\partial\Omega$ (see (3.18)) there is at most one $\psi^P \in \mathrm{DOF}_h(P)$, for every $\psi \in \mathrm{DOF}_h$ and every polygon $P \in \Omega_h$, such that

$$\psi|_{V_{h|P}} = \pm\psi^P.$$

This means that, given an enumaration of the local degrees of freedom $\mathrm{DOF}_h(P)$ just as in Definition ?? for every $P \in \Omega_h$, the function $\iota : \{1, \ldots, \mathcal{N}^{\alpha,m}\} \times \Omega_h \to \mathbb{N}$, with

$$\iota(i, P) = \begin{cases} j \in \{1, \ldots, \mathcal{N}_P^{\alpha,m}\} & \text{if } \psi_i|_{V_{h|P}} = \pm\psi_j^P, \\ 0 & \text{if } \psi_i|_{V_{h|P}} \neq \pm\psi^P \text{ for all } \psi^P \in \mathrm{DOF}_h(P). \end{cases}$$

is well defined.

Remark 3.77. *In other words ι is the function that, given an index i of the global degrees of freedom and a polygon P, returns the corresponding local index of ψ_i now interpreted as an element of* $\mathrm{DOF}_h(P)$*. It returns 0, if ψ_i can not be interpreted as an element of* $\mathrm{DOF}_h(P)$*.*

We also define $\sigma : \{1, \dots, \mathcal{N}^{\alpha,m}\} \times \Omega_h \to \{-1, 0, 1\}$, with

$$\sigma(i, P) = \begin{cases} 1 & \text{if } \psi_i|_{V_{h|P}} \in \mathrm{DOF}_h(P), \\ -1 & \text{if } -\psi_i|_{V_{h|P}} \in \mathrm{DOF}_h(P), \\ 0 & \text{otherwise.} \end{cases}$$

For technical reasons we define $\epsilon_0^P := 0 \in V_{h|P}$ for every $P \in \Omega_h$.

Lemma 3.78. *For all $P \in \Omega_h$ and $i = 1, \dots, \mathcal{N}^{\alpha,m}$ it holds that $\epsilon_i|_P = \sigma(i, P)\epsilon^P_{\iota(i,P)}$.*

Proof. Obviously $\epsilon_i|_P \in V_{h|P}$. Let $j = 1, \dots, \mathcal{N}_P^{\alpha,m}$. Then

$$\psi_j^P(\epsilon_i) = \begin{cases} \sigma(i, P) & \text{if } j = \iota(i, P), \\ 0 & \text{otherwise.} \end{cases} .$$

The unisolvence of $\mathrm{DOF}_h(P)$ thus implies $\epsilon_i|_P = \sigma(i, P)\epsilon^P_{\iota(i,P)}$.

□

Finally we can proof the correctness of the following construction scheme for the global stiffness matrix.

Theorem 3.79. *Let $i, j = 1, \dots, \mathcal{N}^{\alpha,m}$, then the ij-th component of global stiffness matrix is given by*

$$(\mathfrak{A})_{ij} = \sum_{P \in \Omega_h} \sigma(i, P)\sigma(j, P) (\mathfrak{A}_P)_{\iota(i,P)\iota(j,P)} .$$

Here we set $(M_P)_{kl} = 0$, if $k = 0$ or $l = 0$.

Proof. By definition it holds that

$$(M)_{ij} = \mathcal{A}_h(\epsilon_i, \epsilon_j) = \sum_{P \in \Omega_h} \mathcal{A}_{h|P}(\epsilon_i, \epsilon_j) = \sum_{P \in \Omega_h} \mathcal{A}_{h|P}(\epsilon_i|_P , \epsilon_j|_P).$$

Due to Lemma 3.78 and the definition of the local stiffness matrices (Definition 3.66) we can thus conclude

$$(M)_{ij} = \sum_{P \in \Omega_h} \mathcal{A}_{h|P}(\sigma(i, P)\epsilon^P_{\iota(i,P)}, \sigma(j, P)\epsilon^P_{\iota(i,P)}) = \sum_{P \in \Omega_h} \sigma(i, P)\sigma(j, P) (M_P)_{\iota(i,P)\iota(j,P)} .$$

□

3.7 Locally relaxed regularity

Using an approximation scheme of higher regularity (i.e. $\alpha \geq 1$) only makes sense, if we expect the solution to be of higher regularity. In some cases the solution might indeed be of higher regularity almost everywhere in Ω, except for a small subset $\Gamma \subseteq \Omega$.
For Example let $\Gamma \subseteq \Omega$ be a line that seperates Ω into two parts and assume that the solution $u \in C^1(\Omega \setminus \Gamma)$, such that the derivative is discontinuous at the line Γ.

In such a case it might be advantageous, if the approximation u_h mimicked this locally reduced regularity of u. Luckily the VEM offers an easy way to accomplish this.

Definition 3.80. *Let $k = 0, \dots, \alpha$ and $\boldsymbol{e} \in \operatorname{edges}(\Omega_h)$. We define*

$$\operatorname{DOF}_h(\boldsymbol{e}, k) := \left\{ \psi \in \operatorname{DOF}_h \;\middle|\; \psi \text{ is an evaluation of a derivative of order } k \text{ at a point } \boldsymbol{x} \in \boldsymbol{e} \right\}$$

Definition 3.81. *Let $\mathfrak{d} : \{\boldsymbol{e} \in \operatorname{edges}(\Omega_h) \mid \boldsymbol{e} \text{ is an inner edge}\} \to \{0, \dots, \alpha\}$. The corresponding (global) VEM space with locally relaxed regularity is defined by*

$$V_h(\mathfrak{d}) := \Big\{ v \in H_0^1(\Omega) \;\Big|\; \forall P \in \Omega_h : v|_P \in V_{h|P} \text{ and for all inner edges } \boldsymbol{e} \in \operatorname{edges}(\Omega_h) \dots$$
$$\dots \text{ it holds that } \psi(v|_{P_1(\boldsymbol{e})}) = \psi(v|_{P_2(\boldsymbol{e})}) \text{ for all } \psi \in \operatorname{DOF}_h(\boldsymbol{e}, k),\; k = 0, \dots, \mathfrak{d}(\boldsymbol{e}) \Big\}.$$

Hereby $\psi(v|_P)$ means that we interpret ψ as a local degree of freedom (i.e. as an element of $\operatorname{DOF}_h(P)$) and evaluate it at $v|_P \in V_{h|P}$

Using Corollary 3.36, one can easily see that the map $\mathfrak{d}$ governs the differentiability accross the inner edges of the mesh. Thus the space $V_h(\mathfrak{d})$ consists of all functions v, such that $v|_P \in V_{h|P}$ for all $P \in \Omega_h$ and which are at least $\mathfrak{d}(\mathbf{e})$-times continuously differentiable accross every inner edge $\mathbf{e} \in \operatorname{edges}(\Omega_h)$.

Remark 3.82. *It is easy to check that $V_h(\mathfrak{d}) \subseteq C^{\mathfrak{a}}(\overline{\Omega})$, with*

$$\mathfrak{a} := \min \{\mathfrak{d}(\boldsymbol{e}) \mid \boldsymbol{e} \in \operatorname{edges}(\Omega_h) \text{ is an inner edge}\}.$$

Thus $V_h(\mathfrak{d}) \subseteq H^{\mathfrak{a}+1}(\Omega)$.

Furthermore, it is obvious that $V_h \subseteq V_h(\mathfrak{d})$, with $V_h = V_h(\mathfrak{d})$, if and only if $\mathfrak{d}(\boldsymbol{e}) = \alpha$ for all inner edges $\boldsymbol{e} \in \operatorname{edges}(\Omega_h)$.

The (global) degrees of freedom $\operatorname{DOF}_h(\mathfrak{d})$ of the space $V_h(\mathfrak{d})$ are given by the ones of V_h, with the exception, that if $\psi \in \operatorname{DOF}_h(\mathbf{e}, k)$ with $k > \mathfrak{d}(\mathbf{e})$, then ψ is replaced by $\psi|_{V_{h|P_1(\mathbf{e})}}$

and $\psi|_{V_{h|P_2(\mathbf{e})}}$. Thus we define

$$\mathfrak{S}(\mathfrak{d}) := \mathrm{DOF}_h \setminus \bigcup_{\mathbf{e}\in\mathrm{edges}(\Omega_h)} \bigcup_{k>\mathfrak{d}(\mathbf{e})} \mathrm{DOF}_h(\mathbf{e},k),$$

$$\mathrm{DOF}_h(\mathfrak{d}) := \mathfrak{S}(\mathfrak{d}) \ \cup \bigcup_{\mathbf{e}\in\mathrm{edges}(\Omega_h)} \bigcup_{k>\mathfrak{d}(\mathbf{e})} \Big\{ v \mapsto \psi(\,v|_{P_j(\mathbf{e})}) \ \Big| \ \psi \in \mathrm{DOF}_h(\mathbf{e},k), j=1,2 \Big\}.$$

Therefore, the construction of the global stiffness matrix for the space $V_h(\mathfrak{d})$ only requires a change in the assembly of the local stiffness matrices. The local stiffness matrices are constructed just as it was described in subsection 3.6.1.
We will not discuss the construction of the global stiffness matrix as the necessary changes are quite obvious, albeit notationally ugly.

Finally, let us close this chapter by stating that all theorems that were proved for V_h (except those regarding global regularity), can be proved for $V_h(\mathfrak{d})$, using the same proofs.

4 A priori analysis

Throughout this chapter, we denote the weak solution of the Poisson problem (2.1) by $u \in H_0^1(\Omega)$.

Furthermore we denote the solution of the corresponding discrete problem (3.33) by $u_h \in V_h$.

Theorem 4.1. *Let $1 + \alpha \leq s \leq m$. Assume that $u \in H^{1+s}(\Omega)$ and that A is piecewise constant, i.e. $A|_P \in \mathbb{P}_0(P)$ for all $P \in \Omega_h$. Then there exists a constant $C > 0$, which is independent of h, such that*

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq Ch^s \|D^{s+1}u\|_{L^2(\Omega)}. \tag{4.1}$$

Proof. We recall that $u^{\mathrm{I}} \in V_h$ is the interpolant of u (see Definition 3.41) and define $\delta_h := u_h - u^{\mathrm{I}} \in V_h$.

We split the proof into four steps. First, we reduce the task to finding an upper bound of $\mathcal{A}_h(\delta_h, \delta_h)$. In the second step, we split this term up into four parts. In the third step we derive upper bounds for all those parts and in the fourth and final step, we combine the estimates found to prove the theorem.

Step (i): Reducing the error.
By using the triangle inequality and Lemma 3.43, we get

$$\|\nabla u - \nabla u_h\|_{L^2(\Omega)} \leq \|\nabla u - \nabla u^{\mathrm{I}}\|_{L^2(\Omega)} + \|\nabla u_h - \nabla u^{\mathrm{I}}\|_{L^2(\Omega)} \lesssim h^s \|D^{s+1}u\|_{L^2(\Omega)} + \|\nabla \delta_h\|_{L^2(\Omega)}$$

By using the uniform κ-ellipticity of A and the stability of $\mathcal{A}_{h|P}$ (3.21), we get

$$\|\nabla \delta_h\|^2_{L^2(\Omega)} \lesssim \langle A\ \nabla \delta_h, \nabla \delta_h \rangle_{L^2(\Omega)} = \sum_{P \in \Omega_h} \langle A\ \nabla \delta_h, \nabla \delta_h \rangle_{L^2(P)} \lesssim \sum_{P \in \Omega_h} \mathcal{A}_{h|P}(\delta_h, \delta_h) = \mathcal{A}_h(\delta_h, \delta_h)$$

Thus, it remains to dominate $\mathcal{A}_h(\delta_h, \delta_h)$.

Step (ii): Splitting up the remaining error term.
Using the bilinearity of $\mathcal{A}_h$ and that u_h is the solution to the discret problem, we get

$$\mathcal{A}_h(\delta_h, \delta_h) = \mathcal{A}_h(u_h, \delta_h) - \mathcal{A}_h(u^{\mathrm{I}}, \delta_h) = F_h(\delta_h) - \mathcal{A}_h(u^{\mathrm{I}}, \delta_h).$$

Next, we define u_π as the piecewise L^2-orthogonal projection of u onto $\mathbb{P}_m(P)$, i.e. $u_\pi|_P := \pi_m^P(u) \in \mathbb{P}_m(P)$ for all $P \in \Omega_h$.

By inserting $\mathcal{A}_h(u_\pi, \delta_h)$, we get

$$\begin{aligned}\mathcal{A}_h(\delta_h, \delta_h) &= F_h(\delta_h) + \mathcal{A}_h(u_\pi - u^{\mathrm{I}}, \delta_h) - \mathcal{A}_h(u_\pi, \delta_h)\\ &= F_h(\delta_h) + \sum_{P \in \Omega_h} \mathcal{A}_{h|P}(u_\pi - u^{\mathrm{I}}, \delta_h) - \sum_{P \in \Omega_h} \mathcal{A}_{h|P}(u_\pi, \delta_h).\end{aligned}$$

Since $u_\pi|_P \in \mathbb{P}_m(P)$ for every $P \in \Omega_h$, we can use the consistency property of $\mathcal{A}_{h|P}$ (3.20) to rewrite this as

$$\mathcal{A}_h(\delta_h, \delta_h) = F_h(\delta_h) + \sum_{P \in \Omega_h} \mathcal{A}_{h|P}(u_\pi - u^{\mathrm{I}}, \delta_h) - \sum_{P \in \Omega_h} \left\langle \pi^P_{m-1}(A\nabla u_\pi), \nabla \delta_h \right\rangle_{L^2(P)}.$$

By inserting both $\sum_{P \in \Omega_h} \langle A\nabla u_\pi, \delta_h\rangle_{L^2(P)}$ and $\sum_{P \in \Omega_h} \langle A\nabla u, \nabla\delta_h\rangle_{L^2(P)} = \langle A\nabla u, \nabla\delta_h\rangle_{L^2(\Omega)}$ this can be written as

$$\begin{aligned}\mathcal{A}_h(\delta_h, \delta_h) &= F_h(\delta_h) + \sum_{P \in \Omega_h} \mathcal{A}_{h|P}(u_\pi - u^{\mathrm{I}}, \delta_h) + \sum_{P \in \Omega_h} \langle A\nabla(u - u_\pi), \nabla\delta_h\rangle_{L^2(P)} + \cdots\\ &\quad \cdots + \sum_{P \in \Omega_h} \left\langle (\mathrm{I} - \pi^P_{m-1})(A\nabla u_\pi), \nabla\delta_h \right\rangle_{L^2(P)} - \underbrace{\langle A\nabla u, \nabla\delta_h\rangle_{L^2(\Omega)}}_{=\langle f, \delta_h\rangle_{L^2(\Omega)}}\\ &= \underbrace{F_h(\delta_h) - \langle f, \delta_h\rangle_{L^2(\Omega)}}_{:=T_f} + \sum_{P \in \Omega_h} \underbrace{\left\langle (\mathrm{I} - \pi^P_{m-1})(A\nabla u_\pi), \nabla\delta_h \right\rangle_{L^2(P)}}_{:=T_1^P} + \cdots\\ &\quad \cdots + \sum_{P \in \Omega_h} \underbrace{\langle A\nabla(u - u_\pi), \nabla\delta_h\rangle_{L^2(P)}}_{:=T_2^P} + \sum_{P \in \Omega_h} \underbrace{\mathcal{A}_{h|P}(u_\pi - u^{\mathrm{I}}, \delta_h)}_{:=T_3^P}.\end{aligned}$$

We are thus left to find estimates for the four terms T_f, T_1^P, T_2^P and T_3^P.

Step (iii): Dominating the four error terms T_f, T_1^P, T_2^P, T_3^P.

Since $u \in H^{s+1}(\Omega)$ is the weak solution of (2.1) and $s + 1 \geq \alpha + 2 \geq 2$, it holds that $u \in H^2(\Omega)$. Thus u is also a classical solution. Therefore $f = \Delta u \in H^{s-1}(\Omega)$ and we can thus apply Lemma 3.64 to obtain

$$\begin{aligned}|T_f| &\lesssim h^s \left\|D^{s-1} f\right\|_{L^2(\Omega)} \|\nabla\delta_h\|_{L^2(\Omega)} = h^s \left\|D^{s-1}\Delta u\right\|_{L^2(\Omega)} \|\nabla\delta_h\|_{L^2(\Omega)}\\ &= h^s \left\|\Delta D^{s-1} u\right\|_{L^2(\Omega)} \|\nabla\delta_h\|_{L^2(\Omega)} \leq h^s \left\|D^{s+1} u\right\|_{L^2(\Omega)} \|\nabla\delta_h\|_{L^2(\Omega)}.\end{aligned}$$

Here the Laplacian of a vector valued function denotes the application of the Laplacian to every component of that function.

Using the Cauchy–Schwarz inequality (1.6), the triangle inequality, the $L^2(P)$-orthogonality of $\mathrm{I} - \pi_{m-1}^P$, and the boundedness of A, we can estimate T_1^P by

$$\begin{aligned} |T_1^P| &\leq \left\|(\mathrm{I} - \pi_{m-1}^P)(A\nabla u_\pi)\right\|_{L^2(P)} \|\nabla \delta_h\|_{L^2(P)} \\ &\leq \left(\left\|(\mathrm{I} - \pi_{m-1}^P)(A\nabla u)\right\|_{L^2(P)} + \left\|(\mathrm{I} - \pi_{m-1}^P)(A\nabla (u - u_\pi))\right\|_{L^2(P)}\right) \|\nabla \delta_h\|_{L^2(P)} \\ &\leq \left(\left\|(\mathrm{I} - \pi_{m-1}^P)(A\nabla u)\right\|_{L^2(P)} + \|A\nabla (u - u_\pi)\|_{L^2(P)}\right) \|\nabla \delta_h\|_{L^2(P)} \\ &\lesssim \left(\left\|(\mathrm{I} - \pi_{m-1}^P)(A\nabla u)\right\|_{L^2(P)} + \|\nabla (u - u_\pi)\|_{L^2(P)}\right) \|\nabla \delta_h\|_{L^2(P)} . \end{aligned}$$

Since $A|_P \in \mathbb{P}_0(P)$, it holds that $A\nabla u \in H^s(P)^2$ for every $P \in \Omega_h$.
Therefore, we can use Lemma 3.10 and the boundedness of A to derive

$$\left\|(\mathrm{I} - \pi_{m-1}^P)(A\nabla u)\right\|_{L^2(P)} \lesssim h_P^s \|D^s(A\nabla u)\|_{L^2(P)} \lesssim h_P^s \left\|D^{s+1}u\right\|_{L^2(P)}$$

and Lemma 3.11 to get

$$\|\nabla (u - u_\pi)\|_{L^2(P)} = \left\|\nabla (u - \pi_m^P(u))\right\|_{L^2(P)} \lesssim h_P^s \left\|D^{s+1}u\right\|_{L^2(P)} .$$

Thus,

$$|T_1^P| \lesssim h_P^s \left\|D^{s+1}u\right\|_{L^2(P)} \|\nabla \delta_h\|_{L^2(P)} .$$

Due to the Cauchy–Schwarz inequality (1.6) and Lemma 3.11, T_2^P can be bounded by

$$\begin{aligned} |T_2^P| &= \left|\langle A\nabla (u - u_\pi), \nabla \delta_h\rangle_{L^2(P)}\right| \lesssim \|\nabla (u - u_\pi)\|_{L^2(P)} \|\nabla \delta_h\|_{L^2(\Omega)} \\ &\lesssim h_P^s \left\|D^{s+1}u\right\|_{L^2(P)} \|\nabla \delta_h\|_{L^2(P)} . \end{aligned}$$

By using first the continuity of $\mathcal{A}_{h|P}$ (3.49), then the triangle inequality and finally Lemma 3.10 and Lemma 3.43, we get the following upper bound for T_3^P

$$\begin{aligned} |T_3^P| &= \left|\mathcal{A}_{h|P}(u_\pi - u^{\mathrm{I}}, \delta_h)\right| \lesssim \left\|\nabla (u_\pi - u^{\mathrm{I}})\right\|_{L^2(P)} \|\nabla \delta_h\|_{L^2(P)} \\ &\leq \left(\|\nabla (u - u_\pi)\|_{L^2(P)} + \left\|\nabla (u - u^{\mathrm{I}})\right\|_{L^2(P)}\right) \|\nabla \delta_h\|_{L^2(P)} \\ &\lesssim h_P^s \left\|D^{s+1}u\right\|_{L^2(P)} \|\nabla \delta_h\|_{L^2(P)} . \end{aligned}$$

Step (iv): Combining the estimates of T_f, T_1^P, T_2^P, T_3^P
Finally we can use the upper bounds for T_f, T_1^P, T_2^P and T_3^P, to conclude

$$\begin{aligned} \|\nabla \delta_h\|_{L^2(\Omega)}^2 \lesssim \mathcal{A}_h(\delta_h, \delta_h) &\leq |T_f| + \sum_{P \in \Omega_h} \left(|T_1^P| + |T_2^P| + |T_3^P|\right) \\ &\lesssim h^s \left\|D^{s+1}u\right\|_{L^2(\Omega)} \|\nabla \delta_h\|_{L^2(\Omega)} + \sum_{P \in \Omega_h} h_P^s \left\|D^{s+1}u\right\|_{L^2(P)} \|\nabla \delta_h\|_{L^2(P)} \\ &\leq h^s \left\|D^{s+1}u\right\|_{L^2(\Omega)} \|\nabla \delta_h\|_{L^2(\Omega)} . \end{aligned}$$

The first inequality holds due to the stability of $\mathcal{A}_h$ (3.21), the last inequality is a result of the Cauchy–Schwarz inequality for sums (1.7). By dividing through $\|\nabla\delta_h\|_{L^2(\Omega)}$ on both sides, we conclude

$$\|\nabla u - \nabla u_h\|_{L^2(\Omega)} \lesssim h^s\|D^{s+1}u\|_{L^2(\Omega)} + \|\nabla\delta_h\|_{L^2(\Omega)} \lesssim h^s \left\|D^{s+1}u\right\|_{L^2(\Omega)}$$

This concludes the proof. □

Corollary 4.2. *Under the assumptions of Theorem 4.1, there exists a constant $C > 0$ for any $1 + \alpha \leq s \leq m$, which is independent of h, such that*

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^s\|D^{s+1}u\|_{L^2(\Omega)}.$$

Proof. Using the Friedrichs inequality 1.36 and Theorem 4.1, we derive

$$\|u - u_h\|_{L^2(\Omega)} \lesssim \|\nabla u - \nabla u_h\|_{L^2(\Omega)} \lesssim h^s\|D^{s+1}u\|_{L^2(\Omega)}.$$

□

Corollary 4.3. *Under the assumptions of Theorem 4.1 the solution to the discrete problem u_h converges towards the true solution u for $h \to 0$ in $H_0^1(\Omega)$.*

Proof. This is a direct consequence of Theorem 4.1 and Corollary 4.2.

□

5 A posteriori analysis

In this chapter, we allow for VEM spaces with locally relaxed regularity; see Definition 3.81. Let $\mathfrak{d} : \{\mathbf{e} \in \mathrm{edges}(\Omega_h) \mid \mathbf{e} \text{ is an inner edge}\} \to \{0, \ldots, \alpha\}$ be fixed throughout this chapter. We denote the corresponding VEM space by $V_h(\mathfrak{d})$.

As in Chapter 4, $u \in H_0^1(\Omega)$ denotes the solution to the Poisson-type problem (2.1) and $u_h \in V_h(\mathfrak{d})$ denotes the solution to the discrete problem (3.33) throughout this chapter.

Assumption 5. *Throughout this chapter, we assume that the consistency index $m \geq 2$. Furthermore, we assume that the diffusion tensor A is piecewise constant on the mesh, i.e., $A|_P \in \mathbb{P}_0(P)^{2\times 2}$ for all $P \in \Omega_h$.*

Remark 5.1. *The assumptions made here are in line with the assumptions made in [BdV M 15]. To see the* A POSTERIORI *analysis for more general diffusion tensors and for the lowest order case $m = 1$, we refer the reader to [CaGePrSu].*

Before we can define the error estimators, we need an auxiliary definition.

Definition 5.2. *Let $v \in V_h(\mathfrak{d})$. We define the local jump for any $\boldsymbol{e} \in \mathrm{edges}(\Omega_h)$ by*

$$[\![A\nabla v \cdot \boldsymbol{n}]\!]_{\boldsymbol{e}} = \begin{cases} 0 & \text{if } \boldsymbol{e} \subseteq \partial\Omega. \\ (A\nabla v)|_{P_1(\boldsymbol{e})} \cdot \boldsymbol{n}_{P_1(\boldsymbol{e})} + (A\nabla v)|_{P_2(\boldsymbol{e})} \cdot \boldsymbol{n}_{P_2(\boldsymbol{e})} & \text{if } \boldsymbol{e} \text{ is an inner edge.} \end{cases}$$

Remark 5.3. *Note that if $\mathbf{e}$ is an inner edge (see Definition 3.2), A is continuous across $\boldsymbol{e}$ and if v is continuously differentiable across $\boldsymbol{e}$*

$$\begin{aligned} [\![A\nabla v \cdot \boldsymbol{n}]\!]_{\boldsymbol{e}} &= (A\nabla v)|_{P_1(\boldsymbol{e})} \cdot \boldsymbol{n}_{P_1(\boldsymbol{e})} + (A\nabla v)|_{P_2(\boldsymbol{e})} \cdot \boldsymbol{n}_{P_2(\boldsymbol{e})} \\ &= (A\nabla v)|_{P_1(\boldsymbol{e})} \cdot \boldsymbol{n}_{P_1(\boldsymbol{e})} - (A\nabla v)|_{P_2(\boldsymbol{e})} \cdot \boldsymbol{n}_{P_1(\boldsymbol{e})} = 0 \end{aligned}$$

Now, we can define the error estimators.

Definition 5.4. *Let $P \in \Omega_h$. We define*

- *the local residual estimate*

$$\eta_P^{\mathrm{r}} := h_P \left\| f + \pi_m^P(\mathrm{div}(A\nabla u_h)) \right\|_{L^2(P)} \tag{5.1}$$

Note that since $u_h \in V_h$ the term $\mathrm{div}(A\nabla u_h)$ is not necessarily piecewise polynomial.

- *the local loading term estimate*

$$\eta_P^{\mathrm{l}} := h_P \left\| f - f_h \right\|_{L^2(P)} \tag{5.2}$$

- *the local consistency estimate*

$$\eta_P^{\mathrm{c}} := \sqrt{\mathcal{A}_{h|P}(u_h - \Pi_m^P(u_h), u_h - \Pi_m^P(u_h))} \tag{5.3}$$

- *the jump term*

$$\eta_P^{\mathrm{j}} := \sqrt{\frac{1}{2} \sum_{e \in \mathrm{edges}(P)} h_P \left\| [\![A\nabla u_h \cdot \boldsymbol{n}]\!]_e \right\|_{L^2(e)}^2} \tag{5.4}$$

Remark 5.5. *Note that due to Remark 5.3, η_P^{j} sums only over those inner edges of P, accross which u_h is not continuously differentiable, or A is not continuous.*

- *the local error estimator*

$$\eta_P := \sqrt{(\eta_P^{\mathrm{r}})^2 + (\eta_P^{\mathrm{l}})^2 + (\eta_P^{\mathrm{c}})^2 + (\eta_P^{\mathrm{j}})^2} \tag{5.5}$$

Furthermore, we define the global error estimator by

$$\eta := \sqrt{\sum_{P \in \Omega_h} \eta_P^2}. \tag{5.6}$$

Theorem 5.6. *Let $\mathcal{A}_{h|P}$ be defined as in Definition 3.61. Then the global error estimator η is computable using only the degrees of freedom of the discrete solution u_h and known quantities.*

Proof. Since f is known and $f_h|_P := \pi_{m-2}^P(f)$ can be computed through solving the linear system

$$\langle f_h, p \rangle_{L^2(P)} = \langle f, p \rangle_{L^2(P)} \text{ for } p \in \mathcal{M}_{m-2},$$

the local loading term estimate $\eta_P^{\mathrm{l}} := h_P \left\| f - f_h \right\|_{L^2(P)}$ is computable (even without using the degrees of freedom).

The local consistency estimate $\eta_P^{\mathrm{c}} = \sqrt{\mathcal{A}_{h|P}(u_h - \Pi_m^P(u_h), u_h - \Pi_m^P(u_h))}$ can be calculated by using parts of the local stiffness matrix. Let the vector

$$\xi := \begin{pmatrix} \psi_1(u_h) \\ \vdots \\ \psi_{\mathcal{N}_P^{\alpha,m}}(u_h) \end{pmatrix}$$

be the collection of all local degrees of freedom evaluated at u_h. Then, it holds that (see Corollary 3.73)

$$\mathcal{A}_{h|P}(u_h - \Pi_m^P(u_h), u_h - \Pi_m^P(u_h)) = \xi^{\mathrm{T}} \left((\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P \Pi_P)^{\mathrm{T}} (\mathrm{I}_{\mathcal{N}_P^{\alpha,m}} - N_P \Pi_P) \right) \xi.$$

Since the matrices N_P and Π_P are computable by the degrees of freedom and the vector ξ is computable by solving the linear equation (3.33), the local consistency estimate is computable by the degrees of freedom.

The term $\pi_m^P(\mathrm{div}(A\nabla u_h))$ that arises in the local residual estimate η_P^{r} can be computed by solving the linear equation

$$\langle \pi_m^P(\mathrm{div}(A\nabla u_h)), p\rangle_{L^2(P)} = \langle \mathrm{div}(A\nabla u_h), p\rangle_{L^2(P)} \text{ for } p \in \mathcal{M}_m. \tag{5.7}$$

Integrating by parts twice, the right hand side can be written as

$$\begin{aligned} \langle \mathrm{div}(A\nabla u_h), p\rangle_{L^2(P)} &= \langle A\nabla u_h \cdot \mathbf{n}, p\rangle_{L^2(\partial P)} - \langle A\nabla u_h), \nabla p\rangle_{L^2(P)} \\ &= \langle A\nabla u_h \cdot \mathbf{n}, p\rangle_{L^2(\partial P)} - \langle u_h, A\nabla p \cdot \mathbf{n}\rangle_{L^2(\partial P)} + \langle u_h, \mathrm{div}(A\nabla p)\rangle_{L^2(P)}. \end{aligned}$$

Since A is piecewise constant, $\mathrm{div}(A\nabla p) \in \mathbb{P}_{m-2}(P)$ and thus $\langle u_h, \mathrm{div}(A\nabla p)\rangle_{L^2(P)}$ is computable by the local internal degrees of freedom.

Due to Lemma 3.35 and Corollary 3.36, both $u_h|_{\partial P}$ and $\nabla u_h|_{\partial P}$ are computable by using the degrees of freedom. Thus, both

$$\langle A\nabla u_h \cdot \mathbf{n}, p\rangle_{L^2(\partial P)} \text{ and } \langle u_h, A\nabla p \cdot \mathbf{n}\rangle_{L^2(\partial P)}$$

can be computed with the degrees of freedom.

Thus, the right hand side of (5.7) can be computed using the degrees of freedom of u_h. Therefore, the solution to (5.7), and by that the local residual estimate η_P^{r}, is computable by the degrees of freedom.

Finally due to Corollary 3.36 the term

$$[\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}} = (A\nabla u_h)|_{P_1(\mathbf{e})} \cdot \mathbf{n}_{P_1(\mathbf{e})} + (A\nabla u_h)|_{P_2(\mathbf{e})} \cdot \mathbf{n}_{P_2(\mathbf{e})}$$

is computable. Thus, the jump term η_P^{j} is computable by the degrees of freedom. □

As seen in Theorem 5.6, the error estimator can be computed using the results of an iteration of the VEM. In order for it to make sense to use the global error estimator for adaptive mesh refinement, we need to make sure that the following conditions are satisfied

- The estimator should dominate the error up to a scaling constant C_{rel}, that is independent of the mesh size h, i.e.,

  $$\|u - u_h\|_{H^1(\Omega)} \leq C_{\text{rel}}\eta.$$

  We call this condition the reliability of the estimator.

- The upper bound provided by the error estimator should be of similar order as the error (with respect to h). This means that there should be a constant C_{eff}, independent of the mesh size h and a term τ, which is of the same order as the error with respect to h, such that

  $$\eta \leq C_{\text{eff}} \left(\|u - u_h\|_{H^1(\Omega)} + \tau \right)$$

  We call this condition the efficiency of the estimator .

In the next section we discuss some auxiliary results.

5.1 Clement interpolation

Lemma 5.7. *There exist constants $C_1, C_2 > 0$, both independent of h, such that for all $P \in \Omega_h$ and all $v_h \in V_{h|P}$ it holds that*

$$h_P \|\nabla v_h\|_{L^2(P)} \leq C_1 \|v_h\|_{L^2(P)}$$
$$h_P \left\|D^2 v_h\right\|_{L^2(P)} \leq C_2 \|\nabla v_h\|_{L^2(P)}.$$

Proof. For the proof we refer to [BdV M 15, Lemma 4.5].

□

Definition 5.8. *Let $P \in \Omega_h$. We define*

$$\omega_P := \bigcup_{\tilde{P} \in \text{neighbors}(P)} \tilde{P}.$$

Furthermore, let $\boldsymbol{x} \in P$. Then, we define

$$\omega_{\boldsymbol{x}} := \bigcup_{\text{vertices}(\tilde{P}) \ni \boldsymbol{x}} \tilde{P}.$$

Lemma 5.9 ([BdV M 15, Lemma 4.6])**.** *There exists a constant $C > 0$, independent of h, and a Clement type interpolant $\cdot_{\mathfrak{C}} : H_0^1(\Omega) \to V_h \subseteq V_h(\mathfrak{d})$ such that, for all $v \in H_0^1(\Omega)$ and all $P \in \Omega_h$, the interpolant $v_{\mathfrak{C}}$ satisfies that*

$$\|v - v_{\mathfrak{C}}\|_{L^2(P)} + h_P \|\nabla v_{\mathfrak{C}}\|_{L^2(P)} \le C h_P \|\nabla v\|_{L^2(\omega_P)}.$$

□

Lemma 5.10. *Let $P \in \Omega_h$ and $\mathbf{e} \in \operatorname{edges}(P)$. There exists a constant $C > 0$, independent of h, such that*

$$\|v - v_{\mathfrak{C}}\|_{L^2(e)} \le C\sqrt{h_P}\,\|\nabla v\|_{L^2(\omega_P)} \text{ for all } v \in H_0^1(\Omega).$$

Proof. Let $v \in H_0^1(\Omega)$. Let $T \subseteq P$ be the triangle corresponding to the edge $\mathbf{e}$, defined in Lemma 3.12.

By using the trace inequality of Theorem 1.37, we derive that

$$\|v - v_{\mathfrak{C}}\|_{L^2(\mathbf{e})} \le \frac{|\mathbf{e}|}{|T|}\left(\|v - v_{\mathfrak{C}}\|^2_{L^2(T)} + h_T \|(v - v_{\mathfrak{C}})\nabla(v - v_{\mathfrak{C}})\|_{L^1(T)}\right) \tag{5.8}$$

Furthermore we make the following estimates for the terms appearing in (5.8).

- By definition $|\mathbf{e}| \le h_P$ and due to Lemma 3.12, $h_P^2 \lesssim |T|$. Thus we get
$$\frac{|\mathbf{e}|}{|T|} \lesssim \frac{1}{h_P}.$$
- By first using $T \subseteq P$ and then Lemma 5.9, we get
$$\|v - v_{\mathfrak{C}}\|^2_{L^2(T)} \le \|v - v_{\mathfrak{C}}\|^2_{L^2(P)} \lesssim h_P^2 \|\nabla v\|^2_{L^2(\omega_P)}.$$
- By first using $T \subseteq P$, then the Cauchy–Schwarz inequality (1.6), then the triangle inequality, then Lemma 5.9, and finally that $P \subseteq \omega_P$, we get
$$\begin{aligned}
\|(v - v_{\mathfrak{C}})\nabla(v - v_{\mathfrak{C}})\|_{L^1(T)} &\le \|(v - v_{\mathfrak{C}})\nabla(v - v_{\mathfrak{C}})\|_{L^1(P)} \\
&\lesssim \|v - v_{\mathfrak{C}}\|_{L^2(P)} \|\nabla(v - v_{\mathfrak{C}})\|_{L^2(P)} \\
&\le \|v - v_{\mathfrak{C}}\|_{L^2(P)} \left(\|\nabla v\|_{L^2(P)} + \|\nabla v_{\mathfrak{C}}\|_{L^2(P)}\right) \\
&\lesssim h_P \|\nabla v\|_{L^2(\omega_P)} \left(\|\nabla v\|_{L^2(P)} + \|\nabla v\|_{L^2(\omega_P)}\right) \\
&\lesssim h_P \|\nabla v\|^2_{L^2(\omega_P)}.
\end{aligned}$$

By inserting these estimates in (5.8), we get

$$\|v - v_{\mathfrak{C}}\|_{L^2(\mathbf{e})} \lesssim \frac{1}{h_P}\left(h_P^2 \|\nabla v\|^2_{L^2(\omega_P)} + \underbrace{h_T}_{\le h_P} h_P \|\nabla v\|^2_{L^2(\omega_P)}\right) \lesssim h_P \|\nabla v\|^2_{L^2(\omega_P)}.$$

This concludes the proof. □

5.2 Reliability of the error estimate

Theorem 5.11. *There exists a constant $C > 0$, independent of h, such that*

$$\|u - u_h\|_{H^1(\Omega)} + \sqrt{\sum_{P\in\Omega_h} \|\nabla(u - \Pi_m^P(u))\|^2_{L^2(P)}} \leq C\eta.$$

Proof. We define $e := u - u_h \in H_0^1(\Omega)$.

We split the proof into seven steps. First, we decompose the error into four terms. Then (in step two to five), we estimate each term individually. In the sixth step, we estimate $\left\|\nabla(u - \Pi_m^P(u))\right\|_{L^2(P)}$. Finally, we combine the estimates of these terms into an estimate of the error.

Step (i): decomposition of the error
By using the Friedrichs inequality 1.36, the κ ellapticity of A and by inserting $\mathcal{A}_h(u_h, e_{\mathfrak{c}})$ and $\langle A\nabla u, \nabla e_{\mathfrak{c}}\rangle_{L^2(\Omega)}$, we get

$$\begin{aligned}
&\|u - u_h\|^2_{H^1(\Omega)} = \|e\|^2_{H^1(\Omega)} \lesssim \|\nabla e\|^2_{L^2(\Omega)} \lesssim \langle A\nabla e, \nabla e\rangle_{L^2(\Omega)} \\
&= \langle A\nabla e, \nabla(e - e_{\mathfrak{c}})\rangle_{L^2(\Omega)} + \langle A\nabla \underbrace{(u - u_h)}_{=e}, \nabla e_{\mathfrak{c}}\rangle_{L^2(\Omega)} - \mathcal{A}_h(u_h, e_{\mathfrak{c}}) + \mathcal{A}_h(u_h, e_{\mathfrak{c}}) \\
&= \underbrace{\langle A\nabla e, \nabla(e - e_{\mathfrak{c}})\rangle_{L^2(\Omega)}}_{=:T_\mathrm{r}} + \underbrace{\langle A\nabla u, \nabla e_{\mathfrak{c}}\rangle_{L^2(\Omega)} - \mathcal{A}_h(u_h, e_{\mathfrak{c}})}_{=:T_\mathrm{l}} + \underbrace{\mathcal{A}_h(u_h, e_{\mathfrak{c}}) - \langle A\nabla u_h, \nabla e_{\mathfrak{c}}\rangle_{L^2(\Omega)}}_{=:T_\mathrm{c}}.
\end{aligned}$$

Let us look at the residual error T_r in more detail and decompose it into two parts. By first using that u is the weak solution to the Poisson-type problem and then integrating by parts, we can rewrite it as

$$\begin{aligned}
T_\mathrm{r} &= \langle A\nabla u, \nabla(e - e_{\mathfrak{c}})\rangle_{L^2(\Omega)} - \langle A\nabla u_h, \nabla(e - e_{\mathfrak{c}})\rangle_{L^2(\Omega)} \\
&= \langle f, e - e_{\mathfrak{c}}\rangle_{L^2(\Omega)} - \sum_{P\in\Omega_h} \langle A\nabla u_h, \nabla(e - e_{\mathfrak{c}})\rangle_{L^2(P)} \\
&= \langle f, e - e_{\mathfrak{c}}\rangle_{L^2(\Omega)} + \sum_{P\in\Omega_h} \langle \operatorname{div}(A\nabla u_h), e - e_{\mathfrak{c}}\rangle_{L^2(P)} - \sum_{P\in\Omega_h} \langle (A\nabla u_h)\cdot \mathbf{n}, e - e_{\mathfrak{c}}\rangle_{L^2(\partial P)}
\end{aligned}$$

By using that $e, e_{\mathfrak{c}} \in H_0^1(\Omega)$ and that each inner edge is an edge of exactly two polygons we can rewrite the last term as

$$\sum_{P\in\Omega_h} \langle (A\nabla u_h)\cdot \mathbf{n}, e - e_{\mathfrak{c}}\rangle_{L^2(\partial P)} = \sum_{P\in\Omega_h} \sum_{\mathbf{e}\in \mathrm{edges}(P)} \frac{1}{2} \langle [\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}, e - e_{\mathfrak{c}}\rangle_{L^2(\partial P)}.$$

Thus the residual error is equal to

$$T_{\mathrm{r}} = \langle f + \operatorname{div}(A\nabla u_h), e - e_{\mathfrak{c}} \rangle_{L^2(\Omega)} - \sum_{P\in\Omega_h} \sum_{\mathbf{e}\in \mathrm{edges}(P)} \frac{1}{2} \langle [\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}, e - e_{\mathfrak{c}} \rangle_{L^2(\partial P)}$$
$$= \underbrace{\sum_{P\in\Omega_h} \langle f + \operatorname{div}(A\nabla u_h), e - e_{\mathfrak{c}} \rangle_{L^2(P)}}_{:=T_{\mathrm{r}}^1} - \underbrace{\sum_{P\in\Omega_h} \sum_{\mathbf{e}\in \mathrm{edges}(P)} \frac{1}{2} \langle [\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}, e - e_{\mathfrak{c}} \rangle_{L^2(\partial P)}}_{:=T_{\mathrm{r}}^2} .$$

Therefore, it holds that

$$\|\nabla e\|_{L^2(\Omega)}^2 \lesssim \left|T_{\mathrm{r}}^1\right| + \left|T_{\mathrm{r}}^2\right| + |T_{\mathrm{l}}| + |T_{\mathrm{c}}| ,$$

and we are left to dominate $\left|T_{\mathrm{r}}^1\right|, \left|T_{\mathrm{r}}^2\right|, |T_{\mathrm{l}}|$ and $|T_{\mathrm{c}}|$.

Step (ii): estimation of T_{r}^1
By using the Cauchy–Schwarz inequality (1.6), then Lemma 5.9, the Cauchy–Schwarz inequality for sums (1.7), and finally the limited number of neighbors for each polygon $P \in \Omega_h$, we get

$$\begin{aligned}
\left|T_{\mathrm{r}}^1\right| &\le \sum_{P\in\Omega_h} \|f + \operatorname{div}(A\nabla u_h)\|_{L^2(P)} \|e - e_{\mathfrak{c}}\|_{L^2(P)} \\
&\lesssim \sum_{P\in\Omega_h} \|f + \operatorname{div}(A\nabla u_h)\|_{L^2(P)} h_P \|\nabla e\|_{L^2(\omega_P)} \\
&\le \sqrt{\sum_{P\in\Omega_h} h_P^2 \|f + \operatorname{div}(A\nabla u_h)\|_{L^2(P)}} \sqrt{\sum_{P\in\Omega_h} \|\nabla e\|_{L^2(\omega_P)}} \\
&\lesssim \sqrt{\sum_{P\in\Omega_h} h_P^2 \|f + \operatorname{div}(A\nabla u_h)\|_{L^2(P)}^2} \|\nabla e\|_{L^2(\Omega)} .
\end{aligned}$$

Due to the triangle inequality and since

$$\left\|\operatorname{div}(A\nabla u_h) - \pi_m^P(\operatorname{div}(A\nabla u_h))\right\|_{L^2(P)} = \inf_{p\in\mathbb{P}_m} \|\operatorname{div}(A\nabla u_h) - p\|_{L^2(P)} ,$$

we get that

$$\begin{aligned}
h_P \|f + \operatorname{div}(A\nabla u_h)\|_{L^2(P)} &\le \underbrace{h_P \left\|f + \pi_m^P(\operatorname{div}(A\nabla u_h))\right\|_{L^2(P)}}_{\eta_P^{\mathrm{r}}} + h_P \left\|\operatorname{div}(A\nabla u_h) - \pi_m^P(\operatorname{div}(A\nabla u_h))\right\|_{L^2(P)} \\
&= \eta_P^{\mathrm{r}} + h_P \inf_{p\in\mathbb{P}_m} \|\operatorname{div}(A\nabla u_h) - p\|_{L^2(P)} .
\end{aligned}$$

According to our assumptions regarding A and since $\Pi_m^P(u_h) \in \mathbb{P}_m$, it holds that $\operatorname{div}(A\nabla\Pi_m^P(u_h)) \in \mathbb{P}_m(P)$ and thus

$$h_P \|f + \operatorname{div}(A\nabla u_h)\|_{L^2(P)} \le \eta_P^{\mathrm{r}} + h_P \left\|\operatorname{div}(A\nabla u_h) - \operatorname{div}(A\nabla\Pi_m^P(u_h))\right\|_{L^2(P)} .$$

By using Lemma 5.7 and the consistency of $\mathcal{A}_{h|P}$ (3.20), we derive that

$$\begin{aligned}
h_P \|f + \mathrm{div}(A\nabla u_h)\|_{L^2(P)} &\le \eta_P^{\mathrm{r}} + h_P \left\|\mathrm{div}(A\nabla u_h) - \mathrm{div}(A\nabla \Pi_m^P(u_h))\right\|_{L^2(P)} \\
&= \eta_P^{\mathrm{r}} + h_P \left\|\mathrm{div}(A\nabla(u_h - \Pi_m^P(u_h)))\right\|_{L^2(P)} \\
&\lesssim \eta_P^{\mathrm{r}} + h_P \left\|\nabla(A\nabla(u_h - \Pi_m^P(u_h)))\right\|_{L^2(P)} \\
&\lesssim \eta_P^{\mathrm{r}} + \left\|A\nabla(u_h - \Pi_m^P(u_h))\right\|_{L^2(P)} \\
&\lesssim \eta_P^{\mathrm{r}} + \underbrace{\sqrt{\mathcal{A}_{h|P}(u_h - \Pi_m^P(u_h), u_h - \Pi_m^P(u_h))}}_{=\eta_P^{\mathrm{c}}}.
\end{aligned}$$

Thus, we get

$$|T_{\mathrm{r}}^1| \lesssim \sqrt{\sum_{P\in\Omega_h} (\eta_P^{\mathrm{r}} + \eta_P^{\mathrm{c}})^2}\, \|\nabla e\|_{L^2(\Omega)} \le \sqrt{\sum_{P\in\Omega_h} \eta_P^2}\, \|\nabla e\|_{L^2(\Omega)}.$$

Step (iii): estimation of T_{r}^2

By first using the Cauchy–Schwarz inequality (1.6), then Lemma 5.10, and finally the Cauchy–Schwarz inequality for sums (1.7) combined with the uniformly bounded number of edges per polygon $P \in \Omega_h$, we can estimate T_{r}^2 by

$$\begin{aligned}
|T_{\mathrm{r}}^2| &\le \sum_{P\in\Omega_h} \sum_{\mathbf{e}\in\mathrm{edges}(P)} \frac{1}{2} |\langle [\![A\nabla u_h \mathbf{n}]\!]_{\mathbf{e}}, (e - e_{\mathfrak{C}})\rangle_{L^2(\mathbf{e})}| \\
&\le \sum_{P\in\Omega_h} \frac{1}{2} \sum_{\mathbf{e}\in\mathrm{edges}(P)} \|[\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}\|_{L^2(\mathbf{e})} \|e - e_{\mathfrak{C}}\|_{L^2(\mathbf{e})} \\
&\lesssim \sum_{P\in\Omega_h} \frac{1}{2} \sum_{\mathbf{e}\in\mathrm{edges}(P)} \|[\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}\|_{L^2(\mathbf{e})} \sqrt{h_P}\, \|\nabla e\|_{L^2(\omega_P)} \\
&\lesssim \sum_{P\in\Omega_h} \frac{1}{2} \sqrt{\sum_{\mathbf{e}\in\mathrm{edges}(P)} \|[\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}\|_{L^2(\mathbf{e})}^2\, h_P}\, \|\nabla e\|_{L^2(\omega_P)} \\
&\lesssim \sqrt{\sum_{P\in\Omega_h} \frac{1}{2} \sum_{\mathbf{e}\in\mathrm{edges}(P)} h_P \|[\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}\|_{L^2(\mathbf{e})}^2}\, \|\nabla e\|_{L^2(\Omega)} \\
&= \sqrt{\sum_{P\in\Omega_h} (\eta_P^{\mathrm{j}})^2}\, \|\nabla e\|_{L^2(\Omega)} \le \sqrt{\sum_{P\in\Omega_h} \eta_P^2}\, \|\nabla e\|_{L^2(\Omega)}.
\end{aligned}$$

Step (iv): estimation of T_{l}

Since u is the solution of the Poisson problem (2.1) and u_h is the solution to the discrete problem, the loading term error T_{l} can be written as

$$T_{\mathrm{l}} = \langle A\nabla u, \nabla e_{\mathfrak{C}}\rangle_{L^2(\Omega)} - \mathcal{A}_h(u_h, e_{\mathfrak{C}}) = \langle f, e_{\mathfrak{C}}\rangle_{L^2(\Omega)} - F_h(e_{\mathfrak{C}}).$$

Note that this is also the reason why we called T_{l} the loading term error. As $m \geq 2$, $F_h(e_{\mathfrak{c}}) = \langle f_h, e_{\mathfrak{c}} \rangle_{L^2(\Omega)}$ (see Definition 3.62) and thus we can write T_{l} as

$$T_{\mathrm{l}} = \langle f - f_h, e_{\mathfrak{c}} \rangle_{L^2(\Omega)} = \sum_{P \in \Omega_h} \langle f - f_h, e_{\mathfrak{c}} \rangle_{L^2(P))} = \sum_{P \in \Omega_h} \langle f - \pi^P_{m-2}(f), e_{\mathfrak{c}} \rangle_{L^2(P)}.$$

Since $m - 2 \geq 0$, the term $\langle f - \pi^P_{m-2}(f), \pi^P_0(e_{\mathfrak{c}}) \rangle_{L^2(P)} = 0$ for every polygon $P \in \Omega_h$. By adding it to the right side of the equation above and using the Cauchy Schwarz inequality, the loading term can be estimated by

$$T_{\mathrm{l}} = \sum_{P \in \Omega_h} \langle f - \pi^P_{m-2}(f), e_{\mathfrak{c}} - \pi^P_0(e_{\mathfrak{c}}) \rangle_{L^2(P)} \leq \sum_{P \in \Omega_h} \left\| f - \pi^P_{m-2}(f) \right\|_{L^2(P)} \left\| e_{\mathfrak{c}} - \pi^P_0(e_{\mathfrak{c}}) \right\|_{L^2(P)}$$

Due to Lemma 3.10 the estimate $\left\| e_{\mathfrak{c}} - \pi^P_0(e_{\mathfrak{c}}) \right\|_{L^2(P)} \lesssim h_P \left\| \nabla e_{\mathfrak{c}} \right\|_{L^2(P)}$ holds for every $P \in \Omega_h$. By using this estimate, the Cauchy Schwarz inequality, and finally Lemma 5.9, we derive that

$$\begin{aligned} |T_{\mathrm{l}}| &\lesssim \sum_{P \in \Omega_h} \underbrace{\left\| f - \pi^P_{m-2}(f) \right\|_{L^2(P)} h_P}_{= \eta^{\mathrm{l}}_P} \left\| \nabla e_{\mathfrak{c}} \right\|_{L^2(P)} \leq \sqrt{\sum_{P \in \Omega_h} (\eta^{\mathrm{l}}_P)^2} \left\| \nabla e_{\mathfrak{c}} \right\|_{L^2(\Omega)} \\ &\lesssim \sqrt{\sum_{P \in \Omega_h} (\eta^{\mathrm{l}}_P)^2} \left\| \nabla e \right\|_{L^2(\Omega)} \leq \sqrt{\sum_{P \in \Omega_h} \eta^2_P} \left\| \nabla e \right\|_{L^2(\Omega)}. \end{aligned}$$

Step (v): estimation of T_{c}

The term T_{c} can be seen as the inconsistency of the approximation of $\langle A\nabla \cdot, \nabla \cdot \rangle_{L^2(\Omega)}$ by $\mathcal{A}_h$. By inserting the term $\mathcal{A}_h(\Pi^P_m(u_h), e_{\mathfrak{c}})$ and using the consistency of $\mathcal{A}_h$ (3.20), it can be rewritten as

$$\begin{aligned} T_{\mathrm{c}} &= \mathcal{A}_h(u_h, e_{\mathfrak{c}}) - \langle A\nabla u_h, \nabla e_{\mathfrak{c}} \rangle_{L^2(\Omega)} \\ &= \mathcal{A}_h(u_h - \Pi^P_m(u_h), e_{\mathfrak{c}}) + \mathcal{A}_h(\Pi^P_m(u_h), e_{\mathfrak{c}}) - \langle A\nabla u_h, \nabla e_{\mathfrak{c}} \rangle_{L^2(\Omega)} \\ &= \mathcal{A}_h(u_h - \Pi^P_m(u_h), e_{\mathfrak{c}}) + \langle A\nabla \Pi^P_m(u_h), \nabla e_{\mathfrak{c}} \rangle_{L^2(\Omega)} - \langle A\nabla u_h, \nabla e_{\mathfrak{c}} \rangle_{L^2(\Omega)} \\ &= \mathcal{A}_h(u_h - \Pi^P_m(u_h), e_{\mathfrak{c}}) - \langle A\nabla (u_h - \Pi^P_m(u_h)), \nabla e_{\mathfrak{c}} \rangle_{L^2(\Omega)}. \end{aligned}$$

By using first the Cauchy–Schwarz inequality (for both scalar products $\langle A\nabla \cdot, \nabla \cdot \rangle_{L^2(P)}$ and $\mathcal{A}_h$), then the consistency of $\mathcal{A}_h$ (3.20) and finally Lemma 5.9, we derive

$$
\begin{aligned}
|T_{\mathrm{c}}| &\leq \sqrt{\mathcal{A}_h(u_h - \Pi_m^P(u_h), u_h - \Pi_m^P(u_h)}\underbrace{\sqrt{\mathcal{A}_h(e_{\mathfrak{C}}, e_{\mathfrak{C}})}}_{\lesssim \|\nabla e_{\mathfrak{C}}\|_{L^2(\Omega)}} + \cdots \\
&\cdots + \underbrace{\sqrt{\langle A\nabla(u_h - \Pi_m^P(u_h)), \nabla(u_h - \Pi_m^P(u_h))\rangle_{L^2(\Omega)}}}_{\lesssim \sqrt{\mathcal{A}_h(u_h - \Pi_m^P(u_h), u_h - \Pi_m^P(u_h)}} \underbrace{\sqrt{\langle A\nabla e_{\mathfrak{C}}, \nabla e_{\mathfrak{C}}\rangle_{L^2(\Omega)}}}_{\lesssim \|\nabla e_{\mathfrak{C}}\|_{L^2(\Omega)}} \\
&\lesssim \sqrt{\mathcal{A}_h(u_h - \Pi_m^P(u_h), u_h - \Pi_m^P(u_h)} \, \|\nabla e_{\mathfrak{C}}\|_{L^2(\Omega)} = \sqrt{\sum_{P\in\Omega_h} (\eta_P^{\mathrm{c}})^2} \, \|\nabla e_{\mathfrak{C}}\|_{L^2(\Omega)} \\
&\lesssim \sqrt{\sum_{P\in\Omega_h} \eta_P^2} \, \|\nabla e\|_{L^2(\Omega)} .
\end{aligned}
$$

Step (vi): combination of the estimates
As we have seen in step (ii)-(v), it holds that

$$
\|\nabla e\|_{L^2(\Omega)}^2 \lesssim \left|T_{\mathrm{r}}^1\right| + \left|T_{\mathrm{r}}^2\right| + |T_{\mathrm{l}}| + |T_{\mathrm{c}}| \lesssim \eta \, \|\nabla e\|_{L^2(\Omega)} .
$$

Thus, we conclude

$$
\|u - u_h\|_{H^1(\Omega)} \lesssim \|\nabla e\|_{L^2(\Omega)} \lesssim \eta.
$$

Step (vii): estimation of $\sqrt{\sum_{P\in\Omega_h} \|\nabla(u - \Pi_m^P(u))\|_{L^2(P)}^2}$
By using the consistency of the bilinear form $\mathcal{A}_{h|P}$, we get

$$
\begin{aligned}
\left\|\nabla(u - \Pi_m^P(u))\right\|_{L^2(P)} &\leq \|\nabla(u - u_h)\|_{L^2(P)} + \underbrace{\left\|\nabla(\Pi_m^P(u_h) - \Pi_m^P(u))\right\|_{L^2(P)}}_{\lesssim \|\nabla(u-u_h)\|_{L^2(P)}} + \left\|\nabla(u_h - \Pi_m^P(u_h))\right\|_{L^2(P)} \\
&\lesssim \|\nabla(u - u_h)\|_{L^2(P)} + \sqrt{\mathcal{A}_{h|P}(u_h - \Pi_m^P(u_h), u_h - \Pi_m^P(u_h))} \\
&= \|\nabla(u - u_h)\|_{L^2(P)} + \eta_P^{\mathrm{c}}.
\end{aligned}
$$

By squaring and summing over all $P \in \Omega_h$ and using the result obtained in step (vi), we finally derive

$$
\begin{aligned}
\sqrt{\sum_{P\in\Omega_h} \|\nabla(u - \Pi_m^P(u))\|_{L^2(P)}^2} &\lesssim \sqrt{\sum_{P\in\Omega_h} \|\nabla(u - u_h)\|_{L^2(P)}^2} + \sqrt{\sum_{P\in\Omega_h} (\eta_P^{\mathrm{c}})^2} \\
&= \|\nabla(u - u_h)\|_{L^2(\Omega)} + \sqrt{\sum_{P\in\Omega_h} (\eta_P^{\mathrm{c}})^2} \lesssim \eta.
\end{aligned}
$$

This concludes the proof. □

5.3 Efficiency of the error estimate

Assumption 6. *Throughout this section, we make the additional assumption that the inhomogeneity f of the Poisson-type problem (2.1) is piecewise polynomial of a fixed degree. Furthermore we assume the regularity index $\alpha = 2$.*

Before we prove the efficiency of our error estimator, we want to cite the following lemma from [CaGePrSu].

Lemma 5.12 (Inverse inequality)**.** *Let $k \in \mathbb{N}$ and $v \in H^1(\Omega)$ such that*

$$\operatorname{div}(A\nabla v) \in \mathbb{P}_k(P) \text{ for all } P \in \Omega_h.$$

Then there exists a constant C that depends on k, but not on v or h, such that

$$\|\operatorname{div}(A\nabla v)\|_{L^2(P)} \leq C \|\nabla v\|_{L^2(P)}.$$

Proof. For the proof, we refer to [CaGePrSu, Lemma 4]. Note that the source only proved the statement for $A = \mathrm{I}_2$. However the generalization for general constant elliptic A is done easily. □

Theorem 5.13. *There exists a constant $C > 0$ independent of h such that*

$$\eta_P^2 \leq C \left(\|\nabla(u - u_h)\|^2_{L^2(\omega_P)} + \left\|\nabla(u - \Pi_m^P(u))\right\|^2_{L^2(\omega_P)} \right) \text{ for every } P \in \Omega_h.$$

Therefore there exists a constant $C > 0$ independent of h such that

$$\eta \leq C \left(\|u - u_h\|_{H^1(\Omega)} + \sqrt{\sum_{P \in \Omega_h} \|\nabla(u - \Pi_m^P(u))\|^2_{L^2(P)}} \right).$$

If the jump term $\eta_P^{\mathrm{j}} = 0$ for all P, then the last inequality reduces to

$$\eta_P^2 \leq C \left(\|\nabla(u - u_h)\|^2_{L^2(P)} + \left\|\nabla(u - \Pi_m^P(u))\right\|^2_{L^2(P)} \right) \text{ for every } P \in \Omega_h.$$

Proof. As the local error estimator is made up of four parts

$$\eta_P^2 := (\eta_P^{\mathrm{r}})^2 + (\eta_P^{\mathrm{l}})^2 + (\eta_P^{\mathrm{c}})^2 + (\eta_P^{\mathrm{j}})^2,$$

we have to dominate $\eta_P^{\mathrm{r}}, \eta_P^{\mathrm{l}}, \eta_P^{\mathrm{c}}$ and η_P^{j}.

Step (i): estimation of η_P^{c}

Using the consistency of $\mathcal{A}_{h|P}$ and the boundedness of A, we get

$$\eta_P^{\mathrm{c}} = \sqrt{\mathcal{A}_{h|P}(u_h - \Pi_m^P(u_h), u_h - \Pi_m^P(u_h))} \lesssim \sqrt{\langle A\nabla(u_h - \Pi_m^P(u_h)), \nabla(u_h - \Pi_m^P(u_h))\rangle_{L^2(P)}}$$
$$\lesssim \left\|\nabla(u_h - \Pi_m^P(u_h))\right\|_{L^2(P)}.$$

We use the triangle inequality to further estimate η_P^{c} by

$$\eta_P^{\mathrm{c}} \lesssim \|\nabla(u-u_h)\|_{L^2(P)} + \left\|\nabla(u-\Pi_m^P(u))\right\|_{L^2(P)} + \left\|\nabla(\Pi_m^P(u)-\Pi_m^P(u_h))\right\|_{L^2(P)}.$$

Since Π_m^P is a $\langle A\nabla\cdot,\nabla\cdot\rangle_{L^2(P)}$-orthogonal projection onto $\mathbb{P}_m(P)$ we can estimate the last term by

$$\begin{aligned}\left\|\nabla(\Pi_m^P(u)-\Pi_m^P(u_h))\right\|_{L^2(P)} &\lesssim \sqrt{\langle A\nabla(\Pi_m^P(u)-\Pi_m^P(u_h)),\nabla(\Pi_m^P(u)-\Pi_m^P(u_h))\rangle_{L^2(P)}}\\ &\leq \sqrt{\langle A\nabla(u-u_h),\nabla(u-u_h)\rangle_{L^2(P)}} \lesssim \|\nabla(u-u_h)\|_{L^2(P)}\end{aligned}$$

Thus

$$\eta_P^{\mathrm{c}} \lesssim \|\nabla(u-u_h)\|_{L^2(P)} + \left\|\nabla(u-\Pi_m^P(u))\right\|_{L^2(P)}.$$

Step (ii): estimation of η_P^{r}
By using the triangle inequality and the fact that u is the solution to the Poisson-type problem, we get

$$\begin{aligned}\eta_P^{\mathrm{r}} &= h_P\left\|f+\pi_m^P(\operatorname{div}(A\nabla u_h))\right\|_{L^2(P)}\\ &\leq h_P\|f+\operatorname{div}(A\nabla u_h)\|_{L^2(P)} + h_P\left\|\operatorname{div}(A\nabla u_h)-\pi_m^P(\operatorname{div}(A\nabla u_h))\right\|_{L^2(P)}\\ &= h_P\|\operatorname{div}(A\nabla u)-\operatorname{div}(A\nabla u_h)\|_{L^2(P)} + h_P\left\|(\mathrm{I}-\pi_m^P)(\operatorname{div}(A\nabla u_h))\right\|_{L^2(P)}\\ &= h_P\|\operatorname{div}(A\nabla(u-u_h))\|_{L^2(P)} + h_P\left\|(\mathrm{I}-\pi_m^P)(\operatorname{div}(A\nabla u_h))\right\|_{L^2(P)}.\end{aligned}$$

We estimate the two last terms seperately. First we deal with $h_P\|\operatorname{div}(A\nabla(u-u_h))\|_{L^2(P)}$. By applying the triangle inequality two times, it can be dominated by

$$\begin{aligned}\|\operatorname{div}(A\nabla(u-u_h))\|_{L^2(P)} \leq \left\|\operatorname{div}(A\nabla(u-\Pi_m^P(u)))\right\|_{L^2(P)} + \left\|\operatorname{div}(A\nabla(\Pi_m^P(u)-\Pi_m^P(u_h)))\right\|_{L^2(P)} + \cdots\\ \cdots + \left\|\operatorname{div}(A\nabla(\Pi_m^P(u_h)-u_h))\right\|_{L^2(P)}\end{aligned}$$

Since the terms

$$\begin{aligned}\operatorname{div}(A\nabla u) = f \in \mathbb{P}_k(P),\\ \operatorname{div}(A\nabla\Pi_m^P(u)),\operatorname{div}(A\nabla\Pi_m^P(u)) \in \mathbb{P}_{m-2}(P)\end{aligned}$$

are polynomials, we can apply the inverse inequality (Lemma 5.12) to get

$$h_P\left\|\operatorname{div}(A\nabla(u-\Pi_m^P(u)))\right\|_{L^2(P)} \lesssim \left\|A\nabla(u-\Pi_m^P(u))\right\|_{L^2(P)} \lesssim \left\|\nabla(u-\Pi_m^P(u))\right\|_{L^2(P)},$$

and

$$\begin{aligned}h_P\left\|\operatorname{div}(A\nabla(\Pi_m^P(u)-\Pi_m^P(u_h)))\right\|_{L^2(P)} \lesssim \left\|A\nabla(\Pi_m^P(u)-\Pi_m^P(u_h))\right\|_{L^2(P)} \leq \|A\nabla(u-u_h)\|_{L^2(P)}\\ \lesssim \|\nabla(u-u_h)\|_{L^2(P)}.\end{aligned}$$

Hereby the second inequality holds due to the $\langle A\nabla\cdot, \nabla\cdot\rangle_{L^2(P)}$-orthogonality of Π_m^P.

Since $\Pi_m^P(u_h) \in \mathbb{P}_m(P) \subseteq V_{h|P}$ and $u_h \in V_{h|P}$, we can use Lemma 5.7 to derive

$$h_P \left\|\operatorname{div}(A\nabla(\Pi_m^P(u_h) - u_h))\right\|_{L^2(P)} \lesssim h_P \left\|D^2(\Pi_m^P(u_h) - u_h))\right\|_{L^2(P)} \lesssim \left\|\nabla(\Pi_m^P(u_h) - u_h))\right\|_{L^2(P)}.$$

By utilising the stability of $\mathcal{A}_{h|P}$ and the result of step (i), this can be further dominated by

$$\begin{aligned}\left\|\nabla(\Pi_m^P(u_h) - u_h))\right\|_{L^2(P)} &\lesssim \left\|A\nabla(\Pi_m^P(u_h) - u_h))\right\|_{L^2(P)} \lesssim \eta_P^{\mathrm{c}} \\ &\lesssim \|\nabla(u - u_h)\|_{L^2(P)} + \left\|\nabla(u - \Pi_m^P(u))\right\|_{L^2(P)}.\end{aligned}$$

Therefore, we get

$$\|\operatorname{div}(A\nabla(u - u_h))\|_{L^2(P)} \lesssim \|\nabla(u - u_h)\|_{L^2(P)} + \left\|\nabla(u - \Pi_m^P(u))\right\|_{L^2(P)}.$$

The term $h_P \left\|(\mathrm{I} - \pi_m^P)(\operatorname{div}(A\nabla u_h))\right\|_{L^2(P)}$ can be bounded by using the stability of $\mathcal{A}_{h|P}$ and that $\operatorname{div}(A\nabla\Pi_m^P(u_h)) \in \mathbb{P}_{m-2}(P)$.

$$\begin{aligned}h_P \left\|(\mathrm{I} - \pi_m^P)(\operatorname{div}(A\nabla u_h))\right\|_{L^2(P)} &= h_P \left\|(\mathrm{I} - \pi_m^P)(\operatorname{div}(A\nabla(u_h - \Pi_m^P(u_h))))\right\|_{L^2(P)} \\ &\le h_P \left\|\operatorname{div}(A\nabla(u_h - \Pi_m^P(u_h)))\right\|_{L^2(P)} \\ &\lesssim h_P \left\|\nabla(A\nabla(u_h - \Pi_m^P(u_h)))\right\|_{L^2(P)} \\ &\lesssim \left\|A\nabla(u_h - \Pi_m^P(u_h))\right\|_{L^2(P)} \\ &\lesssim \sqrt{\mathcal{A}_{h|P}(u_h - \Pi_m^P(u_h), u_h - \Pi_m^P(u_h))} \\ &= \eta_P^{\mathrm{c}} \lesssim \|\nabla(u - u_h)\|_{L^2(P)} + \left\|\nabla(u_h - \Pi_m^P(u_h))\right\|_{L^2(P)}.\end{aligned}$$

The last inequality is a consequence of step (i).

Thus, we obtain

$$\eta_P^{\mathrm{r}} \lesssim \|\nabla(u - u_h)\|_{L^2(P)} + \left\|\nabla(u_h - \Pi_m^P(u_h))\right\|_{L^2(P)}.$$

<u>Step (iii):</u> estimation of η_P^{l}

By using the definition of f_h, that $\operatorname{div}(A\nabla\Pi_m^P(u)) \in \mathbb{P}_{m-2}(P)$, and the orthogonality of π_{m-2}^P, we get

$$\begin{aligned}\eta_P^{\mathrm{l}} &= h_P \|f - f_h\|_{L^2(P)} = h_P \left\|f - \pi_{m-2}^P(f)\right\|_{L^2(P)} = h_P \left\|(I - \pi_{m-2}^P)(\operatorname{div}(A\nabla u))\right\|_{L^2(P)} \\ &= h_P \left\|(I - \pi_{m-2}^P)(\operatorname{div}(A\nabla(u - \Pi_m^P(u))))\right\|_{L^2(P)} \le h_P \left\|\operatorname{div}(A\nabla(u - \Pi_m^P(u)))\right\|_{L^2(P)}.\end{aligned}$$

Since $\operatorname{div}(A\nabla(u - \Pi_m^P(u))) \in \mathbb{P}_m(P)$, we can use the inverse inequality 5.12 to infer that

$$\eta_P^{\mathrm{i}} \lesssim h_P \frac{1}{h_P} \left\| \nabla(u - \Pi_m^P(u)) \right\|_{L^2(P)} = \left\| \nabla(u - \Pi_m^P(u)) \right\|_{L^2(P)}.$$

Step (iv): estimation of η_P^{j}

As the number of edges per polygon P is uniformly limited (independent of h), it is sufficient to find a bound for $\sqrt{h_P}\, \| [\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}} \|_{L^2(\mathbf{e})}$ in order to estimate

$$\eta_P^{\mathrm{j}} = \sqrt{\frac{1}{2} \sum_{\mathbf{e} \in \mathrm{edges}(P)} h_P \, \| [\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}} \|^2_{L^2(\mathbf{e})}}.$$

Let $\mathbf{e} \in \mathrm{edges}(P)$. If $\mathbf{e} \subseteq \partial\Omega$, then by definition $[\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}} = 0$. Thus we can assume that $\mathbf{e}$ is an inner edge.

Due to Lemma 3.12, there is a triangle $T_j \subseteq P_j(\mathbf{e}), j = 1, 2$ that is contained in each of the two polygons and for which $\mathbf{e}$ is one of the three sides.

We define $\omega_{\mathbf{e}} := T_1 \bigcup T_2$. Note that $\{T_1, T_2\}$ is a triangulation of $\omega_{\mathbf{e}}$. Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ be the set of nodes of this triangulation, i.e., the vertices of both triangles. Here $\mathbf{x}_1, \mathbf{x}_2$ shall be the nodes which are connected by $\mathbf{e}$.

We extend $[\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}$ constantly in the direction orthogonal to the edge $\mathbf{e}$ onto $\omega_{\mathbf{e}}$, i.e., if $\mathrm{p}_{\mathbf{e}} : \omega_{\mathbf{e}} \to \mathbf{e}$ denotes the orthogonal projection onto $\mathbf{e}$, then

$$[\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}(x) := [\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}(\mathrm{p}_{\mathbf{e}}(x)) \text{ for all } x \in \omega_{\mathbf{e}} \setminus \mathbf{e}.$$

Notice that due to Corollary 3.36, $\nabla u_h|_{\mathbf{e}} \in \mathbb{P}_{\alpha_1}(\mathbf{e})$ and therefore $[\![A\nabla u_h \cdot \mathbf{n}]\!]_{\mathbf{e}}$ is a polynomial.

Furthermore we define the two hatfunctions $\zeta_1, \zeta_2 : \omega_{\mathbf{e}} \to \mathbb{R}$ to be the two piecewise linear functions such that for $j = 1, 2$ and $k = 1, \ldots, 4$

$$\zeta_j(\mathbf{x}_k) = \begin{cases} 1 \text{ if } j = k, \\ 0 \text{ if } j \neq k. \end{cases}$$

We define the edge bubble function $b := \zeta_1 \zeta_2$. Note that b is piecewise differentiable and continuous and therefore in $H^1(\omega_{\mathbf{e}})$. Furthermore $b = 0$ on $\partial\omega_{\mathbf{e}} = (\partial T_1 \bigcup \partial T_2) \setminus \mathbf{e}$.

It can be proved (see [CaGePrSu, Lemma 3]) that for all $k \in \mathbb{N}$ and all $p \in \mathbb{P}_k(\omega_{\mathbf{e}})$ it holds that

$$\|p\|^2_{L^2(\mathbf{e})} \lesssim \langle p, b\, p \rangle_{L^2(\mathbf{e})} \lesssim \|p\|^2_{L^2(\mathbf{e})} \tag{5.9}$$

and that

$$|\mathbf{e}|^{-1/2} \|b\, p\|_{L^2(\omega_\mathbf{e})} + |\mathbf{e}|^{1/2} \|\nabla(b\, p)\|_{L^2(\omega_\mathbf{e})} \lesssim \|p\|_{L^2(\mathbf{e})}.$$

Note that due to $h_P \lesssim |\mathbf{e}| \lesssim h_P$ the last inequality implies

$$h_P^{-1} \|b\, p\|_{L^2(\omega_\mathbf{e})} + \|\nabla(b\, p)\|_{L^2(\omega_\mathbf{e})} \lesssim h_P^{-1/2} \|p\|_{L^2(\mathbf{e})}. \tag{5.10}$$

The constants in these inequalities depend on k but not on the polygon, the edge or on h.

Let $v := b[\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}$. Since $b \in H_0^1(\omega_\mathbf{e})$ and since $[\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}$ is a polynom it holds that $v \in H_0^1(\omega_\mathbf{e})$. By using that u is the weak solution to the Poisson problem and by integrating by parts we get the following equality

$$\begin{aligned}
\langle A\nabla(u-u_h), \nabla v\rangle_{L^2(\omega_\mathbf{e})} &= \langle A\nabla u, \nabla v\rangle_{L^2(\omega_\mathbf{e})} - \langle A\nabla u_h, \nabla v\rangle_{L^2(\omega_\mathbf{e})}\\
&= \langle f, v\rangle_{L^2(\omega_\mathbf{e})} - \langle A\nabla u_h, \nabla v\rangle_{L^2(\omega_\mathbf{e})}\\
&= \langle f, v\rangle_{L^2(\omega_\mathbf{e})} - \langle A\nabla u_h, \nabla v\rangle_{L^2(T_1)} - \langle A\nabla u_h, \nabla v\rangle_{L^2(T_2)}\\
&= \langle f, v\rangle_{L^2(\omega_\mathbf{e})} + \langle \operatorname{div}(A\nabla u_h), v\rangle_{L^2(T_1)} + \langle \operatorname{div}(A\nabla u_h), v\rangle_{L^2(T_2)} - \cdots\\
&\quad \underbrace{\cdots - \langle A\nabla u_h \cdot \mathbf{n}_{T_1}, v\rangle_{L^2(\partial T_1)} - \langle A\nabla u_h \cdot \mathbf{n}_{T_2}, v\rangle_{L^2(\partial T_2)}}_{v=0 \text{ on } (\partial T_1 \bigcup \partial T_2)\setminus \mathbf{e}}\\
&= \langle f, v\rangle_{L^2(\omega_\mathbf{e})} + \langle \operatorname{div}(A\nabla u_h), v\rangle_{L^2(T_1)} + \langle \operatorname{div}(A\nabla u_h), v\rangle_{L^2(T_2)} - \cdots\\
&\quad \cdots - \langle A\nabla u_h \cdot \mathbf{n}_{T_1}, v\rangle_{L^2(\mathbf{e})} - \langle A\nabla u_h \cdot \mathbf{n}_{T_2}, v\rangle_{L^2(\mathbf{e})}\\
&= \langle f, v\rangle_{L^2(\omega_\mathbf{e})} + \langle \operatorname{div}(A\nabla u_h), v\rangle_{L^2(\omega_\mathbf{e})} - 2\langle [\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}, v\rangle_{L^2(\mathbf{e})}\\
&= \langle f + \operatorname{div}(A\nabla u_h), v\rangle_{L^2(\omega_\mathbf{e})} - 2\langle [\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}, v\rangle_{L^2(\mathbf{e})}.
\end{aligned}$$

By using (5.9) and the equation above, we get the estimate

$$\begin{aligned}
2\, \|[\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}\|_{L^2(\mathbf{e})}^2 &\lesssim 2\langle [\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}, b[\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}\rangle_{L^2(\mathbf{e})} = 2\langle [\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}, v\rangle_{L^2(\mathbf{e})}\\
&= \langle f + \operatorname{div}(A\nabla u_h), v\rangle_{L^2(\omega_\mathbf{e})} - \langle A\nabla(u-u_h), \nabla v\rangle_{L^2(\omega_\mathbf{e})}
\end{aligned}$$

We can dominate both $\langle f + \operatorname{div}(A\nabla u_h), v\rangle_{L^2(\omega_\mathbf{e})}$ and $\langle A\nabla(u-u_h), \nabla v\rangle_{L^2(\omega_\mathbf{e})}$ by using the Cauchy–Schwarz inequality (1.6) and (5.10).

$$\begin{aligned}
\left|\langle f + \operatorname{div}(A\nabla u_h), v\rangle_{L^2(\omega_\mathbf{e})}\right| &\le \|h_P(f + \operatorname{div}(A\nabla u_h))\|_{L^2(\omega_\mathbf{e})} \left\|h_P^{-1} v\right\|_{L^2(\omega_\mathbf{e})}\\
&= \|h_P(f + \operatorname{div}(A\nabla u_h))\|_{L^2(\omega_\mathbf{e})} \left\|h_P^{-1} b[\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}\right\|_{L^2(\omega_\mathbf{e})}\\
&\lesssim \|h_P(f + \operatorname{div}(A\nabla u_h))\|_{L^2(\omega_\mathbf{e})} \left\|h_P^{-1/2}[\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}\right\|_{L^2(\mathbf{e})}
\end{aligned}$$

$$\begin{aligned}
\left|\langle A\nabla(u-u_h), \nabla v\rangle_{L^2(\omega_\mathbf{e})}\right| &\le \|A\nabla(u-u_h)\|_{L^2(\omega_\mathbf{e})} \|\nabla v\|_{L^2(\omega_\mathbf{e})}\\
&= \|A\nabla(u-u_h)\|_{L^2(\omega_\mathbf{e})} \|\nabla b[\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}\|_{L^2(\omega_\mathbf{e})}\\
&\lesssim \|A\nabla(u-u_h)\|_{L^2(\omega_\mathbf{e})} \left\|h_P^{-1/2}[\![A\nabla u_h \cdot \mathbf{n}]\!]_\mathbf{e}\right\|_{L^2(\mathbf{e})}
\end{aligned}$$

Thus we attain

$$\sqrt{h_P}\,\|[\![A\nabla u_h\cdot\mathbf{n}]\!]_{\mathbf{e}}\|_{L^2(\mathbf{e})} \lesssim \left(\|h_P(f+\operatorname{div}(A\nabla u_h))\|_{L^2(\omega_{\mathbf{e}})} + \|A\nabla(u-u_h)\|_{L^2(\omega_{\mathbf{e}})}\right)$$
$$\leq \left(\sum_{j=1}^{2}\left\|h_{P_j(\mathbf{e})}(f+\operatorname{div}(A\nabla u_h))\right\|_{L^2(P_j(\mathbf{e}))} + \|A\nabla(u-u_h)\|_{L^2(P_j(\mathbf{e}))}\right)$$

Note that we have already estimated $\|h_P(f+\operatorname{div}(A\nabla u_h))\|_{L^2(P)}$ by $\eta_P^{\mathrm{r}}+\eta_P^{\mathrm{c}}$ in step (ii) of the proof of Theorem 5.11.
As both η_P^{r} and η_P^{c} have been bounded in step (i) and (ii), it holds that

$$\|h_P(f+\operatorname{div}(A\nabla u_h))\|_{L^2(P)} \lesssim \eta_P^{\mathrm{r}}+\eta_P^{\mathrm{c}} \lesssim \|\nabla(u-u_h)\|_{L^2(P)} + \left\|\nabla(u-\Pi_m^P(u))\right\|_{L^2(P)}.$$

Furthermore the boundedness of A gives us $\|A\nabla(u-u_h)\|_{L^2(P)} \lesssim \|\nabla(u-u_h)\|_{L^2(P)}$.

Therefore we get

$$\sqrt{h_P}\,\|[\![A\nabla u_h\cdot\mathbf{n}]\!]_{\mathbf{e}}\|_{L^2(\mathbf{e})} \lesssim \sum_{j=1}^{2}\left(\|\nabla(u-u_h)\|_{L^2(P_j(\mathbf{e}))} + \left\|\nabla(u-\Pi_m^P(u))\right\|_{L^2(P_j(\mathbf{e}))}\right),$$

and since all polygons that share an edge with P are part of its neighborhood neighbors(P), we conclude

$$\eta_P^{\mathrm{j}} = \sqrt{\frac{1}{2}\sum_{\mathbf{e}\in\operatorname{edges}(P)} h_P\,\|[\![A\nabla u_h\cdot\mathbf{n}]\!]_{\mathbf{e}}\|^2_{L^2(\mathbf{e})}}$$
$$\lesssim \sqrt{\sum_{\mathbf{e}\in\operatorname{edges}(P)}\sum_{j=1}^{2}\left(\|\nabla(u-u_h)\|_{L^2(P_j(\mathbf{e}))} + \|\nabla(u-\Pi_m^P(u))\|_{L^2(P_j(\mathbf{e}))}\right)^2}$$
$$\lesssim \|\nabla(u-u_h)\|_{L^2(\omega_P)} + \left\|\nabla(u-\Pi_m^P(u))\right\|_{L^2(\omega_P)}.$$

This concludes the proof.

□

List of Symbols

DOF_h	Equation (3.18), page 47	$\mathcal{A}_h$	Equation (3.19), page 48
$\mathcal{A}_{h\|P}$	Equation (3.19), page 48	$\overline{v}^P$	Equation (3.25), page 52
Π_m^P	Definition 3.55, page 53	$\mathcal{S}^P$	Definition 3.30, page 55
F_h	Definition 3.62, page 55	f_h	Definition 3.62, page 57
$\mathcal{M}$	Definition 3.65, page 60	p_j^P	Definition 3.62, page 57
$\mathfrak{A}_P$	Definition 3.66 page 60	ψ_j^P	Definition 3.62, page 57
$\mathfrak{A}$	Definition 3.76 page 66	$V_h(\mathfrak{d})$	Definition 3.81, page 68
$[\![\cdot]\!]_{\mathbf{e}}$	Definition 5.2 page 74	η_P^{r}	Equation (5.1), page 75
η_P^{l}	Equation (5.2) page 75	η_P^{c}	Equation (5.3), page 75
η_P^{j}	Equation (5.4) page 75	η_P	Equation (5.5), page 75
η	Equation (5.6) page 75	$v_{\mathfrak{C}}$	Lemma 5.9, page 78
ω_P	Definition 5.8 page 77		

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