Convergence of adaptive boundary element methods

Markus Aurada, Samuel Ferraz-Leite, and Dirk Praetorius
Vienna University of Technology, Institute for Analysis and Scientific Computing
Wiedner Hauptstr. 8-10, 1040 Wien, Austria
e-mail: {Markus.Aurada, Samuel.Ferraz-Leite, Dirk.Praetorius} @tuwien.ac.at

Abstract

A posteriori error estimators and adaptive mesh-refinement have themselves proven to be an important tool for scientific computing. For error control in finite element methods (FEM), there is a broad variety of a posteriori error estimators available, and convergence as well as optimality of adaptive FEM is well-studied in the literature. This is in sharp contrast to the boundary element method (BEM). Although a posteriori error estimators and adaptive algorithms are also successfully applied to boundary element schemes, even convergence of adaptive BEM is hardly understood mathematically. In our contribution, we present and discuss recent mathematical results which give first positive answers for adaptive BEM.

Keywords: boundary element method, a posteriori error estimate, adaptive mesh-refinement, convergence of adaptive algorithm

1. Introduction

For two reasons, fast and accurate error estimation plays a key role in reliable and efficient scientific computing: First, one may want to check whether the solution of a numerical simulation is accurate enough. Second, if this is not the case, one aims to improve the discretization, e.g., by local refinement of the underlying mesh.

Both subjects are usually covered by so-called a posteriori error estimates and related adaptive mesh-refining algorithms. For error control in finite element methods (FEM), there is a broad variety of a posteriori error estimators available, see e.g., [1], and convergence as well as optimality of adaptive algorithms is well understood, cf. [5] and the references therein. This is in sharp contrast to the boundary element method (BEM), where only a few a posteriori error estimators have been proposed, cf. [3] for an overview. Moreover, even convergence of adaptive BEM is widely open, and first preliminary convergence results have only recently been obtained [2, 7].

2. Model problem

As BEM model problem, we use the simple-layer potential

\[ Vu(x) = -\frac{1}{2\pi} \int_\Gamma \log |x-y| u(y) \, d\Gamma(y) \]

(1)

associated with the 2D Laplacian. Here, \( \Gamma \subseteq \partial \Omega \) is an open and connected piece of the boundary of a Lipschitz domain \( \Omega \subset \mathbb{R}^2 \). Provided \( \text{diam}(\Omega) < 1 \),

\[ \langle u, v \rangle := \int_\Gamma V u(x)v(x) \, d\Gamma(x) \]

(2)

defines an equivalent scalar product on the Sobolev space \( \mathcal{H} := \mathcal{H}^{-1/2}(\Gamma) \). For a given \( \Phi \in \mathcal{H}' \), the Lax-Milgram lemma thus proves the unique existence of (some unknown) \( u \in \mathcal{H} \) with

\[ \langle u, v \rangle = \Phi(v) \quad \text{for all } v \in \mathcal{H}. \]

(3)

To approximate \( u \) by the lowest-order Galerkin scheme, let \( \mathcal{T} \) be a triangulation of \( \Gamma \) and \( X \subset \mathcal{H}^0(\Gamma) := \{ v \in \mathcal{H} : \forall \Gamma \in \mathcal{T}, v|_{\partial \Gamma} \text{ is constant} \} \subset \mathcal{H} \). The (numerically computable) Galerkin solution \( \tilde{u} \in \mathcal{X} \) is the unique solution of

\[ \langle u_0, v_\ell \rangle = \Phi(v_\ell) \quad \text{for all } v_\ell \in X_\ell. \]

(4)

We aim to consider computable quantities \( \eta_e \) which only depend on known and computed data (e.g., on \( u_\ell \) and \( \Phi \)) such that

\[ C^{-1}_\text{eff} \eta_e \leq \| u - u_\ell \| \leq C \text{rel} \eta_e. \]

(5)

Here, \( \| \cdot \| \) denotes the energy norm induced by \( \langle \cdot, \cdot \rangle \). The lower and upper estimate are referred to as efficiency and reliability of the a posteriori error estimator \( \eta_e \), respectively.

3. \((h-h/2)\)-based error estimators

The \((h-h/2)\)-based strategy is a very basic and natural strategy to derive an a posteriori error estimator. Let \( u_\ell \in X_\ell \) and \( \bar{u}_\ell \in \bar{X}_\ell = \mathcal{P}^0(\bar{T}_\ell) \) be Galerkin solutions, where \( \bar{T}_\ell \) is obtained by uniform refinement of \( T_\ell \). One then considers

\[ \eta_e := \| \bar{u}_\ell - u_\ell \|. \]

(6)

to estimate \( \| u - u_\ell \|. \) By Galerkin orthogonality, \( \eta_e \) is always efficient with known constant \( C_\text{eff} = 1 \). Reliability of \( \eta_e \) with \( C\text{rel} = (1 - C^2_\text{sat})^{-1/2} \) follows from the saturation assumption

\[ \| u - \bar{u}_\ell \| \leq C_\text{sat} \| u - u_\ell \| \quad \text{with uniform } C_\text{sat} \in (0, 1). \]

(7)

Unlike the FEM, where (7) is proven for a sufficiently small mesh-size, the saturation assumption is open in the context of BEM but observed in practice, cf. [8] and the references therein.

Since the energy norm \( \| \cdot \| \) is nonlocal, the error estimator \( \eta_e \) does not provide information for a local mesh-refinement. By use of an inverse estimate and an approximation result, there holds

\[ C^{-1}_\text{inv} \eta_e \leq \mu_\ell := \| h^{1/2}_\ell (\bar{u}_\ell - u_\ell) \|_{L^2(T)} \leq C_\text{inv} \eta_e, \]

(8)

where \( h_\ell \in L^\infty(\Gamma) \) denotes the local mesh-size defined by \( h_\ell := \text{diam}(T) \) for \( T \in T_\ell \), cf. [8]. The local contributions

\[ \mu_\ell(T) := \text{diam}(T)^{1/2} \| \bar{u}_\ell - u_\ell \|_{L^2(T)} \]

(9)

of \( \mu_\ell \) are then used to steer the following adaptive algorithm.

4. Adaptive mesh-refining algorithm

Based on the estimator \( \mu_\ell \) from Section 3 and on a fixed parameter \( \theta \in (0, 1) \), the usual adaptive algorithm reads as follows:

Algorithm 1 For a given initial mesh \( T_0 \) with \( \ell = 0 \) do:

(i) Refine \( T_\ell \) uniformly to obtain \( \hat{T}_\ell \).

(ii) Compute discrete solutions \( u_\ell \) and \( \bar{u}_\ell \).
(iii) Find minimal set $M_\ell \subset T_\ell$ such that
\[
\theta \sum_{T \in T_\ell} \mu_T(T)^2 \leq \sum_{T \in M_\ell} \mu_T(T)^2.
\]
(iv) Refine at least marked elements $T \in M_\ell$ to obtain $T_{\ell+1}$.
(v) Increase counter $\ell \rightarrow \ell + 1$ and iterate.

Convergence of this type of algorithms has first been proven in [6], where also the marking criterion (10) is introduced. The latter work considered the residual estimator for a P1-FEM discretization of the Poisson problem, and it is assumed that the given data are sufficiently resolved on the initial mesh. In [9], the resolution of the data is included into the adaptive algorithm. The convergence analysis is based on reliability and the so-called discrete local efficiency of the residual estimator. The main idea of the convergence proof then is to show that the error is contractive up to the data oscillations. In [5], this has been weakened in the sense that it is proven that a weighted sum of error and error estimator yields a contraction property without requiring (discrete local) efficiency.

5. Convergence of $(h - h/2)$-steered adaptive BEM

Only recently, analogous results for adaptive BEM could be derived. A first convergence result for Algorithm 1 steered by $\mu_\ell$ from Section 3, reads as follows [7]:

**Theorem 2 (Ferraz-Leite, Ortner, Praetorius ’08)** Provided that $\mu_\ell$ is reliable and that marked elements are halved, there are constants $\kappa, \gamma \in (0,1)$ such that
\[
\Delta^2_{\ell + 1} := \| u - u_\ell \|^2 + \| u - \tilde{u}_\ell \|^2 + \gamma \mu_{\ell + 1}^2
\]
and error estimator yields a contraction property without requiring (discrete local) efficiency.

and an inductive argument thus proves convergence of $\mu_\ell$ to zero.

Finally, the saturation assumption (7) and Theorem 3 yield $\| u_\ell - u \| \leq C_{\text{sat}} \eta_\ell \rightarrow 0$, whence convergence of $u_\ell \rightarrow u$ as $\ell \rightarrow \infty$. Note that, in contrast to Theorem 2, the saturation assumption (7) now is only used in a second step.

6. Concluding remarks

The crucial step in the proofs of our convergence theorems is the estimator reduction (13), which is based on some $h$-weighting of the refinement indicators. Therefore, the analysis might carry over to adaptive algorithms steered by $h$-weighted residual error estimators or averaging error estimators, whereas the two-level error estimators and the Faermann error estimator seem to need further arguments. For the definition of these different error estimators, see [3, 4, 8] and the references therein.

Our analysis also applies to hypersingular integral equations and mixed formulations in 2D and 3D. For 3D, however, the proof in [7] —as well as the available a posteriori BEM error analysis—is restricted to the case of isotropic mesh-refinement, whereas anisotropic mesh-refinement is needed to resolve edge singularities efficiently. In addition, the new concept of convergence from [2] also seems to apply for certain anisotropic mesh-refining strategies as well as to adaptive FEM-BEM coupling.

Despite of convergence, even the question of optimal convergence rates of adaptive FEM is well-understood. Whereas prior works used an additional coarsening step, recent works prove optimality for Algorithm 1 steered by the residual error estimator. We refer to [5] and the references therein. The latter analysis, however, relies on a discrete local reliability of the error estimator, which remains open for adaptive Galerkin BEM. This will be a major topic for future research.

References