Inverse Problem for Quantum Graphs: Magnetic Boundary Control

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Quantum graph

- **Metric graph**

- **Differential expression** on the edges

\[ \mathcal{L}_{q,a} = \left( i \frac{d}{dx} + a(x) \right)^2 + q(x) \]

- **Matching conditions**
  Via irreducible unitary matrices \( S^m \) associated with each internal vertex \( V_m \)

\[ i(S^m - I)\vec{\psi}_m = (S^m + I)\partial\vec{\psi}_m, \quad m = 1, 2, \ldots, M. \]
**Inverse problem**

**Task**: reconstruct all three members from the family:
- the metric graph
- the potential(s) $q(x)$ and $a(x)$;
- the vertex conditions.

**Contact set** $\partial \Gamma$ - a fixed set of vertices containing all degree one vertices.

**Boundary control** - solution to the wave equation subject to control conditions on $\partial \Gamma$

$$\begin{cases}
(i \frac{d}{dx} + a(x))^2 u(x, t) + q(x)u(x, t) &= -\frac{\partial^2}{\partial t^2} u(x, t) \\
u(x, 0) &= u_t(x, 0) = 0, \\
u(\cdot, t)|_{\partial \Gamma} &= \vec{f}(t)
\end{cases}$$

**Response operator**

$$R^T : \vec{f}(t) \rightarrow \partial \vec{u}(\cdot, t)|_{\partial \Gamma} = \vec{u}(\cdot, t)|_{\partial \Gamma}$$
Response operator and $M$-function

Response operator

$R^T: \vec{f}(t) \mapsto \partial \vec{u}(\cdot, t)|_{\partial \Gamma}.
=\vec{u}(\cdot,t)|_{\partial \Gamma}$

Titchmarsh-Weyl matrix-valued $M$-function:
Consider $\psi(x, \lambda)$ - solution to the stationary equation

$\left(i \frac{d}{dx} + a(x)\right)^2 \psi(x, \lambda) + q(x)\psi(x, \lambda) = \lambda \psi(x, \lambda)$

$M(\lambda): \vec{\psi}|_{\partial \Gamma} \mapsto \partial \vec{\psi}|_{\partial \Gamma}$

Connection

$\left(\hat{R\vec{f}}\right)(s) = M(-s^2)\hat{\vec{f}}(s)$

where $\hat{}$ denotes the Laplace transform.
Two explicit formulas

\[
M_{\Gamma}(\lambda) = -\left( \sum_{n=1}^{\infty} \frac{\langle \psi_n^{st} | \partial \Gamma , \cdot \rangle \ell_2(\partial \Gamma) \psi_n^{st} | \partial \Gamma \rangle}{\lambda_n^{st} - \lambda} \right)^{-1},
\]

where \( \lambda_n^{st} \) and \( \psi_n^{st} \) are the eigenvalues and ortho-normalised eigenfunctions of \( L^{st} \).

\[
M_{\Gamma}(\lambda) - M_{\Gamma}(\lambda') = \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda_n^D - \lambda)(\lambda_n^D - \lambda')} \langle \partial \psi_n^D | \partial \Gamma , \cdot \rangle_{CB} \partial \psi_n^D | \partial \Gamma ,
\]

where \( \lambda_n^D \) and \( \psi_n^D \) are the eigenvalues and eigenfunctions of the Dirichlet Schrödinger operator \( L^D \).

These formulas indicate where zeroes and singularities of the \( M \)-functions may be situated.

Existence of invisible eigenfunctions.
Two explicit formulas

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These formulas indicate where zeroes and singularities of the \( M \)-functions may be situated.

Existence of invisible eigenfunctions.
Limitations

- The exact form of the magnetic potential plays no role \( \Rightarrow \) magnetic fluxes through cycles (if any)

\[
\Phi_j = \int_{C_j} a(y) dy \quad \Rightarrow \quad \text{fluxes } \Phi_j
\]

- Vertex conditions can be determined up to phases corresponding to internal edges:

\[
\psi(x) = e^{i\Theta_n} \hat{\psi}(x), \quad x \in E_n. \quad \Rightarrow \quad \text{phases } \Theta_n
\]

- One has to require that reflection and transmission from the vertices is non-trivial.

Ex.

\[
S^1 = \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0
\end{pmatrix}
\]

\[
(S^m(\infty))_{ij} \neq 0
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\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \]

\[ (S^m(\infty))_{ij} \neq 0 \]
Inverse problems for trees

**Geometric perturbations**
- peeling leaves;
- trimming brunches;
- cutting brunches.

**Cleaning procedure**
- removing potential $q$ on the boundary edges
  \[ \text{Schrödinger} \implies \text{Laplace} \]

All four principles are based on two Lemmas concerning gluing extensions of symmetric operators.
Gluing graphs

Two graphs with 4 and 3 contact vertices are transformed into a graph with 3 contact vertices.

One gluing point:
Gluing extensions of symmetric operators

Let $A_1$ and $A_2$ be two symmetric operators with the boundary values:

$$
\langle A_j^* u_j, v_j \rangle - \langle u_j, A_j^* v_j \rangle = \langle \tilde{u}_j^\partial, \partial \tilde{v}_j \rangle - \langle \partial \tilde{u}_j^\partial, \tilde{v}_j^\partial \rangle.
$$

$M$-functions

$$
M_{1,2}(\lambda) \tilde{\psi}^\partial = \partial \tilde{u}^\partial, \quad \text{where } A_j^* \psi_j = \lambda \psi_j.
$$

Consider the symmetric extension $A = A_1 \oplus A_2$ determined by the coupling conditions:

$$
\tilde{u}^\partial_1 |_{L_1} = \tilde{u}^\partial_2 |_{L_2}; \quad \partial \tilde{u}^\partial_1 |_{L_1} = -\partial \tilde{u}^\partial_2 |_{L_2},
$$

where $L_1, L_2$ are two identified subspaces of the same dimension.

**Lemma 1** (following Schur-Frobenius)

The $M$-functions associated with the operators $A_1, A_2$, and $A$ are related via

$$
M(\lambda) = \begin{pmatrix}
M_{11}^{22} - M_{11}^{21} (M_{11}^{11} + M_{21}^{11})^{-1} M_{11}^{12} & -M_{11}^{21} (M_{11}^{11} + M_{21}^{11})^{-1} M_{11}^{12} \\
-M_{21}^{11} (M_{11}^{11} + M_{21}^{11})^{-1} M_{11}^{12} & M_{22}^{22} - M_{21}^{21} (M_{11}^{11} + M_{21}^{11})^{-1} M_{21}^{12}
\end{pmatrix}.
$$

where $M_{j}^{lm}$ come from the block decomposition of $M_{\Gamma_j}$.

**Lemma 2**

Assume that $\dim L_1 \equiv \dim L_2 = 1$, then any two out of three $M$-functions in the above formula determine uniquely the third one.
Gluing extensions of symmetric operators

Let $A_1$ and $A_2$ be two symmetric operators with the boundary values:

$$\langle A_j^* u_j, v_j \rangle - \langle u_j, A_j^* v_j \rangle = \langle \bar{u}^\partial_j, \partial \bar{v}_j \rangle - \langle \partial \bar{u}^\partial_j, \bar{v}^\partial_j \rangle.$$ 

$M$-functions

$$M_{1,2}(\lambda) \bar{\psi}^\partial = \partial \bar{u}^\partial, \quad \text{where } A_j^* \psi_j = \lambda \psi_j.$$ 

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Let $A_1$ and $A_2$ be two symmetric operators with the boundary values:

$$\langle A^*_j u_j, v_j \rangle - \langle u_j, A^*_j v_j \rangle = \langle \vec{u}^\partial_j, \partial \vec{v}_j \rangle - \langle \partial \vec{u}^\partial_j, \vec{v}_j^\partial \rangle.$$

$M$-functions

$$M_{1,2}(\lambda) \vec{\psi}^\partial = \partial \vec{u}^\partial, \quad \text{where } A^*_j \psi_j = \lambda \psi_j.$$

Consider the symmetric extension $A = A_1 \oplus A_2$ determined by the coupling conditions:

$$\vec{u}_1^\partial|_{L_1} = \vec{u}_2^\partial|_{L_2}; \quad \partial \vec{u}_1^\partial|_{L_1} = -\partial \vec{u}_2^\partial|_{L_2},$$

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**Lemma 2**

Assume that $\dim L_1 \equiv \dim L_2 = 1$, then any two out of three $M$-functions in the above formula determine uniquely the third one.
Three inverse problems

I Reconstruction of the metric graph
   Globally: Every metric tree is uniquely reconstructed from the travelling times between the boundary vertices
   Locally: Travelling times associated with a bunch determine the bunch

II Reconstruction of the potential
   Boundary Control: (the diagonal of) the response operator $R^T$ determines the potential on the boundary edges everywhere at the distance $d < T/2$ from the boundary

III Reconstruction of the vertex conditions
   The response operator for an equilateral bunch with zero potential determines the vertex conditions up to the phase $\theta_n$ associated with the root edge.
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The response operator for an equilateral bunch with zero potential determines the vertex conditions up to the phase $\theta_n$ associated with the root edge.

The kernel of the response operator is of the form:

$$r(t) = -\delta'(t) + 2PS_v(\infty)P\delta'(t-2\ell) + 4P(I-P_{-1})A(I-P_{-1})P\delta(t) + \text{smooth}$$

where

- $S_v(\infty)$ - the limit of the vertex scattering matrix;
- $P$ - the projector on the $d-1$ components associated with the bunch;
- $A$ - the matrix in the quadratic form parameterisation of vertex conditions;
- $P_{-1}$ - the eigenprojector for $S$ corresponding to e.v. $-1$.

$$S_v(\infty) = I - 2P_{-1}$$
Fusion

1. **Selecting a bunch in the tree.**
   Using global or local procedure (Subproblem I)

2. **Reconstruction of the potential on the bunch.**
   Boundary Control determines potential on the boundary edges from the bunch (Subproblem II).

3. **Cleaning of the bunch - removing potential.**
   Based on gluing Lemmas.

4. **Trimming of the bunch.**
   Based on gluing Lemmas.

5. **Recovering vertex conditions at the root.**
   The response operator for equilateral bunch with zero potential determines the vertex condition at the root of the bunch (Subproblem III).

6. **Cutting away bunches - pruning of the tree.**
   Based on gluing Lemmas.

Repeating this procedure sufficiently many times we solve the inverse problem for trees, of course up to the phases $\theta_n$ at internal edges.

Assumptions:
- $q \in L_1(\Gamma)$,
- $S_v(\infty)$ does not have zero entries.
Inverse problems for graphs with cycles

- Magnetic potential is equivalent to change of vertex conditions $\Rightarrow$ only standard vertex conditions (continuity $+$ Kirchhoff).
- The response operator and $M$-matrices are considered as functions of the set of fluxes $\Phi_j$, fixed each time.

Magnetic boundary control

$$M(\lambda, \Phi_j) \mapsto q(x)$$

- Key feature:
  existence of invisible eigenfunctions
  Every such eigenfunction produces ambiguity in the solution of the inverse problem.
Loop graph

\[
M_{\Gamma^\text{loop}_{\ell_1}}(\lambda, \Phi) = M_{11} + 2 \cos \Phi M_{12} + M_{22}
\]

\[
= 2 \cos \Phi - \text{Tr} T(k)
\]

where \( T(k) = \{t_{ij}(k)\} \) is the transfer matrix for \([x_1, x_2]\).

\textbf{NB!} The \( M \)-function depends on \( \cos \Phi \), \textit{i.e.} \( \Phi \) and \(-\Phi\) lead to the same \( M_{\Gamma^\text{loop}_{\ell_1}} \)
Solution using the transfer matrix

Solving inverse problem:

\[ t_{12}(k) = \frac{1}{M_{12}(\lambda)}; \]

\[ u_+(k) := \frac{1}{2} (t_{11}(k) + t_{22}(k)) = \frac{1}{2} \frac{\text{Tr} \, M(\lambda)}{M_{12}(\lambda)}. \]

\( t_{12}(k_j) = 0 \) - the spectrum of the Dirichlet-Dirichlet operator on \([x_1, x_2]\).

At these points we know:

\[ t_{11}(k_j) t_{22}(k_j) = 1 \quad \text{det} \, T(k) = 1; \]

\[ t_{11}(k_j) + t_{22}(k_j) = u_+(k_j) - \text{known function}. \]

\[ \Rightarrow t_{11}(k_j), t_{22}(k_j) \text{ are determined up to a sign} \]

\( t_{ii}(k) \) are analytic functions of exponential type and are uniquely determined by their values at \( k_j \), provided \( \int q(x)dx = 0 \).
Solution using the $M$-function

\[ M_{12}(\lambda) = \frac{1}{4} \left( M_{\Gamma_\ell_1}^{\text{loop}}(\lambda, 0) - M_{\Gamma_\ell_1}^{\text{loop}}(\lambda, \pi) \right), \]

\[ \text{Tr } M(\lambda) = \frac{1}{2} \left( M_{\Gamma_\ell_1}^{\text{loop}}(\lambda, 0) + M_{\Gamma_\ell_1}^{\text{loop}}(\lambda, \pi) \right). \]

Consider $\lambda^D_j = k^2_j$ and check for which (complex) $\Phi$ the $M$-function does not have a singularity at this point.

\[ M_{\Gamma}(\lambda, \Phi) = M_{\Gamma}(\lambda', \Phi) + \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda^D_n - \lambda)(\lambda^D_n - \lambda')} \left\langle \partial \psi^D_n(\Phi)|\partial \Gamma, \cdot \right\rangle_{\ell_2(\partial \Gamma)} \partial \psi^D_n(\Phi)|\partial \Gamma, \]

Find $\Phi_j$ for which the $M$-function is not singular at $\lambda^D_j$. NB! The phase $\Phi_j$ is determined up to a sign, since the formula for $M(\lambda, \Phi)$ contains $\cos \Phi$.

Invisible eigenfunction

\[
\begin{cases}
\psi(x_1) = \psi(x_2) = 0, \\
\psi'(x_1) - e^{i\Phi_j} \psi'(x_2) = 0.
\end{cases}
\]

\[ \Rightarrow \quad \frac{\psi'(x_1)}{\psi'(x_2)} = e^{i\Phi_j} \]
Three scalar $M$-functions

\[
(M(\lambda))_{11} = (M_{\Gamma}(\lambda'))_{11} + \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda_n^D - \lambda)(\lambda_n^D - \lambda')} |\partial \psi^D_n(x_1)|^2
\]

\[
(M(\lambda))_{22} = (M_{\Gamma}(\lambda'))_{22} + \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda_n^D - \lambda)(\lambda_n^D - \lambda')} |\partial \psi^D_n(x_2)|^2
\]

\[
\text{Tr } M(\lambda) = \text{Tr } M(\lambda') + \sum_{n=1}^{\infty} \frac{\lambda - \lambda'}{(\lambda_n^D - \lambda)(\lambda_n^D - \lambda')} (|\partial \psi^D_n(x_1)|^2 + |\partial \psi^D_n(x_2)|^2)
\]

\[
\begin{align*}
\frac{\psi'(x_1)}{\psi'(x_2)} &= e^{i\Phi_j} \\
|\partial \psi^D_n(x_1)|^2 + |\partial \psi^D_n(x_2)|^2
\end{align*}
\]

\[
\Rightarrow |\partial \psi^D_n(x_1)|^2, |\partial \psi^D_n(x_2)|^2
\]

Conclusion:

\[
\text{Tr } M(\lambda) \text{ determines } (M(\lambda))_{11}, (M(\lambda))_{22}
\]

up to a real constant.
Cycle opening procedure

Note that the last formula does not require that $M$ is an $M$-function for an interval - we did not use properties of the transfer matrix.

Each cycle requires a sequence of signs $+$ a real constant.
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