Stieltjes and inverse Stieltjes families of linear relations in Hilbert spaces and their representations

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Nevanlinna families

Recall the definition of a Nevanlinna family, cf. e.g. [Dijksma, de Snoo (1974), Langer, Textorius (1977)].

Definition

A family of linear relations $\mathcal{M}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in a Hilbert space $\mathcal{M}$ is called a Nevanlinna family if:

(i) $\mathcal{M}(\lambda)$ is maximal dissipative for every $\lambda \in \mathbb{C}^+$ (resp. accumulative for every $\lambda \in \mathbb{C}^-$);

(ii) $\mathcal{M}(\lambda)^{*} = \mathcal{M}(\bar{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(iii) for some (and hence for all) $\mu \in \mathbb{C}^{+} (\mathbb{C}^{-})$ the operator family

\[(\mathcal{M}(\lambda) + \mu)^{-1}(\in \mathcal{B}(\mathcal{M})) \text{ is holomorphic for all } \lambda \in \mathbb{C}^{+} (\mathbb{C}^{-}).\]

By (3) the family $\mathcal{M}(\lambda)$ is holomorphic in the resolvent sense of T. Kato, see [Kato (1995)]. $\mathcal{M}(\lambda)$ can also be considered as a pair of bounded holomorphic functions:

\[\mathcal{M}(\lambda) = \{(\mathcal{M}(\lambda) + \mu)^{-1}h, (I - \mu(\mathcal{M}(\lambda) + \mu)^{-1})h \} : h \in \mathcal{M} \quad (\lambda, \mu \in \mathbb{C}^\pm)\]. \hspace{1cm} (1.1)

The class of all Nevanlinna families in a Hilbert space $\mathcal{M}$ is denoted by $\tilde{\mathcal{R}}(\mathcal{M})$. The multi-valued part $M_\infty$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$, cf. (1.1), and $\mathcal{M}$ admits a unique decomposition

\[\mathcal{M}(\lambda) = M_{\text{op}}(\lambda) \oplus M_\infty, \quad M_\infty = \{0\} \times \text{mul} \mathcal{M}(\lambda), \hspace{1cm} (1.2)\]

where $M_{\text{op}}(\lambda)$ is a Nevanlinna family of densely defined operators in $\mathcal{M} \ominus \text{mul} \mathcal{M}(\lambda)$; see [Kreîn, Langer (1971), § 4.3], [Langer, Textorius (1977), Proposition 1.2]. If $M_M := \text{mul} \mathcal{M}(\lambda)(= \text{const})$ and $M_{\text{op}}(\lambda)$ is the operator part of $\mathcal{M}(\lambda)$ in $\mathcal{M} \ominus M_M$, then

\[(\mathcal{M}(\lambda) - \mu I)^{-1} | M_M = 0, \quad (\mathcal{M}(\lambda) - \mu I)^{-1} = (M_{\text{op}}(\lambda) - \mu I)^{-1} P_{M_M} \mathcal{M}.\]
1. Some definitions

Stieltjes and inverse Stieltjes families

Definition

A family of linear relations $\mathcal{M}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, in a Hilbert space $\mathcal{M}$ is said to be a Stieltjes family (respectively, inverse Stieltjes family) if it is a Nevanlinna family for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and, moreover,

(i) for all $x < 0$ the linear relations $\mathcal{M}(x)$ are selfadjoint and $\mathcal{M}(x) \geq 0$ (respectively, $\mathcal{M}(x) \leq 0$),

(ii) the family $\mathcal{M}(\lambda)$ is holomorphic on $\mathbb{R}_-$, i.e., for any $x < 0$ and for some $\xi \in \rho(\mathcal{M}(x))$ (and hence for all $\xi \in \rho(\mathcal{M}(x))$) the resolvent $(\mathcal{M}(\lambda) - \xi I)^{-1}$ exists and is holomorphic in $\lambda$ from a neighborhood of $x$, depending on $\xi$.

The classes of all Stieltjes and inverse Stieltjes families in a Hilbert space $\mathcal{M}$ are denoted by $\tilde{S}(\mathcal{M})$ and $\tilde{S}^{-1}(\mathcal{M})$, respectively.

As in the case of scalar functions these classes are connected to each other. If $\mathcal{M}(\lambda)$ is a Stieltjes (resp. inverse Stieltjes) family, then:

(a) $-\mathcal{M}(1/\lambda)$ is an inverse Stieltjes (resp. Stieltjes) family,

(b) $-\mathcal{M}^{-1}(\lambda)$ is an inverse Stieltjes (resp. Stieltjes) family.
2. Examples

Example 1.

Let $B = B^*$ be nonnegative bounded or unbounded operator in a Hilbert space $\mathcal{M}$. Then

$$M(\lambda) = -\frac{1}{\lambda} B, \quad \lambda \in \mathbb{C} \setminus \{0\},$$

is a Stieltjes function and

$$M(\lambda) = \lambda B, \quad \lambda \in \mathbb{C},$$

is an inverse Stieltjes function.

By applying e.g. the transform $-M(\lambda)^{-1}$ one can obtain multi-valued inverse Stieltjes and Stieltjes families (respectively) with

$$\text{mul} (-M(\lambda)^{-1}) = \ker B.$$
Example 2.

Let $\tilde{A}$ be a selfadjoint relation in a Hilbert space $\mathcal{H}$ and let $M \subset \mathcal{H}$ be a closed subspace. Then

$$P_M(\tilde{A} - \lambda)^{-1} \upharpoonright M \in \tilde{R}(M),$$

(2.1)

is an operator valued Nevanlinna function.

If, in addition, $\tilde{A} \geq 0$, then $P_M(\tilde{A} - \lambda)^{-1} \upharpoonright M$ is a Stieltjes family of bounded operators.

In [Derkach, S.H., Malamud, de Snoo (2006), Theorem 3.9] in the context of the Weyl families of boundary relations it is shown that for an arbitrary Nevanlinna family $M \in \tilde{R}(M)$ there exists (up to unitary equivalence) a unique ($M$-minimal) selfadjoint relation $\tilde{A}$ in $M \oplus \mathbb{R}$, such that

$$P_M(\tilde{A} - \lambda I)^{-1} \upharpoonright M = -(M(\lambda) + \lambda I)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.2}$$

Inverting the formula (2.2) leads to an equivalent expression

$$M(\lambda) = - \left( P_M \left( \tilde{A} - \lambda I \right)^{-1} \upharpoonright M \right)^{-1} - \lambda I_M, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.3}$$

If, in addition, $\tilde{A} \geq 0$, then (b) and (2.1) $\Rightarrow M(\cdot)$ in (2.3) is inverse Stieltjes.

(2.2) is closely related to the description of generalized resolvents by A.V. Shtraus [Shtraus (1954)]. Representations of operator valued Nevanlinna functions and Nevanlinna families as compressed resolvents of selfadjoint exit space extensions has been studied extensively; see e.g. [Arlinskiǐ, Belyi, Tsekanovskiǐ (2011), Derkach, Malamud (1991), Dijksma, de Snoo (1974), Kreǐn, Langer (1971), Kreǐn, Langer (1973), Langer, Textorius (1977)]. In [Arlinskiǐ, Belyi, Tsekanovskiǐ (2011)] Stieltjes/inverse Stieltjes matrix-valued functions appear as the impedance functions of singular $L$-systems.
Example 3. Stieltjes and inverse Stieltjes functions whose values are bounded operator in $\mathcal{M}$

**Proposition**

(For scalar case, see [Kac, Kreǐn (1968/1974)])

1) Any $\mathcal{B}(\mathcal{M})$-valued Stieltjes function $Q$ admits an integral representation of the form

\[
Q(\lambda) = \Gamma_Q + \int_{\mathbb{R}_+} \frac{d\Sigma_Q(t)}{t - \lambda},
\]

where $\Gamma_Q = \Gamma_Q^* \in \mathcal{B}(\mathcal{M})$, $\Gamma_Q \geq 0$, $\Sigma_Q(t) \text{ is a } \mathcal{B}(\mathcal{M})\text{-valued non-decreasing on } \mathbb{R}_+$, $\Sigma_Q(0) = 0$ and \( \int_{\mathbb{R}_+} \frac{(d\Sigma_Q(t)f, f)}{t + 1} < \infty \) for all $f \in \mathcal{M}$.

2) Any $\mathcal{B}(\mathcal{M})$-valued inverse Stieltjes function $R$ admits an integral representation of the form

\[
R(\lambda) = \Gamma_R + \lambda \Pi_R + \int_{\mathbb{R}_+} \left( \frac{1}{t - \lambda} - \frac{1}{t} \right) d\Sigma_R(t),
\]

where $\Gamma_R = \Gamma_R^* \in \mathcal{B}(\mathcal{M})$, $\Gamma_R \leq 0$, $0 \leq \Pi_R^* = \Pi_R \in \mathcal{B}(\mathcal{M})$, $\Sigma_R(t) \text{ is a } \mathcal{B}(\mathcal{M})\text{-valued and non-decreasing on } \mathbb{R}_+$, $\Sigma_R(0) = 0$ and \( \int_{\mathbb{R}_+} \frac{(d\Sigma_R(t)f, f)}{t(t + 1)} < \infty \) for all $f \in \mathcal{M}$.
A selfadjoint relation $\tilde{A}$ in the orthogonal sum $\mathcal{H} = \mathcal{M} \oplus \mathcal{K}$ of Hilbert spaces is called $\mathcal{M}$-minimal ([Langer, Textorius (1977)], [Derkach, S.H., Malamud, de Snoo (2006)]), if

$$\mathcal{H} = \overline{\text{span}} \left\{ \mathcal{M} + (\tilde{A} - \lambda I)^{-1}\mathcal{M} : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\}. \tag{3.1}$$

This definition can be extended to non-selfadjoint relations $\tilde{A}$ in $\mathcal{H} = \mathcal{M} \oplus \mathcal{K}$ with $\rho(\tilde{A}) \neq \emptyset$ by replacing the set $\mathbb{C} \setminus \mathbb{R}$ in (3.1) by the resolvent set $\rho(\tilde{A})$, or by a union of open sets, including one open set from each connected component of $\rho(\tilde{A})$.

Minimality condition in this more general form is applied here e.g to nonnegative and maximal accretive relations with $\lambda$ in (3.1) taken from the left half-plane in $\mathbb{C}$.

Two selfadjoint relations $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ in the Hilbert spaces $\mathcal{M} \oplus \mathcal{K}^{(1)}$ and $\mathcal{M} \oplus \mathcal{K}^{(2)}$, respectively, are said to be unitarily equivalent if there exists a unitary operator $\mathcal{V}$ acting from $\mathcal{K}^{(1)}$ onto $\mathcal{K}^{(2)}$, such that

$$\tilde{A}^{(2)} = \left\{ \left\{ \left( \varphi, \varphi' \right) \right\} : \left\{ \left( \varphi, \varphi' \right) \right\} \in \tilde{A}^{(1)} \right\}, \varphi, \varphi' \in \mathcal{M}, \ f, f' \in \mathcal{K}.$$
Some transforms

\( \mathcal{H} \) a Hilbert space, \( \mathcal{M} \subset \mathcal{H} \) a closed subspace. Decompose \( \mathcal{H} = \mathcal{M} \oplus \mathcal{K} \) with \( \mathcal{K} := \mathcal{H} \ominus \mathcal{M} \).

Define the transformation \( \mathcal{P}_M \) in \((\mathcal{M} \oplus \mathcal{K})^2\) by

\[
\mathcal{P}_M : \left\{ \begin{pmatrix} \varphi \\ f \end{pmatrix}, \begin{pmatrix} \varphi' \\ f' \end{pmatrix} \right\} \mapsto \left\{ \begin{pmatrix} \varphi' \\ f \end{pmatrix}, \begin{pmatrix} \varphi \\ f' \end{pmatrix} \right\}, \quad \varphi, \varphi' \in \mathcal{M}, \ f, f' \in \mathcal{K},
\]

and the transformation \( \mathcal{J}_M \) in \((\mathcal{M} \oplus \mathcal{K})^2\) by

\[
\mathcal{J}_M : \left\{ \begin{pmatrix} \varphi \\ f \end{pmatrix}, \begin{pmatrix} \varphi' \\ f' \end{pmatrix} \right\} \mapsto \left\{ \begin{pmatrix} -i \varphi' \\ f \end{pmatrix}, \begin{pmatrix} i \varphi \\ f' \end{pmatrix} \right\}, \quad \varphi, \varphi' \in \mathcal{M}, \ f, f' \in \mathcal{K}.
\]

These transformations are involutions in \((\mathcal{M} \oplus \mathcal{K})^2\): \((\mathcal{J}_M)^2 = (\mathcal{P}_M)^2 = I_{(\mathcal{M} \oplus \mathcal{K})^2} \).

Fix a fundamental symmetry in \( \mathcal{H} = \mathcal{M} \oplus \mathcal{K} \):

\[
\hat{\mathcal{J}}_M = \begin{bmatrix} -I_M & 0 \\ 0 & I_K \end{bmatrix}.
\]

The adjoint of (the graph of) \( T \) w.r.t. to \((\hat{\mathcal{J}}_M h, k)_{\mathcal{H}}\) is denoted by

\[
T^{[*]} := \hat{\mathcal{J}}_M T^* \hat{\mathcal{J}}_M,
\]

\( T^* \) the Hilbert space adjoint of \( T \) in \( \mathcal{H} \).

Then can define the notions of \( \hat{\mathcal{J}}_M \)-symmetric \((\hat{B} \subset \hat{B}^{[*]}), \hat{\mathcal{J}}_M \)-selfadjoint \((\hat{B} = \hat{B}^{[*]}), \hat{\mathcal{J}}_M \)-dissipative \((\text{Im} (\hat{J}_M u', u) \geq 0, \{u, u'\} \in \hat{B})\) for linear relations \( \hat{B} \) in \( \mathcal{H} \).
Some transforms (continued)

Proposition

Let \( \tilde{A} \) be a l.r. in \( \mathcal{H} = \mathcal{M} \oplus \mathcal{K} \) and let \( \hat{B} = \mathcal{P}_\mathcal{M}(\tilde{A}) \) be defined by (4.1). Then:

1. the transformation \( \mathcal{P}_\mathcal{M} \) preserves adjoints:
   \[ \mathcal{P}_\mathcal{M}(\tilde{A}^*) = \hat{B}^* \] (4.4)

and gives a one-to-one correspondence between symmetric (selfadjoint, (maximal) dissipative) relations \( \tilde{A} \) in \( \mathcal{H} \) and \( \hat{J}_\mathcal{M} \)-symmetric (resp. \( \hat{J}_\mathcal{M} \)-selfadjoint, (maximal) \( \hat{J}_\mathcal{M} \)-dissipative) relations \( \hat{B} \) in \( \mathcal{H} \);

2. if \( \mathcal{h} = \{ h, h' \} \), \( \mathcal{k} = \{ k, k' \} \) \( \in \mathcal{H}^2 \) and \( \{ u, u' \} = \mathcal{P}_\mathcal{M} \mathcal{h}, \{ v, v' \} = \mathcal{P}_\mathcal{M} \mathcal{k} \), then
   \[ (h', k) + (h, k') = (u', v) + (u, v') \quad \text{and} \quad \text{Re} (f', f) = \text{Re} (u', u) \] (4.5)

hence, \( \tilde{A} \) is accretive (\( m \)-accretive, skew-symmetric, skew-selfadjoint) \( \iff \hat{B} = \mathcal{P}_\mathcal{M}(\tilde{A}) \) is accretive (resp. \( m \)-accretive, skew-symmetric, skew-selfadjoint);

3. \( \mathcal{P}_\mathcal{M} \) gives a one-to-one correspondence between nonnegative (nonnegative selfadjoint) \( \tilde{A} \) in \( \mathcal{H} \) and \( \hat{J}_\mathcal{M} \)-symmetric accretive (resp. \( m \)-accretive) relations \( \hat{B} \) in \( \mathcal{H} \);

4. \( \tilde{A} = \tilde{A} \geq 0 \) and \( \hat{J}_\mathcal{M} \)-selfadjoint \( m \)-accretive \( \hat{B} = \mathcal{P}_\mathcal{M}(\tilde{A}) \) are simultaneously \( \mathcal{M} \)-minimal.
Compressed resolvents

The next result gives representations for the functions \( M(\lambda) \in \tilde{R}(\mathcal{M}), -M(\lambda)^{-1} \in \tilde{R}(\mathcal{M}) \), and \( -M(1/\lambda) \in \tilde{R}(\mathcal{M}) \) as compressed resolvents of certain selfadjoint relations.

Lemma

Let \( \mathcal{M}(\cdot) \) be a Nevanlinna family in the Hilbert space \( \mathcal{M} \). Then, up to unitary equivalence, there exists a unique selfadjoint relation \( \tilde{A} \) in the Hilbert space \( \mathcal{M} \oplus \mathbb{R} \) which is \( \mathcal{M} \)-minimal and such that:

1. The Nevanlinna family \( \mathcal{M}(\lambda) \) has the representation

\[
\mathcal{M}(\lambda) = - \left( P_{\mathcal{M}} \left( \tilde{A} - \lambda I \right)^{-1} \right| \mathcal{M} \right)^{-1} - \lambda I_{\mathcal{M}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\] (4.6)

2. If \( \hat{\mathcal{A}} = J_{\mathcal{M}}(\tilde{A}) \) is as defined in (4.2), then

\[
-\mathcal{M}^{-1}(\lambda) = - \left( P_{\mathcal{M}} \left( \hat{\mathcal{A}} - \lambda I \right)^{-1} \right| \mathcal{M} \right)^{-1} - \lambda I_{\mathcal{M}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\] (4.7)

3. If \( \check{\mathcal{A}} = -J_{\mathcal{R}}(\tilde{A}) \), then

\[
-\mathcal{M} \left( \frac{1}{\lambda} \right) = - \left( P_{\mathcal{M}} \left( \check{\mathcal{A}} - \lambda I \right)^{-1} \right| \mathcal{M} \right)^{-1} - \lambda I_{\mathcal{M}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\] (4.8)
Representations via compressed resolvents

Lemma
Let $\tilde{A} = \tilde{A} \geq 0$ in $\mathfrak{H} = \mathcal{M} \oplus \mathcal{K}$. Then:

1. \[ P_M (\tilde{A} - \lambda I)^{-1} | M = -(R(\lambda) + \lambda I_M)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}^+, \quad (4.9) \]
   with $R \in \tilde{S}(-1)(M)$.

2. If $\hat{B}$ is a m-accretive and $\hat{J}_M$-selfadjoint w.r.t. $\hat{J}_M$ in (4.3), then
   \[ P_M (\hat{B} - \lambda I)^{-1} | M = (Q(\lambda) - \lambda I_M)^{-1}, \quad \text{Re} \lambda < 0, \quad (4.10) \]
   with $Q \in \tilde{S}(M)$.

3. If $\hat{A} = \hat{J}_M P_M(\hat{B}) = \left\{ \left[ \begin{array}{cc} \frac{-ih}{f} & \frac{ih'}{f'} \end{array} \right], \left[ \begin{array}{cc} h & f' \end{array} \right] \right\} \in \hat{B} \right\}$, then $\hat{A} = \tilde{\hat{A}}^*$ and
   \[ P_M (\hat{A} - \lambda I)^{-1} | M = -(Q(\lambda) + \lambda I_M)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad (4.11) \]
   and $Q \in \tilde{S}(M)$ is the same function as in (4.10).
Lemma (continued)

Lemma (continued)

(4) If \( \tilde{\mathcal{A}} = -\tilde{\mathcal{J}}(\hat{\mathcal{A}}) \), where \( \hat{\mathcal{A}} \) is as in (3), then \( \tilde{\mathcal{A}} = \hat{\mathcal{A}} \geq 0 \) and

\[
P_{\mathcal{M}} \left( \tilde{\mathcal{A}} - \lambda I \right)^{-1} \upharpoonright \mathcal{M} = - \left( -Q \left( \frac{1}{\lambda} \right) + \lambda I_{\mathcal{M}} \right)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}^+, \quad (4.12)
\]

where \( -Q \left( \frac{1}{\lambda} \right) \in \tilde{\mathcal{S}}^{(-1)}(\mathcal{M}) \) and \( Q \in \tilde{\mathcal{S}}(\mathcal{M}) \) is the same function as in (4.10).

Moreover, if \( \tilde{\mathcal{A}} \) in (4.9) and \( \tilde{\mathcal{B}} \) in (4.10) are connected by \( \tilde{\mathcal{B}} = \Psi_{\mathcal{M}}(\tilde{\mathcal{A}}) \) then \( Q(\lambda) = -\mathcal{R}^{-1}(\lambda) \)

and, furthermore, \( \tilde{\mathcal{A}} = \tilde{\mathcal{A}}^{-1} \).
Characterization of inverse Stieltjes families

The next theorem shows that all inverse Stieltjes families $\mathcal{R} \in \tilde{S}(-1)(\mathcal{M})$ can be characterized by the statement (1) in Lemma 6.

**Theorem**

Let $\mathcal{R}$ belong to the inverse Stieltjes class in $\mathcal{M}$. Then there exists up to unitary equivalence a unique nonnegative selfadjoint relation $\tilde{A}$ in the Hilbert space $\mathcal{M} \oplus \mathbb{R}$ such that $\tilde{A}$ is $\mathcal{M}$-minimal and the relation

$$P_\mathcal{M} \left( \tilde{A} - \lambda I \right)^{-1} | \mathcal{M} = - (\mathcal{R}(\lambda) + \lambda I_\mathcal{M})^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+$$

holds.
Characterization of Stieltjes families

For Stieltjes families $\mathcal{R} \in \tilde{S}(\mathcal{M})$ we have the following characterizations.

**Theorem**

Let $\mathcal{Q}$ belong to the Stieltjes class in $\mathcal{M}$. Then:

1. There exists (up to the unitary equivalence) a unique $m$-accretive $\hat{B}$ in $\mathcal{M} \oplus \mathcal{K}$ such that $\hat{B}$ is $\hat{J}_\mathcal{M}$-selfadjoint w.r.t. $\hat{J}_\mathcal{M}$ in (4.3), such that $\hat{B}$ is $\mathcal{M}$-minimal and

   \[
   P_{\mathcal{M}} \left( \hat{B} - \lambda I \right)^{-1} \upharpoonright \mathcal{M} = (\mathcal{Q}(\lambda) - \lambda I_{\mathcal{M}})^{-1}
   \]

   holds for all $\text{Re} \lambda < 0$.

2. There exists (up to the unitary equivalence) a unique selfadjoint $\hat{A}$ in $\mathcal{M} \oplus \mathcal{K}$, such that $\hat{A}$ is $\mathcal{M}$-minimal and its transform $\hat{J}_\mathcal{M}(\hat{A})$ is nonnegative, and

   \[
   P_{\mathcal{M}} \left( \hat{A} - \lambda I \right)^{-1} \upharpoonright \mathcal{M} = - (\mathcal{Q}(\lambda) + \lambda I_{\mathcal{M}})^{-1}
   \]

   holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Furthermore, one can choose

\[
\hat{A} = \hat{J}_\mathcal{M} P_{\mathcal{M}}(\hat{B}).
\]
Two further characterizations
The next lemma is an extension of [Kac, Kreĭn (1968/1974)] from the scalar case.

**Lemma**

With \( \lambda \in \mathbb{C} \setminus \mathbb{R}_+ \) the following assertions are equivalent:

1. \( Q(\lambda) \in \tilde{S}(\mathcal{M}) \);
2. \( -Q^{-1}(\lambda) \in \tilde{S}(-1)(\mathcal{M}) \);
3. \( \lambda Q(\lambda) \in \tilde{S}(-1)(\mathcal{M}) \).

**Theorem**

1. Let \( Q \in \tilde{S}(\mathcal{M}) \). Then there is a Hilbert space \( \mathcal{H} = \mathcal{M} \oplus \mathcal{K} \) and (up to unitary equivalence) a unique \( \mathcal{M} \)-minimal nonnegative selfadjoint \( \tilde{A} \) in \( \mathcal{H} \) such that

\[
Q(\lambda) = -\frac{1}{\lambda} \left( P_{\mathcal{M}} (\tilde{A} - \lambda I)^{-1} | \mathcal{M} \right)^{-1} - I_{\mathcal{M}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \quad (4.15)
\]

2. Let \( \mathcal{R} \in \tilde{S}(-1)(\mathcal{M}) \). Then there is a Hilbert space \( \mathcal{H} = \mathcal{M} \oplus \mathcal{K} \) and (up to unitary equivalence) a unique \( \mathcal{M} \)-minimal nonnegative selfadjoint \( \tilde{B} \) in \( \mathcal{H} \) such that

\[
\mathcal{R}(\lambda) = I_{\mathcal{M}} - \left( P_{\mathcal{M}} (I - \lambda \tilde{B})^{-1} | \mathcal{M} \right)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+. \quad (4.16)
\]

Moreover, if \( Q(\lambda) \in \tilde{S}(\mathcal{M}) \) is represented by means of \( \tilde{A} \) in (4.15), then \( -Q(\lambda)^{-1} \) admits the representation (4.16) by means of \( \tilde{B} = \tilde{A}^{-1} \).
The mappings $\Phi_+$ and $\Phi_-$ and their fixed points

Recall that by Lemma 10 the transformation

$$
\tilde{S}(M) \ni Q(\lambda) \mapsto \tilde{Q}(\lambda) := -\frac{Q(\lambda)^{-1}}{\lambda} \in \tilde{S}(M)
$$

is well defined mapping in the Stieltjes class. In fact, $\Phi_+$ is an automorphism of the class $\tilde{S}(M)$. Analogously, the transformation

$$
\tilde{S}^{-1}(M) \ni R(\lambda) \mapsto \tilde{R}(\lambda) := -\lambda R(\lambda)^{-1} \in \tilde{S}^{-1}(M)
$$

is an automorphism of the class $\tilde{S}^{-1}(M)$. Here the main purpose is to find the fixed points of these two mappings.

**Proposition**

Let the mappings $\Phi_+ : \tilde{S}(M) \to \tilde{S}(M)$ and $\Phi_- : \tilde{S}^{-1}(M) \to \tilde{S}^{-1}(M)$ be as defined above. Then with $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$:

1. the mapping $\Phi_+$ has a unique fixed point

   $$
   Q_0(\lambda) = \frac{i}{\sqrt{\lambda}} l_M, \quad Q_0(-1) = l_M;
   $$

2. the mapping $\Phi_-$ has a unique fixed point

   $$
   R_0(\lambda) = i\sqrt{\lambda} l_M, \quad R_0(-1) = -l_M.
   $$
Combined Nevanlinna-Schur class

Definition
Let $M$ be a Hilbert space. A $B(M)$-valued Nevanlinna function $\Omega$ which is holomorphic on $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}$ is said to belong to the class $RS(M)$ if

$$
-1 \leq \Omega(x) \leq 1, \quad x \in (-1, 1).
$$

It is proved in [Arlinskiĭ, S.H. (2020)] that the class $RS(M)$ is a subclass of Schur functions $S(M)$. Thus the class $RS(M)$ consists of functions that are Nevanlinna functions in $\mathbb{C} \setminus \mathbb{R}$ and simultaneously Schur functions on the open unit disk.

This class is called a combined Nevanlinna-Schur class of $B(M)$-valued operator functions and explains the notation $RS(M)$.

Observe the following mapping properties

$$
\lambda \in \mathbb{C} \setminus \mathbb{R}_+ \iff z := \frac{1 + \lambda}{1 - \lambda} \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\}, \quad (6.1)
$$

with inverse transform for $\lambda$,

$$
\lambda = \frac{z - 1}{z + 1}, \quad \text{Im} \lambda = \frac{2 \text{Im} z}{|z + 1|^2}. \quad (6.2)
$$
Connection of Stieltjes and inverse Stieltjes families to the combined Nevanlinna-Schur class

Lemma

Let $\Omega \in \mathcal{RS}(\mathcal{M})$. Then for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,

\[
Q(\lambda) = -I + 2 \left( l_\mathcal{M} - \Omega \left( \frac{1 + \lambda}{1 - \lambda} \right) \right)^{-1} \Rightarrow \left\{ \left\{ \left( l_\mathcal{M} - \Omega \left( \frac{1 + \lambda}{1 - \lambda} \right) \right) h, \left( l_\mathcal{M} + \Omega \left( \frac{1 + \lambda}{1 - \lambda} \right) \right) h \right\} : h \in \mathcal{M} \right\}
\]

(6.3)

is a Stieltjes family and

\[
R(\lambda) = I - 2 \left( l_\mathcal{M} + \Omega \left( \frac{1 + \lambda}{1 - \lambda} \right) \right)^{-1} \Rightarrow \left\{ \left\{ \left( l_\mathcal{M} + \Omega \left( \frac{1 + \lambda}{1 - \lambda} \right) \right) h, \left( \Omega \left( \frac{1 + \lambda}{1 - \lambda} \right) - l_\mathcal{M} \right) h \right\} : h \in \mathcal{M} \right\}
\]

(6.4)

is an inverse Stieltjes family.

Conversely, if $Q(\lambda)$ is a Stieltjes family (resp. $R(\lambda)$ is an inverse Stieltjes family) in $\mathcal{M}$, then there exists a function $\Omega \in \mathcal{RS}(\mathcal{M})$ such that (6.3) (resp. (6.4)) holds.

Observe, that the functions $Q$ in (6.3) and $R$ in (6.4) are connected by $R = -Q^{-1}$. 
Inner functions in the Stieltjes and inverse Stieltjes classes

An operator valued Schur function is said to be inner/co-inner/bi-inner if almost everywhere on the unit disk the non-tangential limit values to the unit circle $T$ are, respectively, isometric/co-isometric/unitary. It is proved in [Arlinskiĭ, S.H. (2020)] that the function $\Omega$ of the class $\mathcal{RS}(\mathcal{M})$ is inner if and only if it admits the representation

$$\Omega(z) = (zl + \tilde{D})(l + z\tilde{D})^{-1}, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, +\infty)\},$$

where $\tilde{D}$ is a selfadjoint contraction in $\mathcal{M}$.

The Stieltjes class $\mathcal{S}(\mathcal{M})$ and the inverse Stieltjes class $\tilde{\mathcal{S}}(-1)(\mathcal{M})$ are connected to the class $\mathcal{RS}(\mathcal{M})$ as described in Lemma 14. Notice that (cf. (6.2))

$$\text{Re} \lambda = \frac{|z|^2 - 1}{|z + 1|^2}, \quad \lambda = \frac{z - 1}{z + 1}.$$

In particular, the transform $z \to \frac{z - 1}{z + 1}$ maps $\mathbb{T} \setminus \{1, -1\}$ (nonreal part of the unit circle) bijectively onto the set $\{iy : y \in \mathbb{R}, y \neq 0\}$. This motivates the following definition.

Definition
A family $S$ from the class $\tilde{\mathcal{S}}(\mathcal{M})$ ($\tilde{\mathcal{S}}(-1)(\mathcal{M})$) is said to be inner if holds for all $y \in \mathbb{R} \setminus \{0\}$:

$$\text{Re} (S(iy)f, f) = 0, \quad f \in \text{dom} S(iy).$$
Characterization of inner functions

Proposition

All inner families in the Stieltjes and inverse Stieltjes classes are described as follows:

1. The inner families from the class \( \tilde{S}(M) \) are of the form

\[
Q(\lambda) = -\lambda^{-1} B, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+, \tag{7.2}
\]

where \( B \) runs through the set of all nonnegative selfadjoint relations in \( M \).

2. The inner families from the class \( \tilde{S}^{-1}(M) \) are of the form

\[
R(\lambda) = \lambda C, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}_+, \tag{7.3}
\]

where \( C \) runs through the set of all nonnegative selfadjoint relations in \( M \).

Further details and results can be found in:


