Perturbations of L-systems

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Let $E$ and $\mathcal{H}$ be Hilbert spaces and let $T$ be an unbounded operator in $\mathcal{H}$.

\begin{equation*}
\begin{cases}
(T - zI)x = KJ\varphi_-,
\varphi_+ = \varphi_- - 2iK^*x,
\end{cases}
\text{Im } T = KJK^*.
\end{equation*}

Here $\varphi_- \in E$ is an input vector, $\varphi_+ \in E$ is an output vector, and $x \in \mathcal{H}$ is a vector of the state space. $J = J^* = J^{-1} \in [E, E]$. 
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L-system

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L-system

\[ \Theta = \begin{pmatrix} \mathcal{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & E \end{pmatrix} \]  

(1)

- \( \mathcal{A} \) is a bounded linear operator from \( \mathcal{H}_+ \) to \( \mathcal{H}_- \) 
  \((\ast)-extension of \( T \in \Omega(\dot{\mathcal{A}}) \), i.e., \( \mathcal{A} \supset T \supset \dot{\mathcal{A}}, \mathcal{A}^* \supset T^* \supset \dot{\mathcal{A}} \));
- \( \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \) is a rigged Hilbert space, \( \dim E < \infty \);
- \( \text{Im} \mathcal{A} = KJK^* \);
- \( K \) is a linear bounded operator from \( E \) into \( \mathcal{H}_- \);
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- $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space, $\dim E < \infty$;
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- $K$ is a linear bounded operator from $E$ into $\mathcal{H}_-$;
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L-system

The matrix $\Theta$ is given by

$$\Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & E \end{pmatrix} \quad (1)$$

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- $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is a rigged Hilbert space, $\dim E < \infty$;
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- $J = J^* = J^{-1} \in [E, E]$. 
L-system

\[ \Theta = \begin{pmatrix} A & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & E \end{pmatrix} \]  

- \( A \) is a bounded linear operator from \( \mathcal{H}_+ \) to \( \mathcal{H}_- \) ((\( \ast \))-extension of \( T \in \Omega(\hat{A}) \), i.e., \( A \supset T \supset \hat{A}, A^\ast \supset T^\ast \supset \hat{A} \));
- \( \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \) is a rigged Hilbert space, \( \text{dim } E < \infty \);
- \( \text{Im } A = KJK^\ast \);
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- \( J = J^\ast = J^{-1} \in [E, E] \).
L-system

\[ \Theta = \begin{pmatrix} \mathbb{A} & K & J \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & \mathcal{E} \end{pmatrix} \]  

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\[ \Theta = \begin{pmatrix} A & K & J \\ H_+ \subset H \subset H_- & E \end{pmatrix} \quad (1) \]

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Uniqueness of an L-system

- $T$ is a **main operator** of the L-system.
- $\hat{A}$ is a symmetric operator, the largest common Hermitian part of $T$ and $T^*$. 
- $A$ is a $(\ast)$-extensions of $T$, i.e., $A \supset T$, $A^* \supset T^*$,
- $\hat{A}$ is a quasi-kernel of $\text{Re} A$, a self-adjoint extension of $\hat{A}$ such that $\text{Re} A \supset \hat{A} = \hat{A}^* \supset \hat{A}$.

The triple of operators $\hat{A}$, $T$, and $\hat{A}$ define an L-system uniquely.
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The triple of operators $\hat{A}$, $T$, and $\hat{A}$ define an L-system uniquely.
Transfer and Impedance function of an L-system

Transfer function

\[ W_{\Theta}(z) = I - 2iK^*(A - zI)^{-1}KJ \]

Impedance function of \( \Theta = \text{LFT of } W_{\Theta}(z) \)

\[ V_{\Theta}(z) = i[W_{\Theta}(z) + I]^{-1}[W_{\Theta}(z) - I]J = K^*(\text{Re } A - zI)^{-1}K \quad (2) \]
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Direct and Inverse Realization Problems

Direct Problem  Given an L-system $\Theta$ we need to derive transfer function $W_\Theta(z)$ and classify the impedance function $V_\Theta(z)$

Inverse Problem  Given a function $V(z)$ of a certain class we need to construct an L-system $\Theta$ such that

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$$V(z) = i[W_\Theta(z) + I]^{-1}[W_\Theta(z) - I]J$$
An L-system w/1-D input-output

One-dimensional L-system

\[ \Theta = \begin{pmatrix} A & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \\ C \end{pmatrix} \]
An L-system w/1-D input-output

One-dimensional L-system

\[ \Theta = \begin{pmatrix} A & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & \mathbb{C} \end{pmatrix} \]  (3)
Main operator $T$

In L-system (3) $T \neq T^*$ is a maximal dissipative extension of a symmetric operator $\hat{A}$ with deficiency indices $(1, 1)$,

$$\text{Im}(Tf, f) \geq 0, \quad f \in \text{Dom}(T).$$

Operator $T$ is quasi-self-adjoint that is, $\hat{A} \subset T \subset \hat{A}^*$ and

$$g_+ - \kappa g_- \in \text{Dom}(T) \quad \text{for some } |\kappa| < 1. \quad (4)$$

Operator $T$ is the main operator of L-system (3).
Let $\hat{A}$ be a self-adjoint extension of $A$ such that

$$\text{Re} A \supset \hat{A} = \hat{A}^* \supset A.$$  

By von Neumann’s formula

$$\text{Dom}(\hat{A}) = \text{Dom}(A) \oplus (1 + U) \ker(\hat{A}^* - iI),$$

where $U$ is a unimodular parameter, $|U| = 1$. Operator $\hat{A}$ is the quasi-kernel of the real part $\text{Re} A$ of the state-space operator $A$. 

Quasi-kernel $\hat{A}$ of $\text{Re} A$
A unique L-system w/1-D input-output

A triple \((\hat{A}, T, \hat{A})\) of a symmetric operator, main operator, and a quasi-kernel in a Hilbert space \(\mathcal{H}\) defines an L-system

\[
\Theta = \begin{pmatrix}
\hat{A} & K & 1 \\
\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \\
\mathbb{C}
\end{pmatrix}
\]  \hspace{1cm} (5)

uniquely. This L-system \(\Theta\) has a one-dimensional input-output space \(E = \mathbb{C}\).
Hypothesis 1, $U = -1$

Hypothesis (1)

Suppose that $T \neq T^*$ is a maximal dissipative extension of a symmetric operator $\hat{A}$ with deficiency indices $(1, 1)$ and $\hat{A}$ is a self-adjoint (reference) extension of $\hat{A}$. Let deficiency elements $g_\pm \in \ker(\hat{A}^* \mp il)$ be normalized, $\|g_\pm\| = 1$, and such that

$g_+ - g_- \in \text{Dom}(A)$ and $g_+ - \kappa g_- \in \text{Dom}(T)$ for some $|\kappa| < 1$. 
Hypothesis 1, $U = -1$

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Hypothesis 2 ("Anti-hypothesis"), $U = 1$

Hypothesis (2)

Suppose that $T \neq T^*$ is a maximal dissipative extension of a symmetric operator $\hat{A}$ with deficiency indices $(1, 1)$ and $\hat{A}$ is a self-adjoint (reference) extension of $\hat{A}$. Let deficiency elements $g_\pm \in \ker(\hat{A}^* \mp iI)$ be normalized, $\|g_\pm\| = 1$, and such that

$g_+ + g_- \in \text{Dom}(A)$ and $g_+ - \kappa g_- \in \text{Dom}(T)$ for some $|\kappa| < 1$. 
Hypothesis 2 ("Anti-hypothesis"), \( U = 1 \)

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Suppose that \( T \neq T^* \) is a maximal dissipative extension of a symmetric operator \( \hat{A} \) with deficiency indices \((1, 1)\) and \( \hat{A} \) is a self-adjoint (reference) extension of \( A \). Let deficiency elements \( g_\pm \in \ker(\hat{A}^* \mp il) \) be normalized, \( \|g_\pm\| = 1 \), and such that

\[
g_+ + g_- \in \text{Dom}(A) \quad \text{and} \quad g_+ - \kappa g_- \in \text{Dom}(T) \quad \text{for some} \quad |\kappa| < 1.
\]
Donoghue class $M$

Denote by $M$ the Donoghue class of all analytic mappings $M$ from $\mathbb{C}_+$ into itself that admits the representation

$$M(z) = \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu,$$  \hspace{1cm} (6)

where $\mu$ is an infinite Borel measure and

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = 1, \quad \text{equivalently,} \quad M(i) = i.$$ \hspace{1cm} (7)
Generalized Donoghue classes $\mathcal{M}_\kappa$ and $\mathcal{M}_{-1}\kappa$

An analytic function $M$ from $\mathbb{C}_+$ into itself belongs to the generalized Donoghue class $\mathcal{M}_\kappa$, $(0 \leq \kappa < 1)$ if it admits the representation (6) and

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = a = \frac{1 - \kappa}{1 + \kappa} < 1 \iff M(i) = i \frac{1 - \kappa}{1 + \kappa}$$

(8)

and to the generalized Donoghue class $\mathcal{M}_{-1}\kappa$, $(0 \leq \kappa < 1)$ if it admits the representation (6) and

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(9)

Clearly, $\mathcal{M}_0 = \mathcal{M}_{-1}^{-1} = \mathcal{M}$. 
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and to the generalized Donoghue class $\mathcal{M}_{-\kappa}$, $(0 \leq \kappa < 1)$ if it admits the representation (6) and

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and to the generalized Donoghue class $\mathcal{M}_\kappa^{-1}$, $(0 \leq \kappa < 1)$ if it admits the representation (6) and

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Generalized Donoghue classes $\mathcal{M}_\kappa$ and $\mathcal{M}^{-1}_\kappa$

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and to the generalized Donoghue class $\mathcal{M}^{-1}_\kappa$, $(0 \leq \kappa < 1)$ if it admits the representation (6) and

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} = a = \frac{1 + \kappa}{1 - \kappa} > 1 \iff M(i) = i \frac{1 + \kappa}{1 - \kappa}. \quad (9)$$

Clearly, $\mathcal{M}_0 = \mathcal{M}^{-1}_0 = \mathcal{M}$. 
A scalar Herglotz-Nevanlinna function $V(z)$ belongs to the class $\mathcal{M}^Q$ if it admits the following integral representation

$$V(z) = Q + \int_{\mathbb{R}} \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu, \quad Q = \bar{Q}, \quad (10)$$

and has condition (7) on the measure $\mu$. Similarly, we introduce perturbed classes $\mathcal{M}^Q_\kappa$ and $\mathcal{M}^{-1, Q}_{\kappa}$ if normalization conditions (8) and (9), respectively, hold on measure $\mu$ in (10).
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Donoghue class impedance functions

**Theorem (B., Makarov, Tsekanovskii, '15)**

Let $\Theta$ of the form (5) be an L-system whose main operator $T$ has the von Neumann parameter $\kappa$, $(0 \leq \kappa < 1)$. Then its impedance function $V_\Theta(z)$ belongs to the Donoghue class $\mathcal{M}$ if and only if $\kappa = 0$. 
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Theorem (B., Makarov, Tsekanovskii, ’15)

Let $\Theta_\kappa$, $0 \leq \kappa < 1$, of the form (5) be an L-system with the main operator $T$. Then its impedance function $V_{\Theta_\kappa}(z)$ belongs to the generalized Donoghue class $\mathcal{M}_\kappa$ if and only if the triple $(\hat{\mathcal{A}}, T, \hat{\mathcal{A}})$ satisfies Hypothesis 1.

Theorem (B., Makarov, Tsekanovskii, ’16)

Let $\Theta_\kappa$, $0 \leq \kappa < 1$, of the form (5) be an L-system with the main operator $T$. Then its impedance function $V_{\Theta_\kappa}(z)$ belongs to the generalized Donoghue class $\mathcal{M}_{-1, \kappa}$ if and only if the triple $(\hat{\mathcal{A}}, T, \hat{\mathcal{A}})$ satisfies Hypothesis 2.
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Theorem (B., Makarov, Tsekanovskii, ’16)

Let $\Theta_\kappa$, $0 \leq \kappa < 1$, of the form (5) be an L-system with the main operator $T$. Then its impedance function $V_{\Theta_\kappa}(z)$ belongs to the generalized Donoghue class $\mathcal{M}_{-1}^{-\kappa}$ if and only if the triple $(\dot{A}, T, \hat{A})$ satisfies Hypothesis 2.
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generalized Donoghue class $\mathcal{M}_{-1}^\kappa$ if and only if the triple
$(\hat{A}, T, \hat{A})$ satisfies Hypothesis 2.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} A & K & 1 \\ \mathcal{H}_+ & \mathcal{H} & \mathcal{H}_- \end{pmatrix} \]

Function class

\[ V(z) \in \mathcal{M} \]

\[ T \text{ has } \kappa = 0 \]

\[ U \text{ is an arbitrary unimodular number.} \]
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Function class

$$V(z) \in \mathcal{M}$$

$T$ has $\kappa = 0$

$U$ is an arbitrary unimodular number.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} \mathcal{A} & K & 1 \\ \mathcal{H}_+ & \subset & \mathcal{H} & \subset & \mathcal{H}_- \end{pmatrix} \]

Function class

\[ V(z) \in \mathcal{M} \]

\[ T \] has \( \kappa = 0 \)

\( U \) is an arbitrary unimodular number.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} A \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \end{pmatrix} \begin{pmatrix} K & 1 \\ \mathfrak{c} \end{pmatrix} \]

Function class

\[ V(z) \in \mathcal{M} \]

\[ T \quad \text{has} \quad \kappa = 0 \]

\( U \) is an arbitrary unimodular number.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} A & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 & \mathbb{C} \end{pmatrix} \]

Function class

\[ V(z) \in \mathcal{M}_\kappa \]

\( T \) has von Neumann parameter \( \kappa \)

\( \hat{A} \) is parameterized with \( U = -1 \)

\( (\hat{A}, T, \hat{A}) \) satisfies Hypothesis 1.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} \Lambda & K & 1 \\ \mathcal{H}_+ & \subset & \mathcal{H} & \subset & \mathcal{H}_- \end{pmatrix} \]

Function class

\[ V(z) \in \mathcal{M}_\kappa \]

\( T \) has von Neumann parameter \( \kappa \)

\( \hat{A} \) is parameterized with \( U = -1 \)

\( (\hat{A}, T, \hat{A}) \) satisfies Hypothesis 1.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} A & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & C \end{pmatrix} \]

Function class

\[ V(z) \in \mathcal{M}_\kappa \]

\( T \) has von Neumann parameter \( \kappa \)

\( \hat{\hat{A}} \) is parameterized with \( U = -1 \)

\( (\hat{\hat{A}}, T, \hat{\hat{A}}) \) satisfies Hypothesis 1.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} \hat{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & C \end{pmatrix} \]

- \( V(z) \in M_\kappa \)
- \( T \) has von Neumann parameter \( \kappa \)
- \( \hat{A} \) is parameterized with \( U = -1 \)

\((\hat{A}, T, \hat{A})\) satisfies Hypothesis 1.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} \hat{A} & K & 1 \\ \mathcal{H}_+ & \subset & \mathcal{H} & \subset & \mathcal{H}_- \end{pmatrix} \]

\( T \) has von Neumann parameter \( \kappa \)
\( \hat{A} \) is parameterized with \( U = -1 \)

(\( \hat{A}, T, \hat{A} \)) satisfies Hypothesis 1.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} \hat{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \end{pmatrix} \]

Function class

\[ V(z) \in \mathcal{M}_\kappa \]

\( T \) has von Neumann parameter \( \kappa \)

\( \hat{A} \) is parameterized with \( U = -1 \)

\( (\hat{A}, T, \hat{A}) \) satisfies Hypothesis 1.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} \Lambda & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & \mathbb{C} \end{pmatrix} \]

Function class

\[ V(z) \in \mathfrak{M}_{-1}^{\kappa} \]

\( T \) has von Neumann parameter \( \kappa \)

\( \hat{A} \) is parameterized with \( U = 1 \)

\( (\hat{A}, T, \hat{A}) \) satisfies Hypothesis 2.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} A & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 \end{pmatrix} \]

Function class

\[ V(z) \in M_{\kappa}^{-1} \]

\( T \) has von Neumann parameter \( \kappa \)

\( \hat{A} \) is parameterized with \( U = 1 \)

\( (\hat{A}, T, \hat{A}) \) satisfies Hypothesis 2.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} A & K & 1 \\ H_+ \subset H \subset H_- & C \end{pmatrix} \]

Function class

\[ V(z) \in \mathcal{M}^{-1}_\kappa \]

\[ T \] has von Neumann parameter \( \kappa \)

\( \hat{A} \) is parameterized with \( U = 1 \)

\( (\hat{A}, T, \hat{A}) \) satisfies Hypothesis 2.
Realization of Donoghue classes

\[ \Theta = \left( \begin{array}{ccc} \hat{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 & 1 \end{array} \right) \]

Function class

\[ V(z) \in \mathcal{M}_{\kappa}^{-1} \]

\( T \) has von Neumann parameter \( \kappa \)

\( \hat{A} \) is parameterized with \( U = 1 \)

\( (\hat{A}, T, \hat{A}) \) satisfies Hypothesis 2.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} A & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \end{pmatrix} \]

\( \mathcal{T} \) has von Neumann parameter \( \kappa \)

\( \hat{A} \) is parameterized with \( U = 1 \)

\((\hat{A}, T, \hat{A})\) satisfies Hypothesis 2.
Realization of Donoghue classes

\[ \Theta = \begin{pmatrix} \hat{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 \end{pmatrix} \]

Function class

\[ V(z) \in \mathcal{M}_\kappa^{-1} \]

\begin{itemize}
  \item $T$ has von Neumann parameter $\kappa$
  \item $\hat{A}$ is parameterized with $U = 1$
  \item $(\hat{A}, T, \hat{A})$ satisfies Hypothesis 2.
\end{itemize}
Realization of perturbed Donoghue classes

\[ \Theta = \begin{pmatrix} A & K & 1 \\ \mathcal{H}_+ & \subset & \mathcal{H} & \subset & \mathcal{H}_- \end{pmatrix} \]

Function class
\[ V(z) \in \mathcal{M} \]

Function class
\[ V(z) \in \mathcal{M}_{\kappa_0} \]

Function class
\[ V(z) \in \mathcal{M}_{\kappa_0}^{-1} \]

\[ T \ 	ext{has von Neumann parameter} \ \kappa = ? \]

\[ \hat{A} \ 	ext{is parameterized with} \ U = ? \]
Realization of perturbed Donoghue classes

Function class

\[ V(z) \in \mathcal{M} \]

Function class

\[ V(z) \in \mathcal{M}_{\kappa_0} \]

Function class

\[ V(z) \in \mathcal{M}_{\kappa_0}^{-1} \]

\[ \Theta = \begin{pmatrix} \mathcal{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \end{pmatrix} \]

\( T \) has von Neumann parameter \( \kappa = ? \).

\( \hat{A} \) is parameterized with \( U = ? \).
Realization of perturbed Donoghue classes

**Function class**
\[ V(z) \in M \]

\[ V(z) \in M_{\kappa_0} \]

\[ \Theta = \left( \begin{array}{ccc} A & K & 1 \\ \mathcal{H}_+ & \subset & \mathcal{H} & \subset & \mathcal{H}_- \\ C & \end{array} \right) \]

\[ T \text{ has von Neumann parameter } \kappa = ?. \]

\[ \hat{A} \text{ is parameterized with } U = ?. \]
# Realization of perturbed Donoghue classes

Let \( \Theta = \left( \begin{array}{ccc} A & K & 1 \\ \mathcal{H}_+ & \subset & \mathcal{H} & \subset & \mathcal{H}_- \\ C & \end{array} \right) \)

- **Function class**
  - \( V(z) \in \mathcal{M} \)

- **Function class**
  - \( V(z) \in \mathcal{M}_{\kappa_0} \)

- **Function class**
  - \( V(z) \in \mathcal{M}_{\kappa_0}^{-1} \)

\( T \) has von Neumann parameter \( \kappa = ? \).

\( \hat{A} \) is parameterized with \( U = ? \).
Realization of perturbed Donoghue classes

Function class
\[ V(z) \in \mathcal{M} \]

Function class
\[ V(z) \in \mathcal{M}_{\kappa_0} \]

Function class
\[ V(z) \in \mathcal{M}_{\kappa_0}^{-1} \]

Perturbed function
\[ \hat{A} = Q + V(z) \]

\[ \Theta = \begin{pmatrix} A & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & C \end{pmatrix} \]

\( T \) has von Neumann parameter \( \kappa = ? \).

\( \hat{A} \) is parameterized with \( U = ? \).
Realization of perturbed Donoghue classes

\[ \Theta = \begin{pmatrix} \mathcal{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 & 0 \\ \mathcal{C} & 0 & 1 \end{pmatrix} \]

\( T \) has von Neumann parameter \( \kappa = ? \).

\( \hat{A} \) is parameterized with \( U = ? \).
Realization of perturbed Donoghue classes

Function class
\[ V(z) \in \mathcal{M} \]

Function class
\[ V(z) \in \mathcal{M}_{\kappa_0} \]

Function class
\[ V(z) \in \mathcal{M}_{-1}^{\kappa_0} \]

Perturbed function
\[ Q + V(z) \]

\[ \Theta = \begin{pmatrix} \tilde{A} & K & 1 \\ \mathcal{H} & \subset & \mathcal{H} & \subset & \mathcal{H} & \subset & \mathcal{H} \end{pmatrix} \]

\( T \) has von Neumann parameter \( \kappa = ? \).

\( \tilde{A} \) is parameterized with \( U = ? \).
Realization of perturbed Donoghue classes

Function class
\( V(z) \in \mathcal{M} \)

Function class
\( V(z) \in \mathcal{M}_{\kappa_0} \)

Function class
\( V(z) \in \mathcal{M}^{-1}_{\kappa_0} \)

Perturbed function
\( Q + V(z) \)

\[ \Theta = \left( \begin{array}{ccc} \mathcal{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & & \end{array} \right) \]

\( \hat{A} \) is parameterized with \( U = ? \).

\( T \) has von Neumann parameter \( \kappa = ? \).
Realization of perturbed Donoghue classes

Function class
\( V(z) \in \mathcal{M} \)

Function class
\( V(z) \in \mathcal{M}_{\kappa_0} \)

Function class
\( V(z) \in \mathcal{M}_{\kappa_0}^{-1} \)

Perturbed function
\( Q + V(z) \)

\[ \Theta = \begin{pmatrix} \hat{A} & K & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & \mathbb{C} \end{pmatrix} \]

\( T \) has von Neumann parameter \( \kappa = ? \).

\( \hat{A} \) is parameterized with \( U = ? \).
Realization of class $\mathcal{M}^Q$

**Theorem (B., Tsekanovskii, ’19)**

Let $V(z)$ belong to the class $\mathcal{M}^Q$. Then $V(z)$ can be realized by a minimal L-system $\Theta$ with the main operator $T$ whose von Neumann’s parameter $\kappa$ is determined as a function of $Q$. by the formula

$$\kappa = \frac{|Q|}{\sqrt{Q^2 + 4}}, \quad Q \neq 0. \quad (11)$$

Moreover, the unimodular parameter $U$ of the quasi-kernel $\hat{A}$ of $\Theta$ is also uniquely defined by $Q$.

$$U = \frac{Q}{|Q|} \cdot \frac{-Q + 2i}{\sqrt{Q^2 + 4}}, \quad Q \neq 0. \quad (12)$$
Theorem (B., Tsekanovskii, ’19)

Let $V(z)$ belong to the class $\mathcal{M}^Q$. Then $V(z)$ can be realized by a minimal L-system $\Theta$ with the main operator $T$ whose von Neumann’s parameter $\kappa$ is determined as a function of $Q$ by the formula

$$\kappa = \frac{|Q|}{\sqrt{Q^2 + 4}}, \quad Q \neq 0. \quad (11)$$

Moreover, the unimodular parameter $U$ of the quasi-kernel $\hat{A}$ of $\Theta$ is also uniquely defined by $Q$.

$$U = \frac{Q}{|Q|} \cdot \frac{-Q + 2i}{\sqrt{Q^2 + 4}}, \quad Q \neq 0. \quad (12)$$
Realization of class $\mathcal{M}_κ^Q$

**Theorem (B., Tsekanovskii, ’19)**

Let $V(z)$ belong to the class $\mathcal{M}_κ^Q$ and have a normalization parameter $0 < a < 1$. Then $V(z)$ can be realized by a minimal L-system $\Theta$ with the main operator $T$ whose von Neumann’s parameter $κ$ is uniquely determined as a function of $Q$ and $a$.

\[
κ = \frac{\left(b - 2Q^2 - \sqrt{b^2 + 4Q^2}\right)^2 - a \left(b - \sqrt{b^2 + 4Q^2}\right)^2 + 4Q^2a(a - 1)}{\left(b - 2Q^2 - \sqrt{b^2 + 4Q^2}\right)^2 + a \left(b - \sqrt{b^2 + 4Q^2}\right)^2 + 4Q^2a(a + 1)}
\]

where $Q \neq 0$ and $b = Q^2 + a^2 - 1$. 

(13)
Realization of class \( \mathcal{M}_{\kappa}^{Q} \)

Theorem (B., Tsekanovskii, ’19)

Let \( V(z) \) belong to the class \( \mathcal{M}_{\kappa_0}^{Q} \) and have a normalization parameter \( 0 < a < 1 \). Then \( V(z) \) can be realized by a minimal L-system \( \Theta \) with the main operator \( T \) whose von Neumann’s parameter \( \kappa \) is uniquely determined as a function of \( Q \) and \( a \).

\[
\kappa = \frac{\left( b - 2Q^2 - \sqrt{b^2 + 4Q^2} \right)^2 - a \left( b - \sqrt{b^2 + 4Q^2} \right)^2 + 4Q^2a(a - 1)}{\left( b - 2Q^2 - \sqrt{b^2 + 4Q^2} \right)^2 + a \left( b - \sqrt{b^2 + 4Q^2} \right)^2 + 4Q^2a(a + 1)}
\]

where \( Q \neq 0 \) and \( b = Q^2 + a^2 - 1 \).
Theorem (B., Tsekanovskiĭ, ’19)

Moreover, the quasi-kernel $\hat{\mathbb{A}}$ of $\text{Re} \mathbb{A}$ of the realizing L-system $\Theta$ is uniquely defined with

$$U = \frac{(a + Qi)(1 - \kappa^2) - 1 - \kappa^2}{2\kappa}, \quad Q \neq 0. \quad (14)$$
Theorem (B., Tsekanovskii, ’19)

Moreover, the quasi-kernel $\hat{A}$ of $\text{Re} A$ of the realizing L-system $\Theta$ is uniquely defined with

$$U = \frac{(a + Qi)(1 - \kappa^2) - 1 - \kappa^2}{2\kappa}, \quad Q \neq 0. \quad (14)$$
Realization of class $\mathcal{M}_{\kappa}^{-1,Q}$

Theorem (B., Tsekanovskii, ’19)

Let $V(z)$ belong to the class $\mathcal{M}_{\kappa_0}^{-1,Q}$ and have a normalization parameter $a > 1$. Then $V(z)$ can be realized by a minimal L-system $\Theta$ with the main operator $T$ whose von Neumann’s parameter $\kappa$ is uniquely determined as a function of $Q$ and $a$.

$$\kappa = \frac{a \left( b + \sqrt{b^2 + 4Q^2} \right)^2 - \left( b - 2Q^2 + \sqrt{b^2 + 4Q^2} \right)^2 - 4Q^2 a(a - 1)}{\left( b - 2Q^2 + \sqrt{b^2 + 4Q^2} \right)^2 + a \left( b + \sqrt{b^2 + 4Q^2} \right)^2 + 4Q^2 a(a + 1)}$$

(15)

where $Q \neq 0$ and $b = Q^2 + a^2 - 1$. 
Realization of class $\mathcal{M}_{\kappa}^{-1, Q}$

Theorem (B., Tsekanovskii, ’19)

Let $V(z)$ belong to the class $\mathcal{M}_{\kappa_0}^{-1, Q}$ and have a normalization parameter $a > 1$. Then $V(z)$ can be realized by a minimal L-system $\Theta$ with the main operator $T$ whose von Neumann’s parameter $\kappa$ is uniquely determined as a function of $Q$ and $a$.

$$\kappa = \frac{a \left( b + \sqrt{b^2 + 4Q^2} \right)^2 - \left( b - 2Q^2 + \sqrt{b^2 + 4Q^2} \right)^2 - 4Q^2 a(a - 1)}{\left( b - 2Q^2 + \sqrt{b^2 + 4Q^2} \right)^2 + a \left( b + \sqrt{b^2 + 4Q^2} \right)^2 + 4Q^2 a(a + 1)}$$

(15)

where $Q \neq 0$ and $b = Q^2 + a^2 - 1$. 
Realization of class $\mathcal{M}_{-1,Q}^\kappa$

Theorem (B., Tsekanovskii, ’19)

Moreover, the quasi-kernel $\hat{A}$ of $\text{Re}A$ of the realizing L-system $\Theta$ is uniquely defined with

$$U = \frac{(a + Qi)(1 - \kappa^2) - 1 - \kappa^2}{2\kappa}, \quad Q \neq 0. \quad (16)$$
Realization of class $\mathcal{M}^{-1,Q}_\kappa$

Theorem (B., Tsekanovskii, ’19)

Moreover, the quasi-kernel $\hat{A}$ of $\text{Re} A$ of the realizing L-system $\Theta$ is uniquely defined with

$$U = \frac{(a + Qi)(1 - \kappa^2) - 1 - \kappa^2}{2\kappa}, \quad Q \neq 0.$$ (16)
Direct theorem for L-systems

Theorem (B., Tsekanovskiĭ, ’19)

Let $\Theta$ be a minimal L-system of the form (5) with the main operator $T$ and its von Neumann’s parameter $\kappa$, ($0 \leq \kappa < 1$). Then only one of the following takes place:

1. $V_\Theta(z)$ belongs to class $M^Q$ and $\kappa$ is determined by (11) for some $Q$;

2. $V_\Theta(z)$ belongs to class $M^{\kappa_0}$ and $\kappa$ is determined by (13) for some $Q$ and $a = \frac{1-\kappa_0}{1+\kappa_0}$;

3. $V_\Theta(z)$ belongs to class $M^{-1,\kappa_0}$ and $\kappa$ is determined by (15) for some $Q$ and $a = \frac{1+\kappa_0}{1-\kappa_0}$.

The values of $Q$ and $\kappa_0$ are determined from integral representation (10) of $V_\Theta(z)$. 
Direct theorem for L-systems

Theorem (B., Tsekanovskii, ’19)

Let $\Theta$ be a minimal L-system of the form (5) with the main operator $T$ and its von Neumann’s parameter $\kappa$, $0 \leq \kappa < 1$. Then only one of the following takes place:

1. $V_{\Theta}(z)$ belongs to class $M^Q$ and $\kappa$ is determined by (11) for some $Q$;

2. $V_{\Theta}(z)$ belongs to class $M^{Q}_{\kappa_0}$ and $\kappa$ is determined by (13) for some $Q$ and $a = \frac{1-\kappa_0}{1+\kappa_0}$;

3. $V_{\Theta}(z)$ belongs to class $M^{-1, Q}_{\kappa_0}$ and $\kappa$ is determined by (15) for some $Q$ and $a = \frac{1+\kappa_0}{1-\kappa_0}$.

The values of $Q$ and $\kappa_0$ are determined from integral representation (10) of $V_{\Theta}(z)$. 
Theorem (B., Tsekanovskii, ’19)

Let $\Theta$ be a minimal L-system of the form (5) with the main operator $T$ and its von Neumann’s parameter $\kappa$, $(0 \leq \kappa < 1)$. Then only one of the following takes place:

1. $V_\Theta(z)$ belongs to class $\mathfrak{M}^Q$ and $\kappa$ is determined by (11) for some $Q$;

2. $V_\Theta(z)$ belongs to class $\mathfrak{M}^Q_{\kappa_0}$ and $\kappa$ is determined by (13) for some $Q$ and $a = \frac{1-\kappa_0}{1+\kappa_0}$;

3. $V_\Theta(z)$ belongs to class $\mathfrak{M}^{-1,Q}$ and $\kappa$ is determined by (15) for some $Q$ and $a = \frac{1+\kappa_0}{1-\kappa_0}$.

The values of $Q$ and $\kappa_0$ are determined from integral representation (10) of $V_\Theta(z)$.
Theorem (B., Tsekanovskiĭ, ’19)

Let $\Theta$ be a minimal L-system of the form (5) with the main operator $T$ and its von Neumann’s parameter $\kappa$, ($0 \leq \kappa < 1$). Then only one of the following takes place:

1. $V_{\Theta}(z)$ belongs to class $\mathcal{M}^Q$ and $\kappa$ is determined by (11) for some $Q$;

2. $V_{\Theta}(z)$ belongs to class $\mathcal{M}^{\kappa_0}$ and $\kappa$ is determined by (13) for some $Q$ and $a = \frac{1-\kappa_0}{1+\kappa_0}$;

3. $V_{\Theta}(z)$ belongs to class $\mathcal{M}^{-1,\kappa_0}$ and $\kappa$ is determined by (15) for some $Q$ and $a = \frac{1+\kappa_0}{1-\kappa_0}$.

The values of $Q$ and $\kappa_0$ are determined from integral representation (10) of $V_{\Theta}(z)$. 
Theorem (B., Tsekanovskii, ’19)

Let $\Theta$ be a minimal L-system of the form (5) with the main operator $T$ and its von Neumann’s parameter $\kappa$, $(0 \leq \kappa < 1)$. Then only one of the following takes place:

1. $V_{\Theta}(z)$ belongs to class $\mathcal{M}^Q$ and $\kappa$ is determined by (11) for some $Q$;

2. $V_{\Theta}(z)$ belongs to class $\mathcal{M}_{\kappa_0}^Q$ and $\kappa$ is determined by (13) for some $Q$ and $a = \frac{1-\kappa_0}{1+\kappa_0}$;

3. $V_{\Theta}(z)$ belongs to class $\mathcal{M}_{\kappa_0}^{-1, Q}$ and $\kappa$ is determined by (15) for some $Q$ and $a = \frac{1+\kappa_0}{1-\kappa_0}$.

The values of $Q$ and $\kappa_0$ are determined from integral representation (10) of $V_{\Theta}(z)$.
Direct theorem for L-systems

Theorem (B., Tsekanovskii, ’19)

Let \( \Theta \) be a minimal L-system of the form (5) with the main operator \( T \) and its von Neumann’s parameter \( \kappa \), \( 0 \leq \kappa < 1 \).

Then only one of the following takes place:

1. \( V_\Theta(z) \) belongs to class \( \mathcal{M}^Q \) and \( \kappa \) is determined by (11) for some \( Q \);

2. \( V_\Theta(z) \) belongs to class \( \mathcal{M}^{\kappa_0}_Q \) and \( \kappa \) is determined by (13) for some \( Q \) and \( a = \frac{1-\kappa_0}{1+\kappa_0} \);

3. \( V_\Theta(z) \) belongs to class \( \mathcal{M}^{-1,Q}_{\kappa_0} \) and \( \kappa \) is determined by (15) for some \( Q \) and \( a = \frac{1+\kappa_0}{1-\kappa_0} \).

The values of \( Q \) and \( \kappa_0 \) are determined from integral representation (10) of \( V_\Theta(z) \).
Suppose we are given an L-system $\Theta$ whose impedance function $V_{\Theta}(z)$ belongs to one of the Donoghue classes $\mathcal{M}$, $\mathcal{M}_{\kappa_0}$, or $\mathcal{M}_{-1\kappa_0}$. Let also $Q \neq 0$ be any real number.

Perturbation of an L-system

An L-system $\Theta^Q$ whose construction is based on the elements of a given L-system $\Theta$ (subject to either of Hypotheses 1 or 2) is called the **perturbation of an L-system $\Theta$** if

$$V_{\Theta^Q}(z) = Q + V_{\Theta}(z).$$
Suppose we are given an L-system $\Theta$ whose impedance function $V_{\Theta}(z)$ belongs to one of the Donoghue classes $M$, $M_{\kappa_0}$, or $M_{-1}^{-1}$. Let also $Q \neq 0$ be any real number.

**Perturbation of an L-system**

An L-system $\Theta^Q$ whose construction is based on the elements of a given L-system $\Theta$ (subject to either of Hypotheses 1 or 2) is called the **perturbation of an L-system** $\Theta$ if

$$V_{\Theta^Q}(z) = Q + V_{\Theta}(z).$$
Perturbing an L-system

Unperturbed L-system

\[ \Theta = \begin{pmatrix} A & K & 1 \\ H_+ & H & H_- \end{pmatrix} \]

Given unperturbed L-system.
Perturbing an L-system

Unperturbed L-system

\[ \Theta = \begin{pmatrix} A & K & 1 \\ \mathcal{H}_+ & \subset & \mathcal{H} & \subset & \mathcal{H}_- \\ \mathbb{C} & & & & & & \end{pmatrix} \]

Given unperturbed L-system.
Perturbing an L-system

Unperturbed L-system

\[ \Theta = \begin{pmatrix} \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 \\ \mathbb{C} \end{pmatrix} \]

Keep the symmetric operator \( \dot{A} \) and state space \( \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \).
Perturbing an L-system

Unperturbed L-system

\[ \Theta = \begin{pmatrix} \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- & 1 \\ \mathbb{C} & \mathbb{C} \end{pmatrix} \]

Construct state-space operator $A^Q$ and channel operator $K^Q$. 
Perturbing an L-system

\[ \Theta^Q = \begin{pmatrix} A^Q & K^Q & 1 \\ \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \end{pmatrix} \]

Obtain perturbed L-system \( \Theta^Q \) such that \( V_{\Theta^Q}(z) = Q + V_{\Theta_0}(z) \).
Theorem (B., Tsekanovskii, ’19)

Let $\Theta_0$ be an L-system satisfying the conditions of Hypothesis 1 and such that $V_{\Theta_0}(z) \in M$. Then for any real number $Q \neq 0$ there exists another L-system $\Theta^Q$ with the same symmetric operator $\hat{A}$ as in $\Theta_0$ and such that

$$V_{\Theta^Q}(z) = Q + V_{\Theta}(z).$$

Moreover, the von Neumann parameter $\kappa$ of its main operator $T^Q$ is determined by the formula (11) while the quasi-kernel $\hat{A}^Q$ is defined by $U$ from (12).
Theorem (B., Tsekanovskiĭ, ’19)

Let $\Theta_0$ be an L-system satisfying the conditions of Hypothesis 1 and such that $V_{\Theta_0}(z) \in \mathcal{M}$. Then for any real number $Q \neq 0$ there exists another L-system $\Theta^Q$ with the same symmetric operator $\dot{A}$ as in $\Theta_0$ and such that

$$V_{\Theta^Q}(z) = Q + V_{\Theta}(z).$$

Moreover, the von Neumann parameter $\kappa$ of its main operator $T^Q$ is determined by the formula (11) while the quasi-kernel $\hat{A}^Q$ is defined by $U$ from (12).
Perturbation of class $M_{\kappa}$ systems

Theorem (B., Tsekanovskii, ’19)

Let $\Theta_{\kappa_0}$ be an L-system and such that $V_{\Theta_0}(z) \in M_{\kappa_0}$. Then for any real number $Q \neq 0$ there exists another L-system $\Theta_{\kappa, Q}$ with the same symmetric operator $\hat{A}$ as in $\Theta_{\kappa_0}$ and such that

$$V_{\Theta_{\kappa, Q}}(z) = Q + V_{\Theta_{\kappa_0}}(z).$$

Moreover, the von Neumann parameter $\kappa$ of its main operator $T^Q$ is determined by the formula (13) while the quasi-kernel $\hat{A}^Q$ is defined by $U$ from (14).
Perturbation of class $\mathcal{M}_\kappa$ systems

**Theorem (B., Tsekanovskii, ’19)**

Let $\Theta_{\kappa_0}$ be an L-system and such that $V_{\Theta_0}(z) \in \mathcal{M}_{\kappa_0}$. Then for any real number $Q \neq 0$ there exists another L-system $\Theta^Q_{\kappa}$ with the same symmetric operator $\hat{A}$ as in $\Theta_{\kappa_0}$ and such that

$$V_{\Theta^Q_{\kappa}}(z) = Q + V_{\Theta_{\kappa_0}}(z).$$

Moreover, the von Neumann parameter $\kappa$ of its main operator $T^Q$ is determined by the formula (13) while the quasi-kernel $\hat{A}^Q$ is defined by $U$ from (14).
Theorem (B., Tsekanovskii, ’19)

Let $\Theta_{\kappa_0}$ be an L-system such that $V_{\Theta_0}(z) \in \mathcal{M}_{\kappa_0}^{-1}$. Then for any real number $Q \neq 0$ there exists another L-system $\Theta_Q^\kappa$ with the same symmetric operator $\hat{A}$ as in $\Theta_{\kappa_0}$ and such that

$$V_{\Theta_Q^\kappa}(z) = Q + V_{\Theta_{\kappa_0}}(z).$$

Moreover, the von Neumann parameter $\kappa$ of its main operator $T^Q$ is determined by the formula (15) while the quasi-kernel $\hat{A}^Q$ is defined by $U$ from (16).
Theorem (B., Tsekanovskiĭ, ’19)

Let $\Theta_{\kappa_0}$ be an L-system such that $V_{\Theta_0}(z) \in \mathcal{M}_{\kappa_0}^{-1}$. Then for any real number $Q \neq 0$ there exists another L-system $\Theta^Q_{\kappa}$ with the same symmetric operator $\hat{A}$ as in $\Theta_{\kappa_0}$ and such that

$$V_{\Theta^Q_{\kappa}}(z) = Q + V_{\Theta_{\kappa_0}}(z).$$

Moreover, the von Neumann parameter $\kappa$ of its main operator $T^Q$ is determined by the formula (15) while the quasi-kernel $\hat{A}^Q$ is defined by $U$ from (16).

Thank you!
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