HOW TO COMPUTE THE GRADIENT OF AN AFFINE FUNCTION?

Assume we have a triangle $T = \operatorname{conv}\{z_1, z_2, z_3\}$ with $z_i = (z_{i,x}, z_{i,y}) \in \mathbb{R}^2$ and the corresponding hat-functions $\zeta_i = \zeta_i|_T \in \mathcal{P}^1(T)$. By linear interpolation of the functions $(x, y) \mapsto x$ and $(x, y) \mapsto y$, we obtain

$$x = \sum_{i=1}^{3} z_{i,x}\zeta_i(x, y) \quad \text{and} \quad y = \sum_{i=1}^{3} z_{i,y}\zeta_i(x, y) \quad \text{for all } (x, y) \in T.$$

Moreover, there holds

$$1 = \sum_{i=1}^{3} 1\zeta_i(x, y) \quad \text{for all } (x, y) \in T.$$

Applying the gradient to those three identities shows

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \nabla((x, y) \mapsto 1) \\ \nabla((x, y) \mapsto x) \\ \nabla((x, y) \mapsto y) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ z_{1,x} & z_{2,x} & z_{3,x} \\ z_{1,y} & z_{2,y} & z_{3,y} \end{pmatrix}}_{:=M} \begin{pmatrix} \nabla\zeta_1 \\ \nabla\zeta_2 \\ \nabla\zeta_3 \end{pmatrix}. \tag{1}$$

For non-degenerate T, we see that M is regular by transforming it to

$$\begin{pmatrix} 1 & 0 & 0 \\ z_1 & z_2 - z_1 & z_3 - z_1 \end{pmatrix}.$$

Hence, (1) uniquely determines the gradients $\nabla\zeta_i|_T$, $i = 1, 2, 3$.