Numerische Differentiation mit einseitigen Differenzenquotienten

$$D(h) = \frac{f(x_0 + h) - f(x_0)}{h}$$

h	$n = 0$	$n = 1$	$n = 2$	$n = 3$
h_0	$D(h_0) = D_{00}$	D_{01}	D_{02}	D_{03}
h_1	$D(h_1) = D_{10}$	D_{11}	D_{12}	D_{13}
h_2	$D(h_2) = D_{20}$	D_{21}	D_{22}	D_{23}
h_3	$D(h_3) = D_{30}$	D_{31}	D_{32}	D_{33}
h_4	$D(h_4) = D_{40}$	D_{41}	D_{42}	$\vdots$
h_5	$D(h_5) = D_{50}$	D_{51}	$\vdots$	
h_6	$D(h_6) = D_{60}$	$\vdots$		
$\vdots$	$\vdots$			

$$D_{i0} = P_{i,0}(0) = D(h_i)$$

$$D_{ij} = P_{i,j}(0) = D_{(i+1)\,(j-1)} - \frac{h_{i+j}}{h_{i+j} - h_i}\left[D_{(i+1)\,(j-1)} - D_{i\,(j-1)}\right], \qquad j \geq 1$$

Beispiel: $f(x) = e^x$ und $x_0 = 0$ mit einseitigen Diff.-quotienten

Neville-Schema:

h	$n=0$	$n=1$	$n=2$	$n=3$
1	1.71828182845905	0.87660325434147	1.00747997135508	0.99982039920503
2^{-1}	1.29744254140026	0.97476079210167	1.00077784572378	0.99999046433634
2^{-2}	1.13610166675097	0.99427358231826	1.00008888700977	0.99999944973780
2^{-3}	1.06518762453461	0.99863506083689	1.00001062939680	0.99999996693993
2^{-4}	1.03191134268575	0.99966673725682	1.00000129974704	0.99999999797395
2^{-5}	1.01578903997129	0.99991765912448	1.00000016069559	0.99999999987449
2^{-6}	1.00785334954789	0.99997953530281	1.00000001997713	
2^{-7}	1.00391644242535	0.99999489880855		
2^{-8}	1.00195567061695			

Beispiel: $f(x) = e^x$ und $x_0 = 0$ mit einseitigen Diff.-quotienten

Fehler und Fehlerrate:

h	$n=0$	ρ	$n=1$	ρ	$n=2$	ρ	$n=3$	ρ
1	7.18_{-1}	—	-1.23_{-1}	—	7.48_{-3}	—	-1.78_{-4}	—
2^{-1}	2.97_{-1}	0.41	-2.52_{-2}	0.20	7.78_{-4}	0.10	-9.54_{-6}	0.053
2^{-2}	1.36_{-1}	0.46	-5.73_{-3}	0.23	8.89_{-5}	0.11	-5.50_{-7}	0.058
2^{-3}	6.52_{-2}	0.48	-1.37_{-3}	0.24	1.06_{-5}	0.12	-3.31_{-8}	0.060
2^{-4}	3.19_{-2}	0.49	-3.33_{-4}	0.24	1.30_{-6}	0.12	-2.03_{-9}	0.061
2^{-5}	1.58_{-2}	0.49	-8.23_{-5}	0.25	1.61_{-7}	0.12	-1.26_{-10}	0.062
2^{-6}	7.85_{-3}	0.50	-2.05_{-5}	0.25	1.99_{-8}	0.12		
2^{-7}	3.92_{-3}	0.50	-5.10_{-6}	0.25				
2^{-8}	1.96_{-3}	0.50						
erhofft:	$O(h)$		$O(h^2)$		$O(h^3)$		$O(h^4)$	
		0.5		0.25		0.125		0.0625

Numerische Differentiation mit symmetrischen Differenzenquotienten

$$D_{sym}(h) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

h	h^2	$n=0$	$n=1$	$n=2$	$n=3$
h_0	h_0^2	$D_{sym}(h_0) = D_{00}$	D_{01}	D_{02}	D_{03}
h_1	h_1^2	$D_{sym}(h_1) = D_{10}$	D_{11}	D_{12}	D_{13}
h_2	h_2^2	$D_{sym}(h_2) = D_{20}$	D_{21}	D_{22}	D_{23}
h_3	h_3^2	$D_{sym}(h_3) = D_{30}$	D_{31}	D_{32}	D_{33}
h_4	h_4^2	$D_{sym}(h_4) = D_{40}$	D_{41}	D_{42}	$\vdots$
h_5	h_5^2	$D_{sym}(h_5) = D_{50}$	D_{51}	$\vdots$	
h_6	h_6^2	$D_{sym}(h_6) = D_{60}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$			

$$D_{i0} = P_{i,0}(0) = D_{sym}(h_i)$$

$$D_{ij} = P_{i,j}(0) = D_{(i+1)\,(j-1)} - \frac{h_{i+j}^2}{h_{i+j}^2 - h_i^2}\left[D_{(i+1)\,(j-1)} - D_{i\,(j-1)}\right], \qquad j \geq 1$$

Numer. Differentiation, sym. Diff.-quotienten, $f(x) = e^x$, $x_0 = 0$

Neville-Schema

h	h^2	$n=0$	$n=1$	$n=2$	$n=3$
1	1	1.17520119364380	0.99785375010206	1.00000315726189	0.99999999931903
2^{-1}	2^{-2}	1.04219061098749	0.99986881931440	1.00000004866189	0.99999999999737
2^{-2}	2^{-4}	1.01044926723267	0.99999184682767	1.00000000075775	0.99999999999999
2^{-3}	2^{-6}	1.00260620192892	0.99999949113712	1.00000000001183	1.00000000000000
2^{-4}	2^{-8}	1.00065116883507	0.99999996820716	1.00000000000018	1.00000000000001
2^{-6}	2^{-10}	1.00016276836414	0.99999999801312	1.00000000000001	0.99999999999997
2^{-6}	2^{-12}	1.00004069060087	0.99999999987583	0.99999999999997	
2^{-7}	2^{-14}	1.00001017255709	0.99999999999221		
2^{-8}	2^{-16}	1.00000254313343			

Fehler und Fehlerrate:

h	h^2	$n=0$	ρ	$n=1$	ρ	$n=2$	ρ	$n=3$	ρ
1	1	1.752_{-1}	—	-2.146_{-3}	—	3.157_{-6}	—	-6.81_{-10}	—
2^{-1}	2^{-2}	4.219_{-2}	0.25	-1.312_{-4}	0.0611	4.866_{-8}	0.0154	-2.63_{-12}	0.0039
2^{-2}	2^{-4}	1.045_{-2}	0.25	-8.153_{-6}	0.0622	7.578_{-10}	0.0156	-1.0_{-14}	0.0038
2^{-3}	2^{-6}	2.606_{-3}	0.25	-5.089_{-7}	0.0624	1.183_{-11}	0.0156	0.0	??
2^{-4}	2^{-8}	6.51_{-4}	0.25	-3.179_{-8}	0.0625	1.8_{-13}	0.0152	1.0_{-14}	??
2^{-5}	2^{-10}	1.628_{-4}	0.25	-1.987_{-9}	0.0625	1.0_{-14}	??	-3.0_{-14}	??
2^{-6}	2^{-12}	4.069_{-5}	0.25	-1.242_{-10}	0.0625	-3.0_{-14}	??		
2^{-7}	2^{-14}	1.017_{-5}	0.25	-7.79_{-12}	0.0627				
2^{-8}	2^{-16}	2.543_{-6}	0.25						
			0.25		0.0625		0.015625		.00390625

Numer. Differentiation von $f(x) = |x|^{3/2}$ bei $x = 0$

Fehler bei Extrapolation von einseitigen Differenzenquotienten

h	n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7
2^0	1.00_0	4.14_{-1}	2.52_{-1}	1.68_{-1}	1.15_{-1}	8.06_{-2}	5.66_{-2}	3.99_{-2}
2^{-1}	7.07_{-1}	2.93_{-1}	1.79_{-1}	1.19_{-1}	8.17_{-2}	5.70_{-2}	4.00_{-2}	2.82_{-2}
2^{-2}	5.00_{-1}	2.07_{-1}	1.26_{-1}	8.40_{-2}	5.77_{-2}	4.03_{-2}	2.83_{-2}	
2^{-3}	3.54_{-1}	1.46_{-1}	8.93_{-2}	5.94_{-2}	4.08_{-2}	2.85_{-2}		
2^{-4}	2.50_{-1}	1.04_{-1}	6.31_{-2}	4.20_{-2}	2.89_{-2}			
2^{-5}	1.77_{-1}	7.32_{-2}	4.46_{-2}	2.97_{-2}				
2^{-6}	1.25_{-1}	5.18_{-2}	3.16_{-2}					
2^{-7}	8.84_{-2}	3.66_{-2}						
2^{-8}	6.25_{-2}							
Fehler	$\sqrt{h}$	$\sqrt{h}$	$\sqrt{h}$	$\sqrt{h}$	$\sqrt{h}$	$\sqrt{h}$		

Grund für das "Versagen" der Extrapolation:

wir haben: $D(h) = f'(0) + R(h), \qquad |R(h)| \approx Ch^{1/2}.$

Erinnerung: Für schnelle Konvergenz in Spalte n benötigten wir, daß $D(h) = \sum_{i=0}^{n} a_i h^i + R(h)$ mit $|R(h)| \leq Ch^{n+1}$.

Praktisch bedeutet dies, daß D bei $h = 0$ $n+1$-Mal differenzierbar sein sollte.