problem sheet 9

discussion: Tuesday, 8.12.20

9.1. Consider the equation $y'' + \varepsilon(y')^3 + y = 0$ with $0 < \varepsilon \ll 1$ and $y(0) = a > 0$ and $y'(0) = 0$.

a) Why is the solution bounded?

b) Construct the approximation $y_0(t, T)$ of a multiscale ansatz if $T = \varepsilon t$ is selected.

9.2. Consider the "Duffing equation" (a nonlinear oscillator), i.e., the equation $y'' + y + \varepsilon y^3 = 0$ with $0 < \varepsilon \ll 1$ and $y(0) = a > 0$ and $y'(0) = 0$.

a) Give an "energy" that is conserved. Hence, there are periodic solution where the period depends, of course, on ε and a. Sketch the phase portrait for small ε in the vicinity of $(0, 0)$. Note that, starting from the energy conservation, you have a separable differential equation of first order for y. Using this equation, express the period $T(\varepsilon, a)$ of the oscillation as an integral of the form

$$T(\varepsilon, a) = 4 \int_0^a \cdots .$$

(By symmetric, one only needs to compute a quarter of the period, which explains the factor 4.) Expand the integrand in a suitable way in powers of ε to arrive at the form $T(\varepsilon, a) = c_0(a) + c_1(a)\varepsilon + c_2(a)\varepsilon^2 + \cdots$. Why is the expansion allowed?

b) The classical pendulum equation is $y'' + \sin y = 0$. By Taylor expansion $\sin x = x - \frac{1}{6}x^3 + \cdots$ and for small deflection a one obtains for the frequency $\omega \approx 1 - \frac{1}{16}a^2$.

9.3. In this exercise, we perform a perturbation theory to determine the frequency of a *periodic* orbit.

Consider the Duffing equation $y'' + y + \varepsilon y^3 = 0$ with $y(0) = a$ and $y'(0) = 0$ and $0 < \varepsilon \ll 1$. Make the substition $\tau = \omega t$, where ω is the (unknown) frequency of the periodic solution. The transformed function should then be 2π-periodic. For the transformed function $\widetilde{y}(\tau)$ make the ansatz $\widetilde{y}(\tau) \sim y_0(\tau) + \varepsilon y_1(\tau) + \cdots$ and $\omega = 1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots$ (Why is the leading order term 1?). What are the equations corresponding to ε^0 and ε^1? Determine a condition for ω_1 by supressing resonances. (Why is this a sensible condition?)

9.4. Let $\Omega \subset \mathbb{R}^3$ be a (connected) domain and $\varphi \in C^1(\Omega; \mathbb{R}^3)$. Show: a deformation φ is of the form $\varphi(x) = Qx + a$ for an orthogonal matrix Q and an $a \in \mathbb{R}^3$, if and only if the Cauchy-Green tensor C satisfies $C = (\nabla\varphi)^\top \nabla\varphi \equiv I$. In order to show the assertion "$C = I$ implies $\varphi = Qx + a$" proceed by solving the following subproblems. For simplicity, you may assume that φ is injective, more precisely: $\varphi : \Omega \to \varphi(\Omega)$ is a C^1-diffeomorphismus[1]

a) Show that φ preserves lengths, i.e., $\|\varphi(x) - \varphi(y)\|_2 = \|x - y\|_2$ for all x, $y \in B$, where $B \subset \Omega$ is convex. *hint:* first show $\|\varphi(x) - \varphi(y)\|_2 \le \|x - y\|_2$ and then reason that also $\|x - y\|_2 \le \|\varphi(x) - \varphi(y)\|_2$.

b) Consider the auxiliary function $G(x, y) := \|\varphi(x) - \varphi(y)\|_2^2 - \|x - y\|_2^2$. By differentiating first with respect to y_i and then with respect to x_j show that

$$-\sum_k \frac{\partial\varphi_k(y)}{\partial y_i}\frac{\partial\varphi_k(x)}{\partial x_j} + \delta_{ij} = 0.$$

This implies $(\nabla\varphi(y))^\top \nabla\varphi(x) = I$.

c) Show that $\nabla\varphi$ has to be constant locally. Conclude that $\varphi(x) = Qx + a$.

[1] otherwise, one argues locally and combines the local assertions