problem sheet 7

discussion: Tuesday, 24.11.20

7.1. In high energy explosions there is a rapid release of energy E that produces an approximately spherical shock waves that expands in time (see Fig. 1).

a) Assume that the radius R (at time t) depends on the energy E, the time t, and the density ρ (of air). Use dimensional reduction to derive the so-called "Taylor-Sedov" formula for R. Determine the missing constant with the observation that for $E = 1J$ and $\rho = 1kg/m^3$ one has the relation $R = t^{2/5}m/s^{2/5}$.

b) Estimate the energy that is released in the nuclear explosion shown in Fig. 1.

c) Also a supernova can be modelled with the Taylor-Sedov formula. When did the "Tycho supernova" explode? The current radius is 7.5 light years, the density is $\rho = 2 \cdot 10^{-21}kg/m^3$, and the estimated energy release is $10^{44}J$.

7.2. An elastic ball (radius R, density ρ and elastic modulus [1] E) is dropped from a height h_0 and bounces back to height h_r. Assume that h_r is a function of h_0, R, E, ρ, and the gravity of earth g.

a) Apply dimensional reduction to derive a formula for h_r.

b) Experimentally, it is found out that h_r depends linearly on h_0 and that $h_r = 0$ for $h_0 = 0$. What is the structure of the formula now?

c) Assume that the density ρ is doubled. How do you have to modify E so that the ball bounces back to the same height?

7.3. Consider the following (regularly perturbed) equation

$$-y'' + \varepsilon y' + y = f(x), \qquad x \in (0,1), \qquad y(0) = y(1) = 0$$

for small ε.

a) Make the ansatz $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$ and derive suitable equations for the functions y_i.

b) Try to justify the expansion, i.e., show that for small ε (and under suitable smoothness assumptions on f) the finite series is a good approximation to the exact solution. *Hint:* You may use that by the Lax-Milgram lemma the solution of the problem

$$-u'' + u = g, \qquad x \in (0,1), \qquad u(0) = u(1) = 0$$

satisfies

$$\|u'\|^2_{L^2(0,1)} + \|u\|^2_{L^2(0,1)} \le \|g\|^2_{L^2(0,1)}.$$

c) (*, optional) Derive pointwise estimates for the error.

[1] this is the "spring constant" of the material and measured in units of pressure

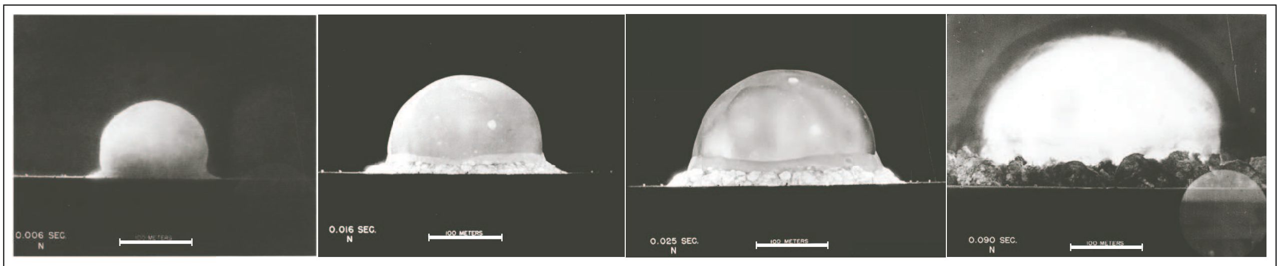

Abbildung 1: shock wave produced by a nuclear explosion (at time 6 msec, 16 msec, 25 msec, 90 msec). The white bar indicates 100m.

d) Let $f \in C^\infty(\mathbb{R})$. Then all terms y_i of the "outer expansion" exist. Does the series $\sum_{i=0}^\infty \varepsilon^i y_i$ converge?

7.4. A model for an upward throw of a ball (with *small* air resistance) that has already been non-dimensionalized is

$$y'' = -1 - \varepsilon(y'(t))^2, \qquad y(0) = 0, \quad y'(0) = 1.$$

The model describes the location of the ball up to the maximal height that is reached.

a) Compute the coefficients $y_0(t)$, $y_1(t)$ of the formal asymptotic expansion

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$$

b) Compute the maximal height the ball reaches using the terms up to order ε of the asymptotic expansion. As this is a function of ε, you may compute this up to order $O(\varepsilon^2)$.

c) An alternative view point for the asymptotic expansion $y(t) \sim y_0(t) + \varepsilon y_1(t) + \cdots$ is obtained by studying the Taylor expansion around $\varepsilon = 0$ of the function $y(t, \varepsilon)$ for fixed t:

$$y(t, \varepsilon) = y(t, 0) + \varepsilon \partial_\varepsilon y(t, 0) + \cdots$$

What are the initial value problems that are solved by $y(\cdot, 0)$ and $\partial_\varepsilon y(\cdot, 0)$? Solve them.